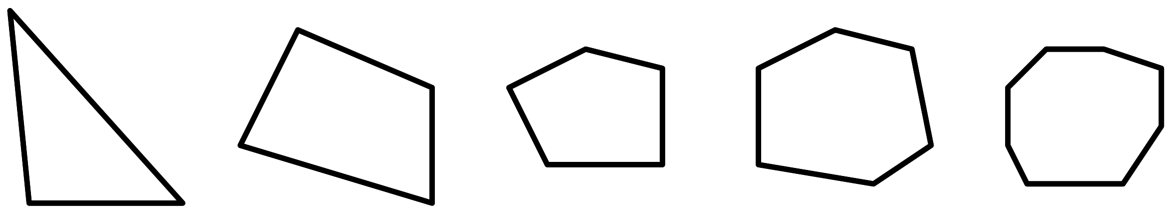
Sect 8.3 Triangles and Hexagons

Objective 1: Understanding and Classifying Different Types of Polygons.

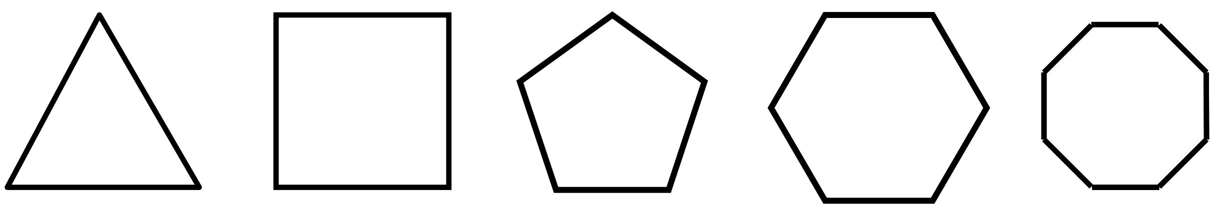
A **Polygon** is closed two-dimensional geometric figure consisting of at least three line segments for its sides. If all the sides are the same length (the angles will also have the same measure), then the figure is called a **Regular Polygon**. The points where the two sides intersect are called vertices.

Triangle	**Quadrilateral**	**Pentagon**	**Hexagon**	**Octagon**
3 sides	4 sides	5 sides	6 sides	8 sides



Regular Polygons

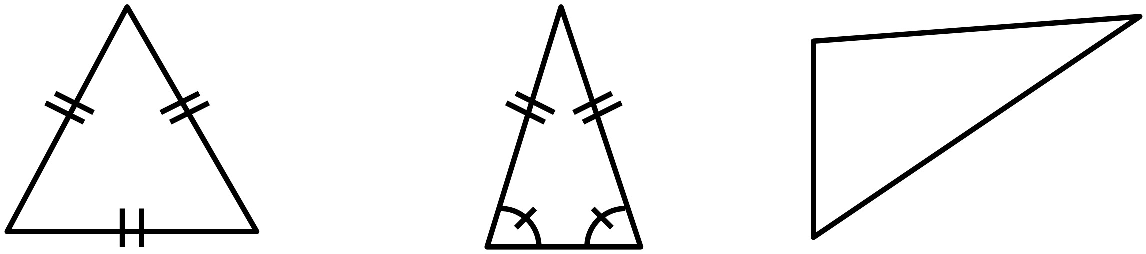
Triangle	**Quadrilateral**	**Pentagon**	**Hexagon**	**Octagon**



Objective 2: Understanding and Classifying Different Types of Triangles.

We can classify triangles by their sides:

Equilateral Triangle	**Isosceles Triangle**	**Scalene Triangle**
All sides are equal.	Two sides are equal.	No sides are equal.

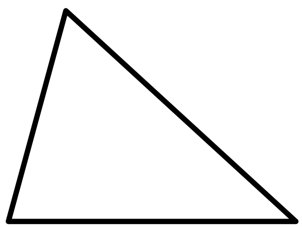


All angles measure 60˚.	The two base angles are equal.	No angles are equal.

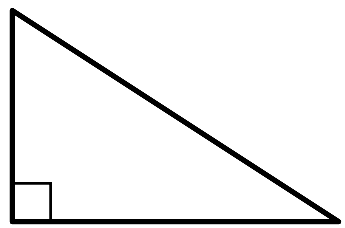
An equilateral triangle is also called an **equiangular triangle**. The third angle in an isosceles triangle formed by the two equal sides is called the **vertex angle**.

