FAQ'S ON WAVELET THEORY AND APPLICATIONS.

1) What is a Fourier series? Why it is labeled as "mathematical prism?

Ans:

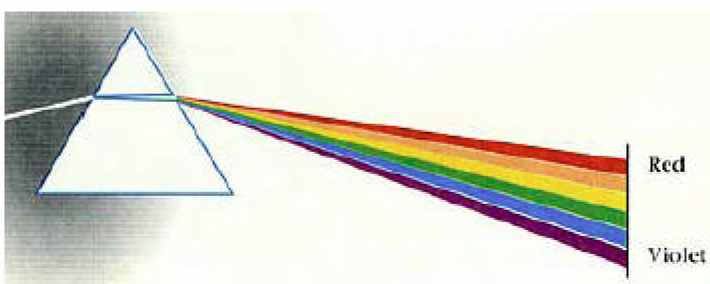
Main Question

Given a 2π-periodic function f, we want to know when the equality

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

holds, where a_n and b_n are defined as above. Also we want to study the convergence properties of the series.

This series is called the **Fourier Series** of f, and can be viewed as the spectral decomposition of a signal, in terms of pure frequencies (in this case the sine and cosine functions): If we think of these functions as representing the "pure" colors (red, green, blue, etc.), then if f is a light ray, the coefficients represent the intensity of each of the "pure" colors.



2) Why we need a transform? What is a transform anyway?

Ans:

Mathematical transforms are applied to signals to obtain further information from the signal that is not readily available in the raw signal. There are a number of transforms that can be applied, among which the Fourier transforms are probably by far the most popular. In general, a transform is a mathematical operation that takes a function or a sequence and maps it into another one.

1) Transformation of a function (signal) may give additional/hidden information about the original function, which may not be available otherwise.
2) Transformation of an equation may be easier to solve than the original equation.
3) Transformed signal may require less storage and hence provide data compression/reduction.

3) What is a Fourier transform? What are the advantages and limitations of Fourier transform?

Ans: The Fourier transform of f$\in L_2(\mathbb{R})$ is defined by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

The Fourier transform analyses the "frequency contents" of a signal. Its many properties make it suitable for studying linear time invariant operators, such as differentiation. FT can be applied for non-stationary and periodic signals if we are only interested in knowing what spectral components exist in the signal, but, not interested in where(when) these exist(occur).However, if this information is needed i.e., if we want to know what spectral components occur at what time(interval) then FT is not the right transform. FT gives what frequency components exist in the signal, nothing more, nothing less. When the time localization of the spectral components are needed, a transform giving the time-frequency representation of the signal is needs to be invented.

4)What factors limit Time-Frequency localization and due to what?

Ans:

The Fourier transform can be viewed as a representation of a function as a sum of sinusoidal waves. These sinusoids are very well localized in the frequency, but not in time, since their support has an infinite length. This is a consequence of periodicity. To represent the frequency behavior of a signal locally in time, the signal should be analyzed by functions which are localized both in time and frequency, for instance, signals that are compactly supported in the time *and* Fourier domains. This time-frequency localization is limited by the following two results:

1) The Heisenberg's Uncertainty Principle 2) Compact support

5) What are the strategies which make Time-Frequency localization achievable?

Ans: The drawback of poor time localization and near-perfect frequency localization can be effectively and satisfactorily handled through, two transforms:

1) Windowed Fourier Transform 2) Wavelet Transform

These time frequency localization strategies are presented in parallel; the first one leads to the windowed Fourier transform, while the other one leads to the wavelet transform. The windowed Fourier transform replaces the Fourier transform's sinusoidal wave by the product of a sinusoid and a window which is localized in time. It takes two arguments: time and frequency.

The goal is to increase the resolution in time (space) for sharp discontinuities while keeping a good frequency resolution at high frequencies. Of course if the signal is composed of high frequencies of long duration (as in a very noisy signal), this strategy does not pay off, but if the signal is composed of relatively long smooth areas separated by well-localized sharp discontinuities (as in many real or computer-generated images and scenes) then this approach will be effective.

The wavelet transform replaces the Fourier transform's sinusoidal waves by a family, generated by translations and dilations of a window called a wavelet. It takes two arguments: time and scale.

6) What is a wavelet, a wavelet transform, a wavelet series and a wavelet basis?

Ans: A *wavelet* is just an L_2 function y which satisfies the following property:

If we dilate and translate the function, $y_{j,k}(x) = 2^{\frac{j}{2}} y(2^j x - k)$ then $\{y_{j,k}\}_{j,k}$ is an orthonormal basis of L_2.

