# Locally Optimal Tests for multivariate parameter with order-restricted alternatives

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## Problem set-up

Statistical experiment

$$X_1, \dots, X_n \sim P_\theta, \quad \theta \in \Theta \subset \mathbb{R}^m$$

Hypotheses

$$H_0 : \theta = \theta_0$$

$$H_1 : \theta \geq \theta_0, \ |\theta - \theta_0| > 0$$

- $m = 1$: In many cases the uniformly most powerful (UMP) tests exist.
- $m > 1$: Typically no UMP
  - For this type of order-restricted alternatives there is no evidence of superiority of Likelihood Ratio Test (LRT), at least in "large sample" set-up (see examples of inferiority of LRT below).
  - Bayesian methods do not deal *directly* with the maximization of power function (to be considered at a later time).
  - There is a method based *directly* on power function in this problem: Most Stringent Somewhere Most Powerful Tests (MSSMP) (Shi & Kudô, 1987; Shi, 1987).

## Local Optimization

Let $\varphi$ be any (randomized) statistical test based on the sample $X = (X_1, \dots, X_n)$.

Power function:

$$P(\theta; \varphi) = E_\theta \varphi(X)$$

Gradient of the power function (we will need it only at $\theta = \theta_0$):

$$P'_\theta(\theta, \varphi) = (E_\theta \varphi(X))'_\theta \, .$$

Regularity condition:

C1. There exists a statistics $T(X)$ such that

$$P'_\theta(\theta_0, \varphi) = E_{\theta_0} \varphi(X) T(X)$$

Typically $T(X) = (\log f_\theta(X))'_\theta \big|_{\theta=\theta_0}$.

Let $\gamma$ be any direction. The derivative of the power function in this direction is

$$\gamma^T P'_\theta(\theta_0; \varphi).$$

Let's consider the tests such that

$$P(\theta_0; \varphi) \leq \alpha.$$

It is easy to see that, in this class, the test giving the maximum to the directional derivative has the following form:

$$\varphi_\gamma = I_{\{\gamma^T T > c\}} + \epsilon I_{\{\gamma^T T = c\}},$$

where $c$ is some critical constant and $\epsilon$ is some randomization constant defined by $\alpha$.

Now, we would like to maximize the minimum value of the directional derivatives of these tests.

So, our problem is to *maximize*

$$\inf_\beta \beta^T P'_\theta(\theta_0; \varphi_\gamma)$$

*over all directionally optimal tests* $\varphi_\gamma$ *subject to*

$$P(\theta_0; \varphi_\gamma) \leq \alpha.$$

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Below we give some examples of feasibility of this approach for some parametric families.

Bivariate normal distribution

For simplicity, without loss of generality, suppose that the covariance matrix is $\mathbb{I}$.

$$X \sim \mathcal{N}(\theta, \mathbb{I}), \quad \theta \in \mathbb{R}^2$$

$$H_0 : \theta = 0$$

$$H_1 : \theta \geq 0, \ |\theta| > 0$$

Locally optimal test (in the above sense) for the bivariate normal distribution has the form:

$$X_1 + X_2 > c,$$

where

$$c = \Phi^{-1}(1-\alpha)\sqrt{2}$$

This test coincides with MSSMP for this parametric family (Shi & Kudô, 1987).
This test is locally superior to LRT, as shown by Tsai & Sen, 1993.
We show that, more than that, for any $\alpha$ its efficiency

$$\mathcal{E}(\alpha) = \frac{\beta^t P'_\theta(\theta_0, \varphi_\beta)}{\beta^t P'_\theta(\theta_0, \varphi_{\mathrm{LRT}})} > \frac{\pi}{2\sqrt{2}} \approx 1.11072$$

with

$$\lim_{\alpha \to 0} \mathcal{E}(\alpha) = \frac{\pi}{2\sqrt{2}}$$

## Asymptotically Locally Optimal Tests

Locally Asymptotically Normal (LAN) Families

$$\prod_{j=1}^{n} \frac{f_{\theta+\frac{u}{\sqrt{n}}}(X_j)}{f_\theta(X_j)} = e^{(\Delta_n,u)-\frac{1}{2}(I(\theta)u,u)+\psi_n}$$

$\Delta_n \Rightarrow \mathcal{N}(0, I(\theta))$, and $\psi_n \xrightarrow{P_\theta} 0$, as $n \to \infty$.