We can also classify triangles by their angles:

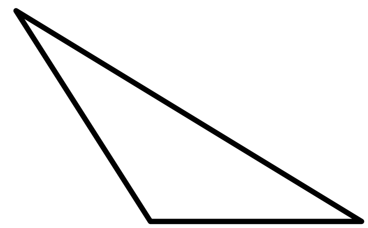
Acute Triangle
All angles are acute.



Right Triangle
Has on right angle.



Obtuse Triangle
Has one obtuse angle.

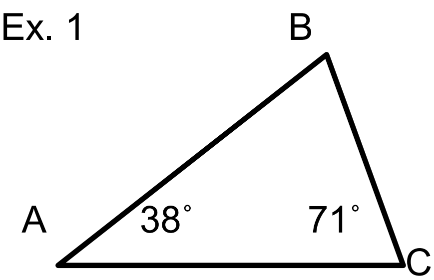


In a right triangle, the two sides that intersect to form the right angle are called the **legs** of the right triangle while the third side (the longest side) of a right triangle is called the **hypotenuse** of the right triangle.

Recall that the sum of the measures of the angles of a triangle is equal to 180˚.

<u>Determine what type of triangle is picture below:</u>

Ex. 1

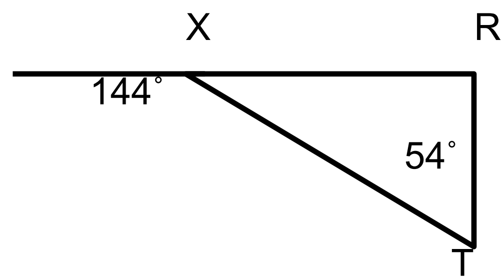


<u>Solution:</u>
Since the measures of the angles of a triangle total 180˚, then

$38^\circ + m\angle B + 71^\circ = 180^\circ$
$m\angle B + 109^\circ = 180^\circ$
$\underline{-109^\circ = -109^\circ}$
$m\angle B = 71^\circ$.

So, ΔABC is an Isosceles and an Acute Triangle.

Ex. 2



<u>Solution:</u>
Since $\angle TXR$ and 144˚ are supplementary angles, then
$m\angle TXR = 180^\circ - 144^\circ = 36^\circ$.
But, $m\angle TXR + m\angle T = 54^\circ + 36^\circ = 90^\circ$. So, $\angle R = 180^\circ - 90^\circ = 90^\circ$.
Since the angles are different, then none of the sides are equal.
Thus, ΔXRT is a scalene and a right triangle.

Objective 3: Right Triangles and the Pythagorean Theorem

To find the square root of a number on a scientific calculator, we will use the $\sqrt{x}$ key.

Simplify the following (round to the nearest thousandth):

Ex. 3a $\sqrt{49}$

Ex. 3b $\sqrt{144}$

Ex. 3c $\sqrt{0}$

Ex. 3d $\sqrt{\frac{625}{64}}$

Ex. 3e $\sqrt{15}$

Ex. 3f $3\sqrt{7}$

Solution:

a) $\sqrt{49} = 7.$

b) $\sqrt{144} = 12.$

c) $\sqrt{0} = 0.$

d) $\sqrt{\frac{625}{64}} = \frac{25}{8}.$

e) Use your calculator: $\sqrt{15} = 3.8729833... \approx 3.873.$

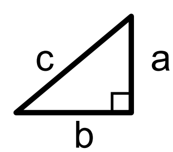
f) Use your calculator. First, find $\sqrt{7}$ and then multiply by 3:
$3\sqrt{7} = 3(2.64575...) = 7.93725... \approx 7.937.$

In a right triangle, there is a special relationship between the length of the legs (a and b) and the hypotenuse (c). This is known as the Pythagorean Theorem:

Pythagorean Theorem

In a right triangle, the square of the hypotenuse (c^2) is equal to the sum of the squares of the legs ($a^2 + b^2$)

$$c^2 = a^2 + b^2$$



Some alternate forms of the Pythagorean Theorem are:

$$c = \sqrt{a^2 + b^2}$$

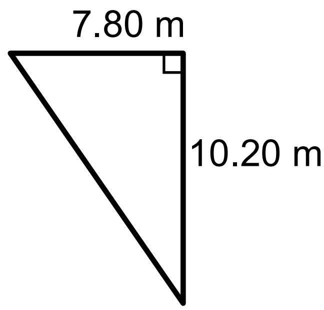
$$a = \sqrt{c^2 - b^2}$$

$$b = \sqrt{c^2 - a^2}$$

Keep in mind that the hypotenuse is the longest side of the right triangle.