Hence, we can now define the *wavelet transform* by assigning to each function *f* the coefficients

$f_{j,k} = \int_{0}^{\infty} f(x)\psi_{j,k}(x)dx$ and we can recover the function back by means of the *wavelet series*

$$f(x) = \sum_{0}^{\infty} \sum_{-\infty}^{\infty} f_{j,k}\psi_{j,k}$$

if $f \in L_2$. Further,

$$\{2^{\frac{j}{2}}\psi(2^j x - k)\},\ j,k \in \mathbb{Z}$$

is called a *wavelet basis*.

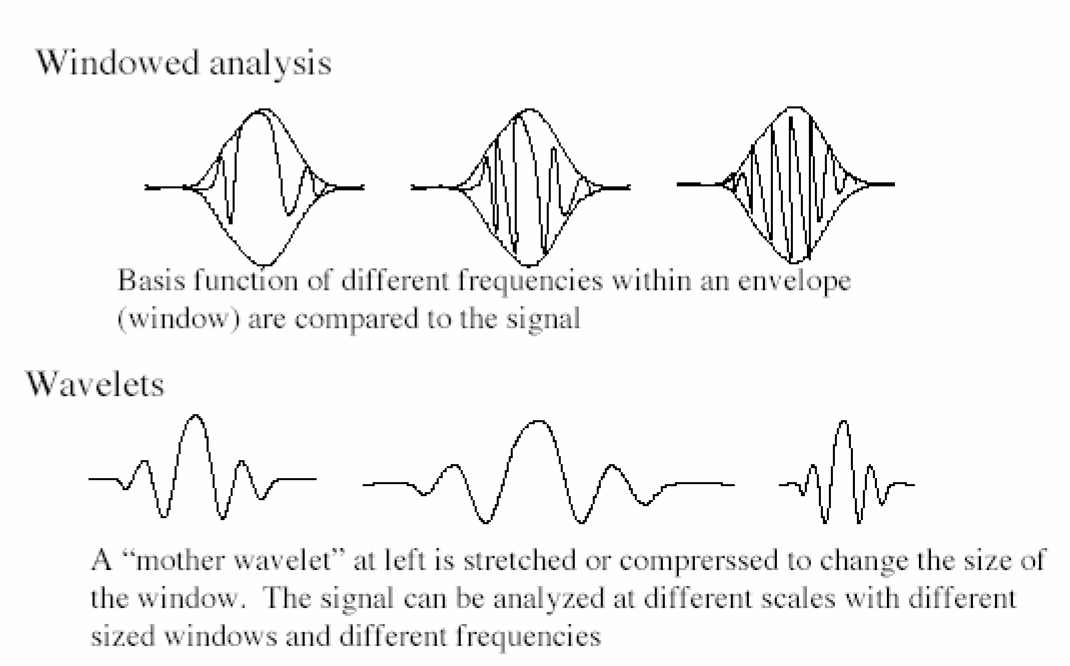
7) **What is the difference between Windowed analysis and Wavelet analysis?**

Ans:

Windowed analysis of a signal is carried out by Fourier Windowed Transform(FWT), more appropriately called Short Time Fourier Transform(STFT). In STFT the time and frequency resolutions are determined by the width of the analysis window, which is selected once and for all. As a result, both time and frequency resolutions are constant. This is the main reason why we switch over to wavelet analysis. We basically need Wavelet Transform(WT) to analyze non-stationary signals i.e., whose frequency response varies with time.

The problem rather dilemma with STFT is narrow (wide) window leads to good (poor) time resolution and poor (good) frequency resolution.

The difference is illustrated pictorially as follows:



8) **How to remove white noise in the signal? Or what is denoising?**

Ans:

- Convert image function to wavelet coefficnets using an orthogonal basis
- Threhold all coefficents below a particular size
- Reconstruct new signal

9) What is thresholding?

Ans:

Once DWT is performed, the next task is thresholding which is neglecting certain wavelet coefficients. For doing this one has to decide the value of the threshold and how to apply the same. This is an important step, which affects the quality of the compressed image. The basic idea is to truncate the insignificant coefficients, since the amount of information contained in them is negligible. It can be carried out using the following steps.

- Given a mother wavelet and signal, many coefficients are close or equal to zero
- Hard thresholding:
  - Given tolerance level *t* set all wavelet coefficients below *t* to zero
- Soft thresholding:
  - Select tolerance *h*, absolute value of a coefficient is less than *h* it is set to zero
  - All other entries, *d*, are replace with *sign(d)||d-h|*.
  - Thought of as a translation of the signal toward zero by the amount *h*
- Quantile thresholding
  - Percentage *p* of entries to be elimated is set, smallest in absolute value *p* percent are set to zero.

