I-WAVELETS AND THEIR APPLICATIONS

A Birds Eye View

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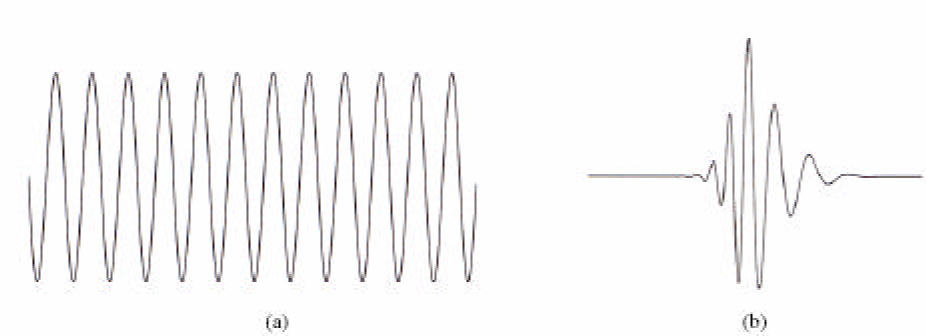
Abstract:

Wavelet analysis is a time or space-scale representation and analysis of signals which has found a wide range of applications in applied mathematics, physics and engineering in the last few years. In order to get a feeling for it and to understand its success, some basic facts are provided. A glimpse of the chronological evolution and a survey of the various applications are given. Finally, an attempt is made to compare and contrast wavelet transform with Fourier transform, which is usually seen as a universal tool.

1.1 A BRIEF HISTORY:

The history of wavelets can be traced to many ideas developed in pure and applied mathematics, Physics and Engineering. Way back in 1910, the mathematician Alford Haar was the first to produce a complete orthonormal set for the Hilbert Space $L^2(R)$ the elements of which are the in a sense building blocks of wavelet theory. However the interest in the field activated only during early 1980's, beginning with the work of J. Morlet (1982). The results obtained by him though encouraging were not well received by the mathematical community. It was A. Grossman (1984) who laid a firm foundation to the theory. His work besides gaining mathematical respectability triggered active research in the field. The main breakthrough came only in the late 1980's with an axiomatic treatment of Multiresolution analysis by Mallat and Meyer (1986) and the method of construction of orthonormal wavelets having compact support by Ingrid Daubeachies (1987). It is because of the contributions made by these scholars and many others wavelets theory stands today as a discipline in its own right sharing borders with scientific computing, signal and image processing, Data compression (to name a few). It has been one of the major research domains in science and Engineering in the last decade and is still undergoing rapid growth.

1.2 BASIC FACTS:



Demonstration of (a) a wave and (b) a wavelet

Basic wavelet theory includes concepts from real and complex analysis linear algebra, Fourier analysis and Numerical analysis. In this respect it mimics traditional as well as modern mathematics, which is becoming increasingly interdisciplinary. The approximation (the representation) of an arbitrary known or unknown function f by means of special functions can be viewed as a central theme of wavelet theory. So one of the FAQ's (frequently asked questions) in wavelet theory is when and in what sense it is true that $f = \sum_{k \in Z} c_k f_k$ ------- (1) This equation, figuratively speaking is a decomposition as well as reconstruction formula and forms a basis of many applications of wavelet theory. What we mean by this is "given a function f we can encode it by means of $\{c_k\}$ and equation (1) allows us to reconstruct it from c_k's and f_k's the so called basis functions. Some basis functions, in particular wavelet bases are found to do this job more efficiently than others. There are two ways of representing a function (in wavelet theory function and signal are used interchangeably) analog and digital. Analog refers to continuous and digital refers to discrete. Any function in analog format can be converted to digital form by sampling at evenly spaced points. Physically we can think of an audio signal e.g. a piece of speech or music and a. visual-signal (still or moving) e.g. B&W (color) photograph, a finger print or moving image on a T.V. screen.