<u>Find the length of the missing sides:</u>

Ex. 4



<u>Solution:</u>

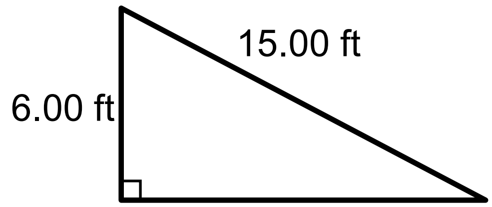
In this problem, we have the two legs of the triangle and we are looking for the hypotenuse:

$$c^2 = a^2 + b^2$$
$$c^2 = (7.8)^2 + (10.2)^2$$
$$c^2 = 60.84 + 104.04$$
$$c^2 = 164.88$$

To find c, take the square root* of 164.88:

$$c = \pm\sqrt{164.88} = \pm\ 12.8405...$$
$$c \approx 12.8 \text{ m}$$

Ex. 5



<u>Solution:</u>

In this problem, we have one leg and the hypotenuse and we are looking for the other leg:

$$c^2 = a^2 + b^2$$
$$(15)^2 = (6)^2 + b^2$$
$$225 = 36 + b^2 \quad \text{(solve for } b^2\text{)}$$
$$\underline{-\ 36 = -\ 36}$$
$$189 = b^2$$

To find b, take the square root of 189:

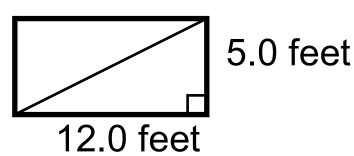
$$b = \pm\sqrt{189} = \pm\ 13.7477...$$
$$b \approx 13.7 \text{ ft}$$

* - The equation $c^2 = 164.88$ actually has two solutions ≈ 12.8 and ≈ – 12.8, but the lengths of triangles are positive so we ignore the negative solution.

<u>Find the length of the diagonal of the following rectangle:</u>

Ex. 6 A rectangle that is 12 feet by 5 feet.

<u>Solution:</u>



First, draw a picture. Since the angles of a rectangle are right angles, then the two sides are the legs of a right triangle while the diagonal is the hypotenuse of the right triangle. Using the Pythagorean Theorem, we get:

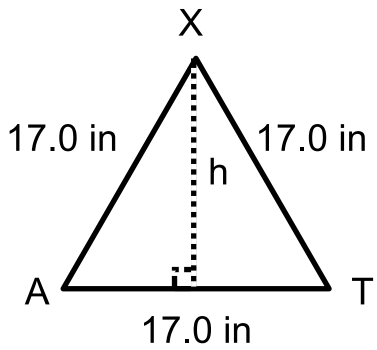
$$c^2 = a^2 + b^2$$
$$c^2 = (12)^2 + (5)^2$$
$$c^2 = 144 + 25 = 169$$
$$c = \pm\sqrt{169} = \pm\ 13.$$

So, the diagonal is 13 feet.

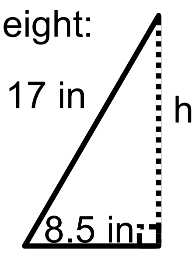
Solve the following:

Ex. 7 Find the height of ΔAXT if each side has a length of 17.0 in.

Solution:



The altitude (height) of the triangle forms a right angle with the base of the triangle and divides the base into two equal parts. Thus, the distance from point A and this point of intersection is 17 in $\div$ 2 = 8.5 in. The right half of the triangle together with the altitude form a a right triangle with a hypotenuse of 17 in, one leg equal to 8.5 in, and the other leg equal to the height of the equilateral triangle being the length of the other leg. Using the Pythagorean Theorem, we can find the height:



$17^2 = (8.5)^2 + h^2$

$289 = 72.25 + h^2$ (– 72.25 from both sides)

$216.75 = h^2$ (solve for h)

$h = \pm\sqrt{216.75} = \pm 14.7224...$

So, the height is about 14.7 inches.