10) **What are the steps needed in image compression?**

Ans:

1. Digitize the source image into a signal s, which is a string of numbers.
2. Decompose the signal into a sequence of wavelet coefficients w.
3. Use thresholding to modify the wavelet coefficients from w to another sequence w'.
4. Use quantization to convert w' to a sequence q.
5. Apply entropy coding to compress q into a sequence e.

11) What is entropy coding?

Ans: This is the last component in the compression model. This is a lossless technique which aims at eliminating the coding redundancies. A few coding techniques are 1)Run length encoding 2) Differential Pulse coding 3)Huffman coding In short,

- Thresholding prepares the data for compression but must be re-coded to achieve compression
- Run-Length-Encoding
- Probability of occurrence encoding: Huffman

12) What is quantization?

Ans: High compression ratios can be achieved by quantizing the non-zero wavelet coefficients, before they are encoded. To achieve best results, a separate quantization should be designed foe each scale.

- Convert sequence of floating numbers w to a sequence of integers q
  - Simplest form: round to nearest
  - Multiply each number in w by constant k followed by rounding.
- Quantization is a lossy process
  - w to q is not a one-to-one mapping

13) How are CWT and DWT related and why DWT is needed?

Ans:

The Continuous Wavelet Transform (CWT) is provided by equation 2.1, where x(t) is the signal to be analyzed. ψ(t) is the mother wavelet or the basis function. All the wavelet functions used in the transformation are derived from the mother wavelet through translation (shifting) and scaling (dilation or compression).

$$X_{WT}(\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t) \cdot \psi^{*}\left(\frac{t-\tau}{s}\right) dt \qquad 2.1$$

The Wavelet Series is just a sampled version of CWT and its computation may consume significant amount of time and resources, depending on the resolution required. The Discrete Wavelet Transform (DWT), which is based on sub-band coding is found to yield a fast computation of Wavelet Transform. It is easy to implement and reduces the computation time and resources required.

In CWT, the signals are analyzed using a set of basis functions which relate to each other by simple scaling and translation. In the case of DWT, a time-scale representation of the digital signal is obtained using digital filtering techniques. The signal to be analyzed is passed through filters with different cutoff frequencies at different scales.

14) What is a wavelet approximation? How it differs from Fourier approximation?

Ans:

We define:

$$\left(P_i f\right)(t) = \sum_{q=i+1}^{\infty} \sum_{k \in \mathbb{Z}} \left\langle f, y_{i,k} \right\rangle y_{i,k},$$

to be the orthogonal approximation to *f* at resolution 2^{-i}. It can be shown that, L^2-norm of the projection error has the asymptotic form: $\left\| f - P_j f \right\| = O\left(2^{-jp}\right)$ (i.e. it decays exponentially) on the function to be approximated.

The fundamental idea behind wavelet approximation is to analyze a function according to a scale. This is made possible through multiresolution technique.

In contrast, Fourier approximation uses Fourier basis (sinusoids) to approximate functions.

15) What is wavelet - Sampling approximation?

Ans: It differs from the usual L_2 - approximation, in that the coefficients are the samples, rather than L_2 inner products of the function and the scaling function. I t has been shown that, for a Coifman wavelets using samples as the expansion coefficients gives an excellent approximation.

Wavelet sampling approximation is,what is used in most applications, because it is easiest to compute and the degree of approximation is comparable to that obtained by Daubechies wavelets.

16) What are vanishing moments?

Ans:

Wavelets are usually designed with vanishing moments:

$$\int_{-\infty}^{+\infty} \psi(x) x^m dx = 0, \qquad m = 0, \ldots, M-1,$$

which makes them orthogonal to the low degree polynomials, and so tend to compress non-oscillatory functions.

For example, we can expand in a Taylor series

$$f(x) = f(0) + f'(0)x + \cdots + f^{(M-1)}(0)\frac{x^{M-1}}{(M-1)!} + f^{(M)}(\xi(x))\frac{x^M}{M!}$$

and conclude

$$|\langle f, \psi \rangle| \leq \max_x \left| f^{(M)}(\xi(x)) \frac{x^M}{M!} \right|.$$

17) What is a frame? What is a Riesz basis? How do they differ from each other?

Ans:

Under certain conditions, one can recover any signal f in a Hilbert space from the scalar products that it forms with a family of vectors. Such a family of vectors, a finite number of which spans the Hilbert space, is termed a frame. It can be shown that if there exist scalars A and B both positive and such that for any f :

$$A\|f\|^2 \le \sum_n |f, \phi_n|^2 \le B\|f\|^2$$

then the sequence ϕ_n is a frame of the space that f belongs to. When $A = B = 1$, the frame forms an orthonormal basis. Thus, wavelet frames are special cases.