In order to fix ideas, we consider a function $f : \Re \to C$ assuming that f is differentiable infinitely many times in an nhd of $a \in \Re$. Such a function can be approximated (can infact be represented accurately) under suitable conditions using Taylor's series. In the general set up depending on the situation at hand one chooses a family of basis functions, $\{f_a\}_{a \in I}$ I may be discrete or continuous. An approximation of

f by means of f_a 's then has the form $f(t) = \sum_{k=1}^{N} c_k f_{ak}(t)$ with coefficients c_k to be determined and a representation of f has the form $f(t) = \sum_{a \in I} c_a f_a(t)$ --------------------- (2)

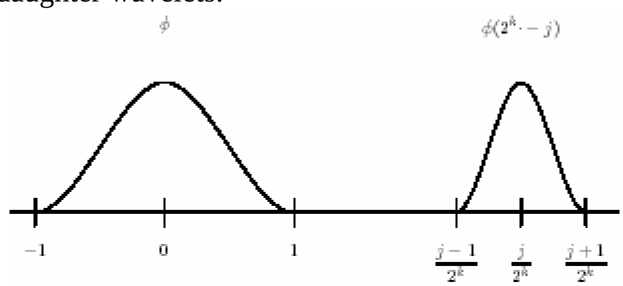
or it appears as an integral $f(t) = \int_I da\, c(a) f_a(t)$ ---------------------------- (3)

In addition to Taylor's expansion Tchebyschev's approximation also deserves mention in this context. It may be noted that the coefficients are easy to determine if the basis functions are orthonormal. Before we proceed any further another issue needs to be addressed namely "discretization of the function , from the stand point of application" ,It goes without saying that for the numerical work discretization becomes essential and can be accomplished by computing the values of the function at discrete places t = kr ($k \in Z, r > 0$ fixed). The most important tool in the construction of wavelet theory is Fourier –analysis. Any reasonable function ($2p$ -periodic) is actually represented by its Fourier –series. $\sum_{-\infty}^{\infty} c_k e^{ikt}$ The system $e_k(t) = e^{ikt}, k \in Z$ is already discrete. Incidentally, in wavelet theory $f : \Re \to C$ is referred to as time –frequency signal. Using this terminology there are only integer frequencies k. If we discretise with respect to the time t also the discrete Fourier transform (DFT) is obtained. The DFT has received an enormous boost especially after the invention of a fast algorithm, known by the name Fast Fourier Transform (FFT).

1.3 FOURIER TRANSFORM V/S WAVELET TRANSFORMS:

Fourier transform on R has its goal, the analysis of the signal $f : \Re \to C$ using $\{e_a\}$ where $e_a(t) = e^{iat}$ as basis functions. If $f \in L^2(R)$ then $\hat{f}(a) = \frac{1}{\sqrt{2p}} \int_{-\infty}^{\infty} f(t) e^{-iat} dt$ is called the Fourier transform of f. An individual value $\hat{f}(a)$ may be viewed as the complex amplitude by which the frequency a is present in the signal. But there is no localization w.r.t. 't' meaning, one cannot tell from $\hat{f}(a)$ at which time the ' note 'a was played . Thus signal analysis using FT is for from satisfactory. In the field of image processing, one would like to make use of two – dimensional FT. Imagine e.g. a picture of landscape. In different areas of the image you see different textures (a forest, a lake, clouds and so on). These textures can be analyzed by using $\hat{f} : \Re^2 \to C$ of this image. Again from looking at $\hat{f}$ you might be able to tell which kinds of texture occur in the original picture, but definitely not where. So what is clear from the above discussion is, FT based analysis of signal is global in character. It is unable to provide information regarding different aspects of the signal locally. Our ultimate goal should therefore be to search for a transform which gives local analysis of time / space and frequency. Localized analysis of a signal $f : \Re \to C$ can be achieved through the so called Windowed (Gabor) Fourier transform (WFT). The WFT can be described as follows.