In general, the formula for the height of an equilateral triangle with side of length a is $h = \frac{a\sqrt{3}}{2}$.

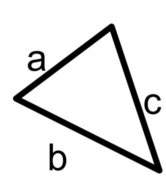
Objective 4: Understanding Heron's Formula.

Recall the area of a triangle is equal to one-half the base times the height. This formula works well if we have the base and the height of the triangle. But, if we only know the lengths of the sides of the triangle, we can use Heron's formula to find the area. The key to using Heron's Formula is to first find the perimeter of the triangle and then divide the rest by two to get the semi-perimeter. We then plug the values into the formula.

Heron's Formula

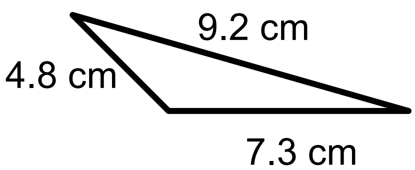
Let a, b, and c be the lengths of the triangle and s be the semi-perimeter of the triangle. Then the area of the triangle is

$$A = \sqrt{s(s-a)(s-b)(s-c)} \qquad \text{where } s = \frac{a+b+c}{2}$$

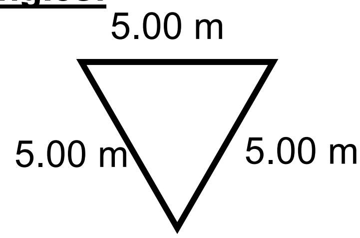


<u>Find the area of the following triangles:</u>

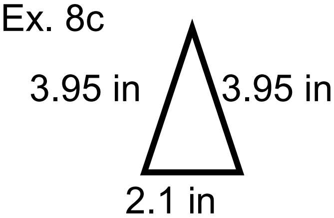
Ex. 8a



Ex. 8b



Ex. 8c



<u>Solution:</u>

a) First, calculate the semi-perimeter:

$s = (4.8 + 7.3 + 9.2)/2 = 21.3/2 = 10.65$ cm

Now, plug into Heron's Formula:

$A = \sqrt{10.65(10.65-4.8)(10.65-7.3)(10.65-9.2)}$ (subtract)

$= \sqrt{10.65(5.85)(3.35)(1.45)}$ (multiply)

$= \sqrt{302.6334...}$ (square root)

$= 17.3963\ldots$ (round to two significant digits)

$\approx 17\ \text{cm}^2$

b) First, calculate the semi-perimeter:

$s = (5 + 5 + 5)/2 = 15/2 = 7.5$ cm

Now, plug into Heron's Formula:

$A = \sqrt{7.5(7.5-5)(7.5-5)(7.5-5)}$ (subtract)

$= \sqrt{7.5(2.5)(2.5)(2.5)}$ (multiply)

$= \sqrt{117.1875}$ (square root)

$= 10.825\ldots$ (round to three significant digit)

$\approx 10.8\ \text{m}^2$

c) First, calculate the semi-perimeter:

$s = (3.95 + 3.95 + 2.1)/2 = 10/2 = 5$ in

Now, plug into Heron's Formula:

$A = \sqrt{5(5-3.95)(5-3.95)(5-2.1)}$ (subtract)

$= \sqrt{5(1.05)(1.05)(2.9)}$ (multiply)

$= \sqrt{15.98625}$ (square root)

$= 3.9982\ldots$ (round to two significant digits)

$\approx 4.0\ \text{in}^2$

In general, we can use Heron's Formula to derive the formulas for the area of special types of triangles. For instance, suppose we want to find the formula for the area of an equilateral triangle. Let a = the length of one side of an equilateral triangle. Then the semi-perimeter is:

$$s = (a + a + a)/2 = \frac{3}{2}a$$

and the area is:

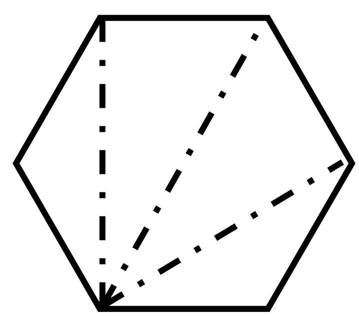
$$A = \sqrt{\frac{3}{2}a(\frac{3}{2}a - a)(\frac{3}{2}a - a)(\frac{3}{2}a - a)} \qquad \text{(subtract)}$$

$$= \sqrt{\frac{3}{2}a(\frac{1}{2}a)(\frac{1}{2}a)(\frac{1}{2}a)} \qquad \text{(multiply)}$$