If the frame of vectors ϕ_n is linearly independent, it is termed a Riesz basis. Because the vectors are linearly independent, any finite number of ϕ_n form a subspace, S, in which the best approximation of a function f is:

$$\sum_{n\epsilon[1,dim(S)]} \phi_n^* f.\phi_n$$

The function f can be recovered from the frame coefficients, f_n, using an inverse frame:

$$f = \sum_{n\epsilon[1,dim(S)]} \tilde{\phi_n} f_n$$

Sometimes frames are used as a substitute for a Riesz basis. The important difference between frames and Riesz bases is that if $\{j_n\}$ is a frame, the vectors $\{j_n\}$ need not be linearly independent. It is important to be noted that Riesz bases are generalizations of orthonormal bases.

18) What benefits does wavelet analysis offer over those supplied by traditional Fourier analysis?

Ans: In many of its aspects, wavelet theory can be regarded as an extension of Fourier formalism. Wavelet theory offers a very flexible and computationally efficient alternative to the short-time Fourier transform.

Instead of projecting the signal on sines and cosines, the signal is projected on a set of generally well-localised wavelet functions. Wavelets allow good resolution in both time and frequency, and, in contrast to the short-time Fourier transform, the wavelet transform is easily invertible.

19) What new activities/applications does wavelet theory it make possible?

Ans: Multiresolution analysis has become a quite standard tool in signal processing. Wavelet theory has been applied to basically all scientific fields, including domains as different as quantum mechanics, econometrics and social sciences.

Despite the large variety of wavelet applications, the main domain of applications is still in image processing. The new standard JPEG 2000, for instance, is based on wavelet data compression schemes.

20) What new difficulties do wavelets present for the practitioner?

Ans: The main difficulty is the mathematics behind wavelet theory in order to apply it correctly. Once the theory is mastered, multiresolution analysis is easily implemented. As wavelet analysis offers a greater flexibility than Fourier analysis, there is also more room for optimisation.

21) Is wavelet analysis less computationally expensive than Fourier analysis?

Ans: The fast Fourier transform reduces from N 2 to N log 2 N the number of necessary operations for a Fourier transform of a signal with N values. A wavelet decomposition using the fast wavelet decomposition algorithm is slightly more efficient, as O(N) operations are necessary.

22) What is the motivation for combining fuzzy logic with multiresolution analysis in a commerccial field?

Ans: Price and power consumption are important issues in sensorics. Combining wavelet theory and fuzzy logic in a single method furnishes a very efficient means to conduct both a spectral analysis and a classification task in a single step. This would not have been practically feasible using conventional methods

23) What does multiresolution analysis have in common with soft computing?

Ans: Multiresolution analysis is of central importance in the mechanisms of perception and decision. Humans are particularly good at such tasks. For instance, image processing in the brain relies heavily on the analysis of the signals at several levels of resolution. Extracting details of importance out of a flow of information is an essential part of any decision process. Soft computing covers a range of methods that are somewhat tolerant of imprecision, uncertainty and partial truth. Hybrid methods combining soft computing methods with wavelet theory have therefore the potential to accommodate two central elements of the human brain: the ability to select an appropriate resolution to the description of a problem and to be somewhat tolerant of imprecision.

Multiresolution analysis and wavelet theory are a natural complement to soft computing methods. Soft computing deals with solving computationally intensive problems with a limited amount of computing power and memory by giving up some precision. Multiresolution analysis can be used to determine how and where to give up the precision.

24) Where do the chief synergies between wavelet analysis and fuzzy logic lie?

Ans: There are two very distinct domains in which fuzzy logic and wavelet theory complement each other. The combination of wavelet theory and fuzzy logic is a very efficient method for spectral analysis, as it furnishes a natural method to extend fuzzy logic to the frequency domain. An example of a fuzzy rule in the frequency domain may be: if high-frequency component of the signal is large then.... The degree of membership is computed from the wavelet coefficients.

The second chief synergy is in learning. Wavelet modelling can be used to develop a fuzzy system automatically from a set of data. It takes advantage of the connection between wavelet-based spline modelling and the Takagi-Sugeno fuzzy model. One starts from a dictionary of pre-defined membership functions forming a multiresolution. The membership functions are dilated and translated versions of a scaling function. Appropriate fuzzy rules and membership functions are determined either through a classical wavelet method or using a wavelet-network.