For regular families $\Delta_n = \frac{1}{\sqrt{n}} T_n$ (score function), and $I(\theta)$ is the Fisher information matrix.

We are interested in testing

$$H_0 : \theta = \theta_0$$

vs

$$H_1 : \theta \geq \theta_0, \ |\theta - \theta_0| > 0$$

To construct an Asymptotically Locally Optimal Test in this problem, we need the Locally Optimal Test for the following problem of testing hypotheses in an "auxiliary" multivariate normal distribution family.

Let $X \sim \mathcal{N}(I(\theta)u, I(\theta))$. Let us construct the Locally Optimal Test $\varphi^*_{\gamma^*} = \varphi^*_{\gamma^*}(X)$ (as above) for testing $H_0 : \ u = 0$ vs $H_0 : \ u \geq 0$, $|u| > 0$ and let $c_\alpha$ be its critical constant. Now, for the LAN family, let

$$\varphi^*_n = I_{\{\gamma^{*T}\Delta_n > c_\alpha\}}.$$

Then, for any "directionally locally optimal" test $\varphi_{n,\gamma} = \varphi_{n,\gamma}(X_1, \cdots, X_n)$ such that

$$\overline{\lim}_{n\to\infty} E_{\theta_0} \varphi_{n,\gamma} \leq \alpha$$

we have

$$\underline{\lim}_{n\to\infty}\frac{\inf_\beta \beta^T E_{\theta_0}\varphi_n^* T_n}{\inf_\beta \beta^T E_{\theta_0}\varphi_{n,\gamma} T_n} \geq 1.$$

This property can be considered as a form of the asymptotic local optimality of the test $\varphi_n^*$.

## <u>An example: Generalized Gamma Distribution</u>

This is a family with p.d.f.

$$f(x;d,\delta) = \frac{1+\delta}{\Gamma\left(\frac{1+d}{1+\delta}\right)}\, x^d e^{-x^{1+\delta}},\ d > -1, \delta > -1$$

$$\theta = (d,\delta)$$

Important: it is *not* an exponential family. So even in a given direction, there may not exist a UMP test!

Now, let

$$X_1,\ldots,X_n \sim G.G.D.(d,\delta)$$

The hypothesis

$$H_0 : d = 0, \delta = 0$$

against

$$H_1 : d \geq 0, \delta \geq 0,\ |d| + |\delta| > 0$$

is tested.

In this case, the score function

$$T = \left(\sum \log X_i, \sum X_i \log X_i\right).$$

Following the procedure above, we compare the Locally Optimal Test for the G.G.D. with the Locally Optimal Test in the "auxiliary" bivariate normal problem. The results are in the table below.

The numbers given in the table are the minimum of the directional derivative corresponding to the optimal direction. The last column $n = \infty$ is for the "limiting" bivariate normal case.

**The results of numerical (Monte Carlo) comparison of the two tests.**

| $\alpha$ | n=10 | n=20 | n=30 | n=50 | n=100 | n=200 | n=$\infty$ |
|---|---|---|---|---|---|---|---|
| 0.05 | 0.0505 | 0.0581 | 0.0588 | 0.0553 | 0.0510 | 0.0617 | 0.0678 |
| 0.02 | 0.0201 | 0.0238 | 0.0233 | 0.0265 | 0.0262 | 0.0333 | 0.0316 |
| 0.01 | 0.0133 | 0.0143 | 0.0154 | 0.0107 | 0.0160 | 0.0153 | 0.0172 |
| 0.005 | 0.0036 | 0.0060 | 0.0053 | 0.0076 | 0.0063 | 0.0101 | 0.0093 |

**References**


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