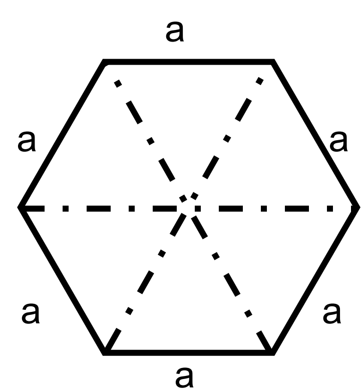
$$= \sqrt{\frac{3}{16}a^4} \qquad \text{(square root)}$$

$$= \frac{a^2\sqrt{3}}{4}$$

Objective 5: Understanding Properties of Regular Hexagons.



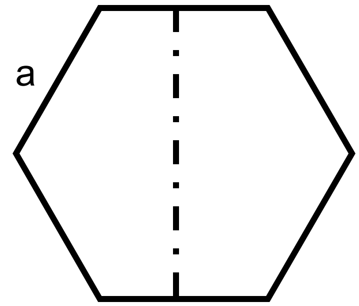
In a regular hexagon, all the sides are equal and all the angles are equal. To find the measure of the angles of a regular hexagon, we can split the regular hexagon into a series four of triangles as shown. This means that the sum of the measures of the angles of a hexagon is equal to four times the sum of the measures of the angles of a triangle or 4•180˚ = 720˚. Since the angles of a regular hexagon are equal, then each angle is equal to 720˚ ÷ 6 = 120˚.



To find the area of a regular hexagon, first draw the diagonals of the regular hexagon. Each diagonal bisects the angle of the hexagon into two 60˚ angles. Thus, the regular hexagon consists of six equilateral triangles. Since the length of of each side is equal to a, then the area of a regular hexagon is six times the area of an equilateral triangle or $6 \bullet \frac{a^2\sqrt{3}}{4} = \frac{3a^2\sqrt{3}}{2}$.

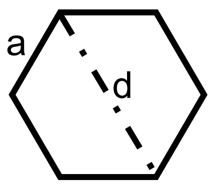
This also implies that the length of the diagonal of a regular hexagon, d, is equal to 2a. This is also referred to as the distance across the corners.

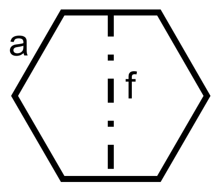
One final measurement of a regular hexagon we will need is the distance across the flats, f. Notice that f is twice the height of an equilateral triangle formed by the diagonals. As noted in example #7, the formula for finding the height of an equilateral triangle is $h = \frac{\sqrt{3}}{2}a$.



Hence, $f = 2 \bullet h = 2 \bullet \frac{a\sqrt{3}}{2} = a\sqrt{3}$. Let's summarize our findings:

<u>Properties of Regular Hexagons with side of length a:</u>





1) Each angle measures 120˚.
2) Distance across the corners (diagonal): $d = 2a$
3) Distance across the flats: $f = a\sqrt{3}$
4) The area: $A = \frac{3a^2\sqrt{3}}{2}$

<u>Complete the following table:</u>

Ex. 9

	a	**d**	**f**	**A**
a)	12 in			
b)		$\frac{5}{8}$ cm		
c)			15.32 mm	

<u>Solution:</u>

a) Since a = 12 in, then

$d = 2a = 2 \bullet 12 = 24$ in

$f = a\sqrt{3} = 12\sqrt{3} = 20.7846\ldots \approx 21$ in

$A = \frac{3a^2\sqrt{3}}{2} = \frac{3(12)^2\sqrt{3}}{2} = 374.1229\ldots \approx 370 \text{ in}^2$

b) Since $d = \frac{5}{8}$ cm = 0.625 cm, then

$d = 2a$ implies $0.625 = 2a$ (divide by 2)