Multiresolution-based fuzzy methods furnish a new approach to the problem of transparency and linguistic interpretability. Linguistic interpretability is included per design by using predefined membership functions forming a multiresolution. Membership functions are chosen among a dictionary and describe terms such as "very small" or "large" that do not change during learning. A fuzzy system developed with this method consists of a number of rules using membership functions with clear linguistic interpretations.

Linguistic interpretability and transparency are slightly different concepts. Linguistic interpretability refers to the ease with which rules can be interpreted in natural language, for example: "if temperature is low then heater is on". Transparency refers to the ease with which a system may be understood by the human operator. A preliminary condition for transparency is a natural linguistic interpretability of rules. A second condition is that the number of rules and the number of different levels in a hierarchical fuzzy system is still manageable by human experts. The fuzzy-wavelet approach furnishes innovative solutions to developing transparent fuzzy systems.

25) What can wavelets be used for, especially in the field of earth and environmental sciences?

Ans:

Wavelets have been used successfully in areas of geophysical study. Orthonormal wavelets, for instance, have been applied to the study of atmospheric layer turbulence.

Wavelets have also been used to analyze seafloor bathymetry, or the topography of the ocean floor. Several other geophysical applications such as analysis of marine seismic data and characterization of hydraulic conductivity distributions have also been used. The usefulness of wavelets in data analysis is clear, particularly in the field of geophysics, where large and cumbersome data sets abound. Studies such as the atmospheric layer turbulence and corn crop turbulence have further shown the proficiency of wavelets in the analysis and synthesis of geoseismic signals which in the long run, help create “warning systems” to tsunamis (sea-quakes).

26) In what way the hierarchical structure of wavelets has benefited Numerical analysis or more precisely scientific computing?

Numerical Mathematics has changed a lot over the last 50 years. One sign of these changes is the fact that often the name Scientific Computing is used instead of Numerical Mathematics. It should be noted that Numerical Mathematics or Scientific Computing combine the following parts which mutually dependent: modelling, algorithms, analysis. Modelling is (a) the appropriate mathematical formulation of a problem from a field outside Mathematics and (b) the discretisation process with translates infinitely dimensional **problems** into finitely dimensional ones. The heart of Numerical Mathematics is the algorithms enabling the solution process. Numerical Analysis comes into play to judge the consequences of the discretisation process, to control the algorithms (reliability, quality, costs, etc.). Since after the discretisation process, the arising systems of equations can be as large as the computer memory allows, the computer technology obviously has an important influence. It may be self-evident that we would like the algorithms to be as efficient as possible, i.e., they should yield the desired results for lowest computational costs. This vague request can be made more precise. This explains why the development of the computer technology directly leads to the need of algorithms with linear complexity, i.e., the computational work for performing an algorithm with n input data must be proportional to n. The algorithms satisfying this requirement often show a hierarchical structure. An early example is the Fast Fourier Transform (FFT) which is almost of linear complexity and has a typical recursive structure. **Wavelets** and algorithms exploiting their features are a more recent example. For the solution of discrete elliptic partial differential equations, wavelet-based multi-grid methods have been developed and turned out to be very flexible. They essentially make use of a hierarchy of wavelets and wavelet bases.

27) What is zoom-in and zoom-out property of CWT by which it earns the title "mathematical microscope'?

Just like in Fourier analysis we can generalize a discrete integer wavelet series to a continuous integral wavelet transform where squeeze-shift integers j,k are replaced by real parameters a, b

$$\hat{f}_{\psi(a,b)} \equiv \langle \psi(a,b) | f \rangle \equiv \frac{1}{\sqrt{|a|}} \int_{-\infty}^{+\infty} \psi^* \left(\frac{x' - b}{a} \right) f(x') dx'$$

The notation is that $\hat{f}$ is the integral wavelet transform, whilst $\tilde{f}$ is the orthodox integral Fourier transform. We can, of course, still keep as a special case

$$a \rightarrow \frac{1}{2^j}$$

$$b \rightarrow ka = \frac{k}{2^j}$$

$$\frac{x'-b}{a} \rightarrow 2^j x' - k \equiv x$$

Therefore, $j > 0$ squeezes the domain of support where $\psi(x') \neq 0$ into a smaller region as well as decreasing the shift. This is a zoom-out transformation like decreasing the magnification of a microscope showing less detail, less resolution of the image. On the contrary, $j < 0$ does the opposite zoom-in increasing the resolution of the image or increasing the magnification of the microscope. Note that ordinary Fourier analysis in physics does not have this capability that is particularly suitable for nonstationary statistical processes with fast changes especially for open non-equilibrium systems. The integral wavelet transform $\hat{f}_{\psi(a,b)}$ relative to some basic wavelet $\psi(t)$ has this … zoom-in and zoom-out capability."