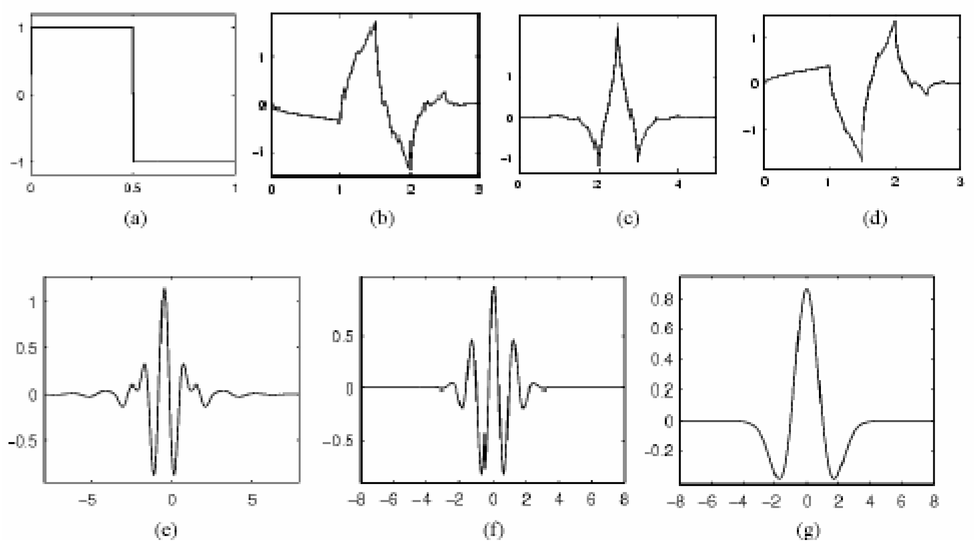
One begins by choosing a window function $g : \Re \to \Re \geq 0$ defined by $g(t) = \frac{1}{\sqrt{2p}\, s} e^{\frac{-t^2}{2s^2}}$ s being a fixed parameter. (Gaussian function).The function g should have "total mass" 1 and be more or less concentrated around t = 0. This simply means, it should have compact support containing 0 or at least a maximum at t = 0 and fast decay when $|t| \to \infty$, for a given $s \in \Re$ the function $g_s(t) = g(t - s)$ represents the window translated by the amount s. We define the window transform of f by $G_f(a, s) = \frac{1}{\sqrt{2p}} \int_{-\infty}^{\infty} f(t) g(t - s) e^{-iat} dt$ for practical purposes; one of course has to resort to a discrete version of WFT. Though WFT provides local analysis it is not quite adequate since it uses a window of fixed width. Another serious limitation of WFT according to the famous Heisenberg Uncertainty Principle [C] a theorem in Fourier analysis that plays an important role in quantum mechanics is that a signal f and its FT $\hat{f}$ cannot be simultaneously localized at $t = a = 0$. It is against this background wavelets came into the picture. Wavelets are structured for fast algorithms from the outset and can be tailored suitably to meet the requirements. This explains why wavelets become a powerful tool in various application fields within a span of 10-15 years. The basic model of the wavelet transform works on $f : \Re \to C$ also. One begins by choosing a suitable analyzing wavelet also called the mother wavelet. Dilated and translated copies $f(2^k - j)$ of j are called daughter wavelets.



The continuous wavelet transform (CWT) $w_f : \Re \times \Re \to C$ of f is defined by $w_f(a, b) = \frac{1}{|a|^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(t) y\left(\frac{t - b}{a}\right) dt$ when it comes to the field of applications; we discretise the domain by choosing $a_r = 2^r$ $b_r = k2^r (r, k \in Z)$. This corresponds to sampling (dynamic sampling) co-ordinates (a, b) on a grid which allows in an optimal way the precise localization of the high frequency occurring in the processed time signal f. The systematic exploitation of DWT leads to the so called multiresolution analysis (MRA) and Fast Wavelet Transform (FWT) that goes into it. MRA is the essential ingredient for efficiently extracting information about the signal.

Wavelets are grouped into families (selected few). These are in order of appearance:

1) Haar wavelet
2) Morlet wavelet
3) Symmlet
4) Meyer Wavelet
5) Daubeachies Wavelet.
6) Coiflet



Wavelet families: (a) Haar (b) Daubechies-4 (c) Coiflet (d) Symlet (e) Morlet (f) Meyer (g) Mexican Hat

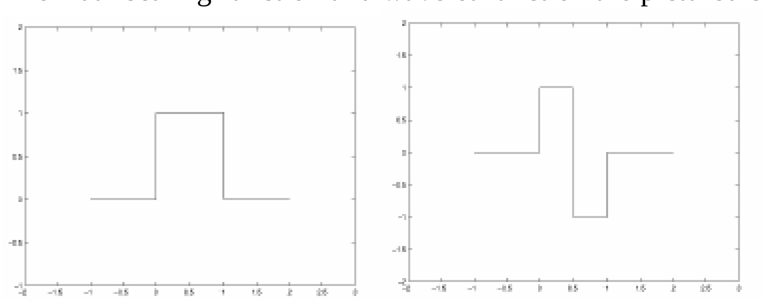
1.4 HAAR WAVELET:

(By way of illustration)
Many aspects of WaveletTheory can be observed and understood by studying the oldest and indeed the simplest wavelet of all and so will serve as a handy tool for illustrative and educational purposes.