$a = 0.3125 \approx 0.313$ cm

$f = a\sqrt{3} = 0.3125\sqrt{3} = 0.54126\ldots \approx 0.541$ cm

$$A = \frac{3a^2\sqrt{3}}{2} = \frac{3(0.3125)^2\sqrt{3}}{2} = 0.253718\ldots \approx 0.254 \text{ cm}^2$$

c) Since f = 15.32 mm, then

$f = a\sqrt{3}$ implies $15.32 = a\sqrt{3}$ (divide by $\sqrt{3}$)

$a = 8.845006\ldots \approx 8.845$ mm

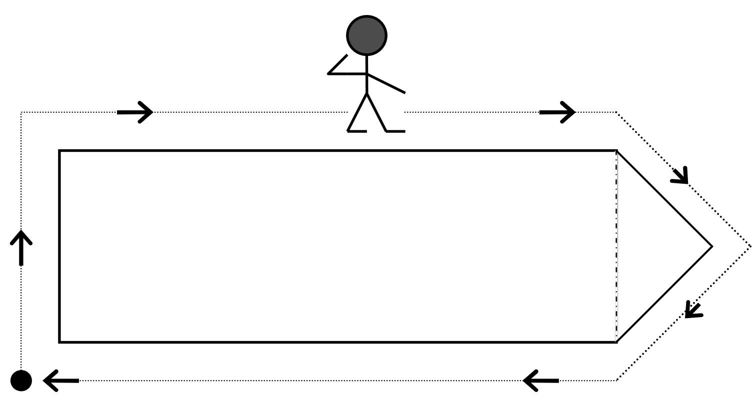
$d = 2a = 2 \bullet 8.845006\ldots = 17.690012\ldots \approx 17.69$ mm

$$A = \frac{3a^2\sqrt{3}}{2} = \frac{3(8.845006\ldots)^2\sqrt{3}}{2} = 203.258\ldots \approx 203.3 \text{ mm}^2$$

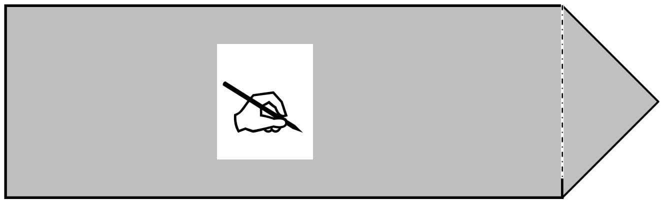
	a	**d**	**f**	**A**
a)	12 in	24 in	21 in	370 in^2
b)	0.313 cm	$\frac{5}{8}$ cm	0.541 cm	0.254 cm^2
c)	8.845 mm	17.69 mm	15.32 mm	203.3 mm^2

Objective 6: Understanding Composite Figures

A **<u>Composite Figure</u>** is a geometric figure that consists of two or more basic geometric figures. We want to find the perimeter and area of composite figures. The perimeter of any figure is the total length around the outside of the figure. Think of a person walking along the outside of the figure; how far does the person have to travel to get back to the beginning?

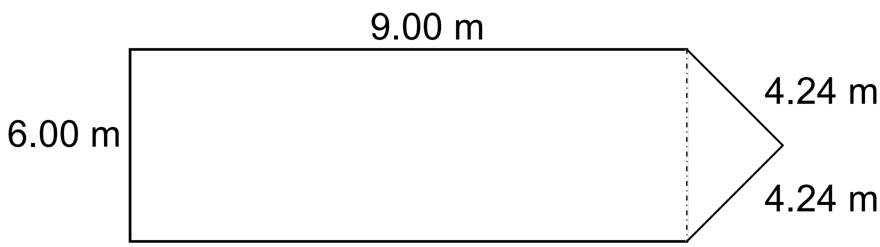


In terms of the area of the composite figure, think of how much paint is needed to paint the inside of the figure.



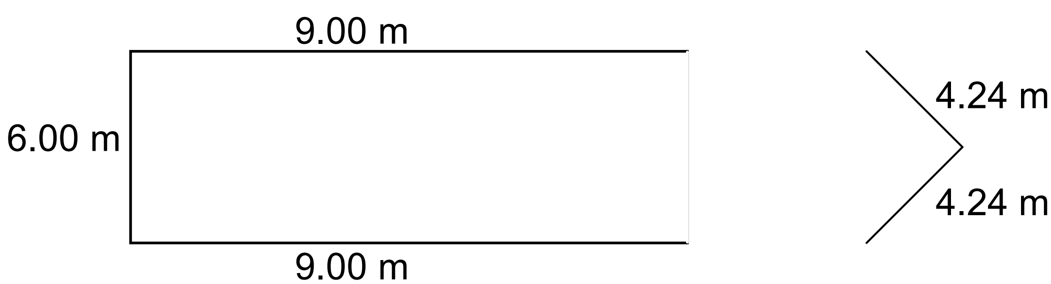
The key to solving a composite figure problem is to split it apart into a set of basic figures. Let's look at some examples.