28) What are audio applications of wavelets?

Ans:

DWT can be used to analyze temporal and spectral properties of non-stationary signals such as audio. Unlike the Fourier transform, whose basic functions are sinusoids, wavelet transforms are based on small waves, called wavelets, of varying frequency and limited duration. That reveals not only what notes (or frequencies) to play but also when to play them. Conventional Fourier transforms, on the other hand, provide only the notes or frequency information; temporal information is lost in transformation process.

Some of audio applications where DWT could offer considerable improvement are extraction of beat attributes from music signals and automatic classification of non-speech audio signal using statistical pattern recognition. Shrinking of transform coefficients towards zero in wavelet domain is one of the wavelet techniques, which offers advantage of removal of noise in wide variety of signal types while preserving non-smooth features.

29) What are video applications of wavelets?

Ans:

Wavelet basis functions are obtained from single wavelet by transformation and scaling of mother wavelets. Also, multi-resolution concept, satisfied by almost all useful wavelet functions, makes it very useful in analyzing "real world" signals.

Multi-resolution theory is concerned with the representation and analysis of signals at more than one resolution. The multi-resolution of videos has an advantage of scalability. i.e. possibility to transmit the same sequence at different resolution as high-resolution television, videophone and videoconferencing. DWT offers better approximation at half the width and half as wide translation steps. This is conceptually similar to improving frequency resolution by doubling the number of harmonics in Fourier series expansion.

While DCT-based image coders like JPEG perform very well at moderate bit rates, at low bit rates the image quality degrades rapidly because of the blocking artifacts introduced by the block based DCT transform. JPEG-2000 is an emerging standard in image processing that uses DWT to achieve far superior image quality at very low bit rates because of overlapping basis functions and better energy compaction property of wavelet transformation.

30) What is Cascade Algorithm?

Ans:

The Cascade Algorithm, is an iterative procedure for computing successive approximations to the scaling function. It arises from iteration of the *dilation equation,* which is the counterpart of the repeated products on the Fourier side.

The dilation equation is

$$\varphi(t) = \sum_k h_k \varphi(2t-k).$$

A sequence of functions $\varphi^{(i)}(t)$ is generated by iterating the sum on the right hand side of the dilation equation:

$$\varphi^{(i+1)} := \sum_k h_k \varphi^{(i)}(2x-k). \tag{1}$$

If $\varphi^{(i)}(t)$ converges to a limit function $\varphi(t)$, then this $\varphi(t)$ automatically satisfies the dilation equation. That is, $\varphi(t)$ is a fixed point of the iterative scheme. On the Fourier transform side, the Fourier transform of equation (1) is

$$\hat{\varphi}^{(i+1)}(\omega) := m_0(\omega/2)\hat{\varphi}^{(i)}(\omega/2).$$

31) What is lifting sceme?

Ans:

Lifting scheme is just another approach to the calculation of DWT and to the calculation of the scaling functions and wavelets themselves. Although it is related to several other schemes such as Cascade and Mallat's algorithms, the idea was first explained by Wim Sweldens as a time-domain construction based on interpolation. Lifting does not use Fourier methods and can be applied to more general problems.

32) What are the notable success stories of wavelet applications?

Ans:

The FBI's fingerprint standard i.e. JPEG-2000 is just one of the success stories of wavelet applications. One problem with fingerprints is that there are so many of them. The FBI has approximately 200 million fingerprint cards. According to a rough estimate, the files occupy an acre of office space. By digitizing the files and storing them electronically, they can fit in a 20x20 foot room. But the digitized images have to be of high quality. At a resolution of 500 pi.
And with 256 levels of gray-scale, a single card contains 10 MB of data. At a standard modem rate of 9600 bits per second, a single card would take three hours to transmit it on a telephone line. That is where wavelet compression comes in. The compression ratio of JPEG-2000 is 20:1 and gives reconstructions that are hardly distinguishable from the originals.

Music is not the only domain in which noise is a problem. Statisticians have long grappled with the problem of noisy data. It appears that wavelets hold promise in denoising the data by "wavelet shrinkage" a technique that works wonders on a variety of data sets.

Whether wavelets will have an impact in specific areas say clinical trials in medical imaging and analyzing turbulence remains to be seen.

33) What is Mallat's tree or Pyramidal algorithm?

Filters are one of the most widely used signal processing functions. Wavelets can be realized by iteration of filters with rescaling. The resolution of the signal, which is a measure of the amount of detail information in the signal, is determined by the filtering operations, and the scale is determined by upsampling and downsampling (subsampling) operations[5].