The Haar scaling function is the box function $f(x) = \begin{cases} 1 & x \in [0,1) \\ 0 & \text{elsewhere} \end{cases}$

The Haar wavelet is the following simple step function $y(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise} \end{cases}$

The Haar scaling function and wavelet function are pictured below :



This has compact support. Obviously $\int_{-\infty}^{\infty} y(x)dx = 0, \quad \int_{-\infty}^{\infty} |y(x)|^2 = 1$ it is well localized in the time domain but unfortunately discontinuous. The FT of y is computed as follows

$$\hat{y}(a) = \frac{1}{\sqrt{2p}}\left(\int_0^{1/2} e^{-iax}dx - \int_{1/2}^{1} e^{-iax}dx\right) = \frac{i}{\sqrt{2p}} \frac{\sin^2 a/4}{a/4} e^{\frac{-ia}{2}}$$

One might observe that it is reasonably localized but the discontinuity of $\hat{y}$ causes slow decay as $a \to \infty$ Using y as a template we now generate $y_{r,k}(t) = 2^{-r/2} y\left(\frac{t-k2^r}{2^r}\right)$ and state a theorem

(For proof see [c]) $y_{r,k}$ form an orthonormal basis of $L^2(R)$. Had the Haar family been found satisfactory, other wavelet construction together with MRA framework would have been superfluous. However the frequency localization of this wavelet is so bad, the improvements had been sought for. After successive attempts the simplest wavelet constructed from Daubachies family though not peace-wise linear is related to Haar family in some weak sense. It is therefore considered next in line to Haar because it is continuous orthonormal with shortest support. Haar wavelet has compact support but discontinuous. The Shannon wavelet is smooth, all its derivatives exist and continuous but its support is R. Since these wavelet families are at the far ends of support and continuity spectra neither is ideal for use in applications. Rather some sort of compromise between compact support and smoothness is needed and one was discovered by Ingrid Daubauchies (1987) [A].

1.5 WAVELETS IN GENERAL:

Wavelets have certain properties that distinguish them from more traditional representation of functions.

1) Wavelets series approximates much more accurately than Fourier series

2) Wavelet approximation does not cost more to calculate than an ordinary Fourier approximations.

3) The terms in wavelet series are orthogonal to one another (just like the terms of Fourier series). This means that information carried by one term is independent of the information carried by any other term. Numerically it means neither computing cycles nor storage are wasted when wavelet series is calculated or stored in a computer.

4) Compactly supported wavelet basis functions can model local behavior efficiently, because they are not constrained by properties of the data far away from the location of interest.

5) Multiresolution or scalable mathematical representation may provide a simpler and more efficient representation than conventional representations.

6) Computational complexity of

   1) DFT is O (N^2)

   2) FFT is O ($N \log N$)

   3) DWT is O (N) which is measure of number of elementary operations needed to solve a problem.

1.6 APPLICATIONS OF WAVELET THEORY:

The invention of wavelets is directly connected with practical applications. The fact that analytic properties of wavelets are decidedly more intricate than those of pure harmonics e_a , renders them less useful for the working mathematicians (but things are beginning to change). The two applied fields where wavelets have been used with greatest success are signal analysis and image processing. Under the term processing, purification, filtering (de-noising), efficient storage, retrieval and transmission of time-signal and image data and above all their compression. Recent advances in Wavelet theory, have shown that, wavelets due to their coherent properties, can be successfully used to find numerical solutions to differential and integral equations.

1.7 CONCLUSION:

Wavelets have gained enormous popularity in Mathematics and Engineering. It is sufficient to note that there are currently more than 10,000 subscribers to the monthly e-magzine **“WAVELET DIGEST".** There are already hundreds, perhaps thousands of papers relating to wavelets. It is also not necessarily up-to-date because papers in the field are still being published at a rapid-pace. Vast amounts of more current information can be found on entering the query “Wavelets" into Google, the favorite internet search engine. At the same time tailoring concrete wavelet systems to specific applications is still a challenge especially in more than one dimension to the current and future researchers. As a general conclusion, it is fair to say that the wavelet techniques have become part and parcel of the mathematicians as well as engineers tool-kit. Thus we may safely bet that wavelets are here to stay, and that they have a bright future.

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