Ex.10 a) Find the perimeter of the following.
b) Find the area of the following.



Solution:

a) We begin by splitting the figure apart:



The perimeter of this figure is the sum of the lengths of the three sides of the rectangle plus the two sides of the triangle. Thus, the perimeter is:

P = (9 + 6 + 9 + 4.24 + 4.24) meters = 32.48 meters

b) We begin by splitting the figure apart:

The area of this figure is the sum of the area of the rectangle and the area of the triangle. The area of the rectangle is L•w = 6• 9 = 54 m^2. To find the area of the triangle, we will use Heron's Formula:

s = (6 + 4.24 + 4.24) ÷ 2 = 14.48 ÷ 2 = 7.24 m

$A = \sqrt{7.24(7.24-6)(7.24-4.24)(7.24-4.24)}$ (subtract)

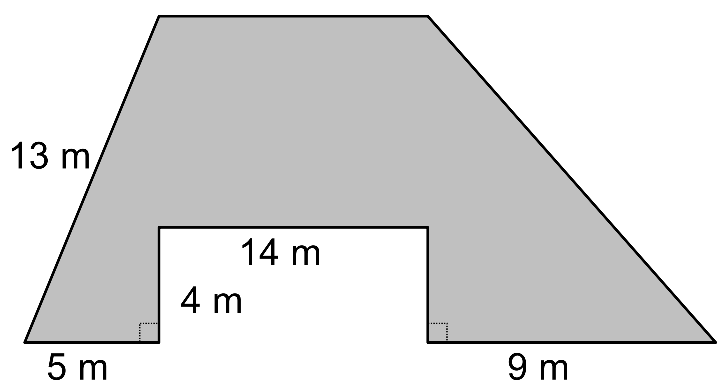
$= \sqrt{7.24(1.24)(3)(3)}$ (multiply)

$= \sqrt{80.7984}$ (square root)

$= 8.98879\ldots\ m^2$

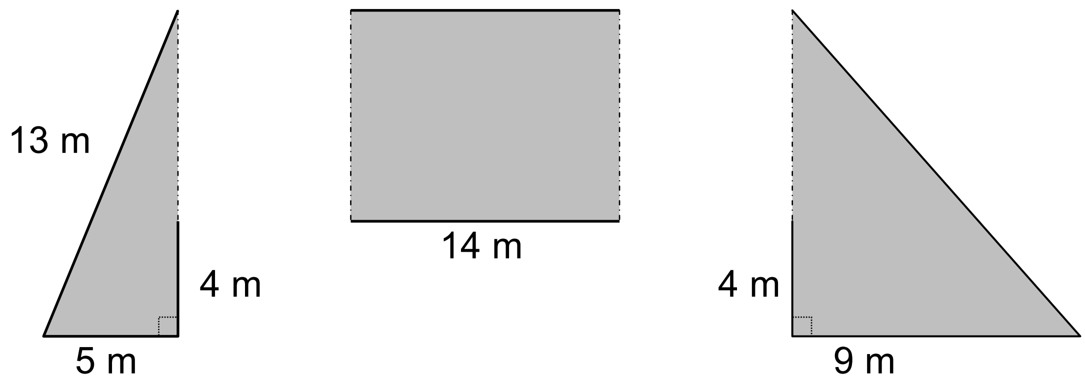
Thus, the total area = 54.0 + 8.988… = 62.98… ≈ 63.0 m^2.