The DWT is computed by successive lowpass and highpass filtering of the discrete time-domain signal as shown in figure 2.2. This is called the Mallat algorithm or Mallat-tree decomposition. Its significance is in the manner it connects the continuous-time mutiresolution to discrete-time filters. In the figure, the signal is denoted by the sequence x[n], where n is an integer. The low pass filter is denoted by G_0 while the high pass filter is denoted by H_0. At each level, the high pass filter produces detail information, d[n], while the low pass filter associated with scaling function produces coarse approximations, a[n].

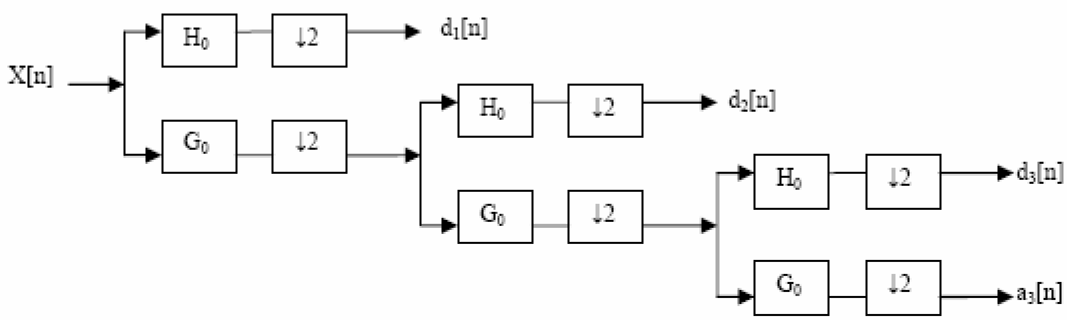


Figure 2.2 Three-level wavelet decomposition tree.

At each decomposition level, the half band filters produce signals spanning only half the frequency band. This doubles the frequency resolution as the uncertainity in frequency is reduced by half. In accordance with Nyquist's rule if the original signal has a highest frequency of ω, which requires a sampling frequency of 2ω radians, then it now has a highest frequency of ω/2 radians. It can now be sampled at a frequency of ω radians thus discarding half the samples with no loss of information. This decimation by 2 halves the time resolution as the entire signal is now represented by only half the number of samples. Thus, while the half band low pass filtering removes half of the frequencies and thus halves the resolution, the decimation by 2 doubles the scale.

With this approach, the time resolution becomes arbitrarily good at high frequencies, while the frequency resolution becomes arbitrarily good at low frequencies. The time-frequency plane is thus resolved as shown in figure 1.1(d) of Chapter 1. The filtering and decimation process is continued until the desired level is reached. The maximum number of levels depends on the length of the signal. The DWT of the original signal is then obtained by concatenating all the coefficients, a[n] and d[n], starting from the last level of decomposition.

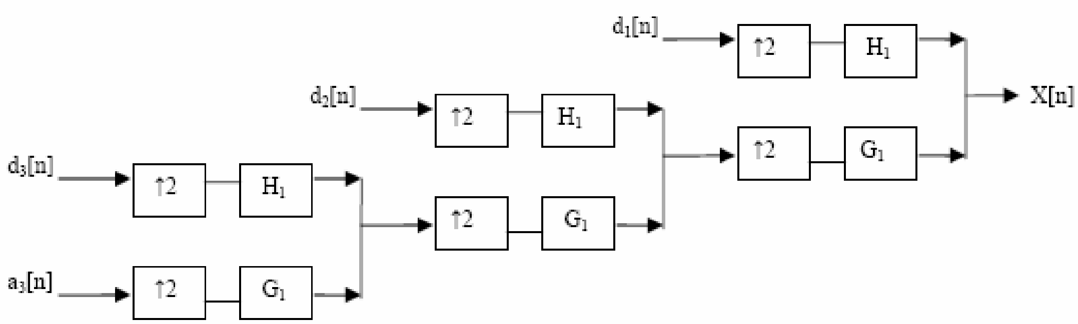


Figure 2.3 Three-level wavelet reconstruction tree.

Figure 2.3 shows the reconstruction of the original signal from the wavelet coefficients. Basically, the reconstruction is the reverse process of decomposition. The approximation and detail coefficients at every level are upsampled by two, passed through the low pass and high pass synthesis filters and then added. This process is continued through the same number of levels as in the decomposition process to obtain the original signal.

34) What are wavelet packets and Wavelet Packet Transform (WPT)? What is the "Best –Basis criterion"?

