Math1107B   Test #2   **Feb 11th 2005**

Each question is worth 10 marks.

1. Consider the function $T : \mathbb{R}^3 \to \mathbb{R}$ defined for $x \in \mathbb{R}^3$ by

$$T(x) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = 2 + x_1 + 2x_2 + 4x_3$$

(a) Let $c \neq 0$ be some non-zero number. What is the image of cx under T?

(b) What is $cT(x)$?

(c) Is T linear? Why?

2. With **the least amount of work** determine if the following sets are independent.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

3. Suppose B is a 3×4 matrix with columns $B = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}$.

(a) Suppose A is $m \times n$, for what values of m and n does AB make sense?

(b) Suppose m and n are such that AB makes sense, what are the columns of AB?

(c) What does it mean for the columns of B to be dependent?

(d) Suppose the columns of B are dependent, use your answers to the previous parts to show that the columns of AB are also dependent.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^5$ be a linear transformation.

(a) Let A be the standard matrix for T; how many rows and columns does A have?

(b) Is it possible for T to be onto? Why?

(c) How many pivots does A need in order to be one-to-one? Why?

1. Q1

   (a) $T(cx) = 2 + cx_1 + 2cx_2 + 4cx_3$

   (b) $cT(x) = 2c + cx_1 + 2cx_2 + 4cx_3$

   (c) No, T is not linear, since $T(cx) \neq cT(x)$.

2. Q2

   (a) Independent since the two vectors are not multiples of each other.

   (b) Dependent since there are more vectors in the set than the size of the vectors.

   (c) Dependent since any set containing the zero set is dependent.

3. Q3

   (a) m can be anything, but $n = 3$.

   (b) The columns of AB are Ab_1, Ab_2, Ab_3, Ab_4.

   (c) The columns of B are dependent if there exist numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ not all zero such that $\alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 + \alpha_4 b_4 = 0$

   (d) Multiplying thru' by A we have

   $$\alpha_1 Ab_1 + \alpha_2 Ab_2 + \alpha_3 Ab_3 + \alpha_4 Ab_4 = 0$$

   where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are not all zero. Therefore the columns of AB are dependent.

4. Q4

   (a) A is 5×2.

   (b) No, in order for T to be onto the columns of A must span $\mathbb{R}^5$. This requires A to have a pivot in every row, but A can have at most two pivots.

   (c) A must have pivots in every column OR no free variables in $Ax = 0$. This means two pivots.