Ex. 11 Find the area of the following figure:



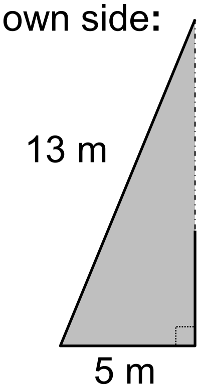
Solution:

First, we will need to split the figure apart and determine if there any dimensions we will need to find:

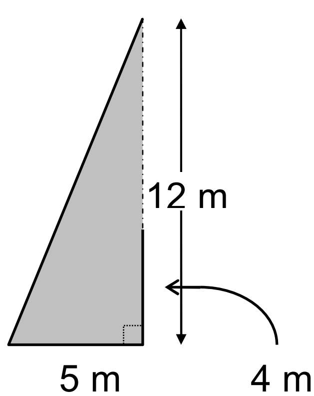


We need to find the height of the triangles and the width of the rectangle. Notice that the width of the rectangle is 4 m less than the height of the triangles.

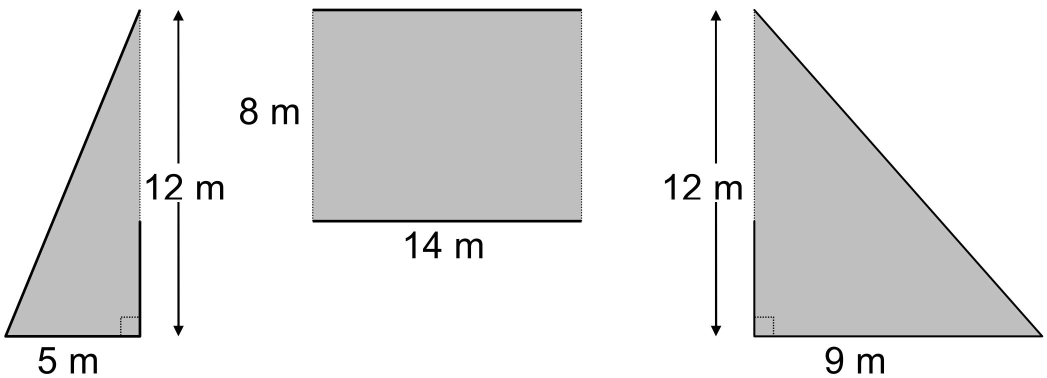
Consider the triangle on the left. We have the hypotenuse and a leg of a right triangle. We can use the Pythagorean Theorem to solve for the unknown side**:**



$a^2 + 5^2 = 13^2$
$a^2 + 25 = 169$
$a^2 = 144$
$a = \pm\sqrt{144}$, so a = 12 m since a > 0.
Thus, the heights of the two triangles are 12 m. Now, we can find the width of the rectangle.



The dashed side corresponds to the width of the rectangle in the middle. Since the height of the triangle is 12 m and the length of side from the base of the triangle to the rectangle is 4 m, then the width of the rectangle is
12 – 4 = 8 m.
We can find the area of the composite figure by adding the areas of the triangle and the area of the rectangle**:**

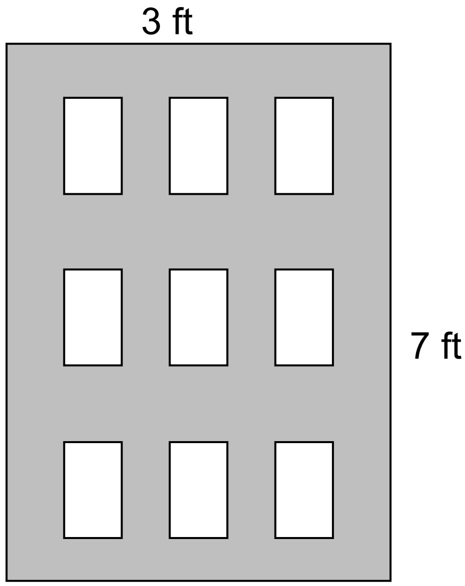


$A = \frac{1}{2}bh + Lw + \frac{1}{2}bh$

$= [\frac{1}{2}(5)(12) + 14(8) + \frac{1}{2}(9)(12)]\ m^2$

$= [30 + 112 + 54]\ m^2 = 196\ m^2 \approx 200\ m^2$.

Ex. 12 A wooden door has 9 rectangular pieces of glass embedded in it. If each piece of glass is 1/3 ft by 1/2 ft, find the area of the wood. Round to the nearest half-foot. (See figure)



Solution:

The area of the wood is equal to the area of the door minus the area of the nine pieces of glass:

$A = Lw - 9Lw$

$A = 3(7) - 9(1/3)(1/2) = 21 - 1\frac{1}{2} = 19\frac{1}{2}\ ft^2$