Using a pair of lowpass and highpass filters to split a space corresponds to splitting the frequency content of a signal into roughly a low-frequency and a high-frequency component. In wavelet decomposition we leave the high-frequency part alone and keep splitting the low-frequency part. In wavelet packet decomposition, we can choose to split the high-frequency part also into a low-frequency part and a high-frequency part. So in general, wavelet packet decomposition divides the frequency space into various parts and allows better frequency localization of signals.

The transformation of data into wavelet packet basis presents no extra difficulties. We can simply do a convolution using filters h and g on the details $\{d_k^j\}$ as well as on the trend $\{f_k^j\}$. As in the wavelet transform, we can keep doing the decomposition until we cannot go any further. On the other hand, we can also choose not to decompose a particular subspace while decomposing others. So there are many choices for decomposing a signal. We can keep all the coefficients at all decomposition levels and generate a table of coefficients of wavelet packet decomposition.

The table of coefficients is a representation of the original signal in different wavelet packet bases. It is highly redundant and we need to choose from among all the representations the one that represents the signal most efficiently. By "efficient" we mean that a signal can be represented by a small number of wave forms or wavelet packets, that is, the basis for the decomposition is chosen such that the weight of the coefficients is concentrated on a small number of wavelet packets and a large number of coefficients are close to zero.

The commonly used criterion for choosing the most efficient or best basis for a given signal is the minimum entropy criterion (see Coifman and Wickerhauser (1992) and Wickerhauser (1994)). That is, let $\{\nu_i\}$ be the decomposition coefficients of a signal for a particular choice of the wavelet packet basis. For each set of decomposition coefficients $\{\nu_i\}$ we associate a nonnegative quantity $\epsilon^2(\{\nu_i\})$ called entropy defined by

$$\epsilon^2(\{\nu_i\}) = -\sum_i \frac{\nu_i^2}{\|\nu\|^2} \log_2 \frac{\nu_i^2}{\|\nu\|^2}$$

where $\|\nu\|^2 = \sum_i \nu_i^2$. The best basis is the one which produces the least entropy. Intuitively, the entropy defined above gives a measure of how many effective components are needed to represent the signal in a specific basis. For example, if in a particular basis the decomposition produces all zero coefficients except one (*i.e.*, the signal coincides with a wave form), then the entropy reaches its minimum value of zero. On the other hand, if in some basis the decomposition coefficients are all equally important, say $\nu_i = 1/\sqrt{N}$ where N is the length of the data, the entropy in this case is maximum, $\log_2 N$. Any other decomposition will fall in between these two extreme cases. In general, the smaller the entropy the fewer significant coefficients needed to represent the signal.

35) How are wavelets classified in general? What are their characteristic features?
Ans:

We can classify wavelets into two classes: (a) orthogonal and (b) biorthogonal. Based on the application, either of them can be used.

(a)Features of orthogonal wavelet filter banks

The coefficients of orthogonal filters are real numbers. The filters are of the same length and are not symmetric. The low pass filter, G_0 and the high pass filter, H_0 are related to each other by

$$H_0 (z) = z^{-N} G_0 (-z^{-1}) \qquad 2.4$$

The two filters are alternated flip of each other. The alternating flip automatically gives double-shift orthogonality between the lowpass and highpass filters [1], i.e., the scalar product of the filters, for a shift by two is zero. i.e., $\sum G[k]\, H[k-2l] = 0$, where $k,l \in Z$ [4]. Filters that satisfy equation 2.4 are known as Conjugate Mirror Filters (CMF). Perfect reconstruction is possible with alternating flip.

Also, for perfect reconstruction, the synthesis filters are identical to the analysis filters except for a time reversal. Orthogonal filters offer a high number of vanishing moments. This property is useful in many signal and image processing applications. They have regular structure which leads to easy implementation and scalable architecture.

(b)Features of biorthogonal wavelet filter banks

In the case of the biorthogonal wavelet filters, the low pass and the high pass filters do not have the same length. The low pass filter is always symmetric, while the high pass filter could be either symmetric or anti-symmetric. The coefficients of the filters are either real numbers or integers.

For perfect reconstruction, biorthogonal filter bank has all odd length or all even length filters. The two analysis filters can be symmetric with odd length or one symmetric and the other antisymmetric with even length. Also, the two sets of analysis and synthesis filters must be dual. The linear phase biorthogonal filters are the most popular filters for data compression applications.

36) What are popular wavelets? What wavelets should we use?

The simplest wavelet also referred to as Haar wavelet, turns out to be the only orthogonal wavelet that has symmetric analysis and synthesis filters. This particular wavelet has been studied extensively in the image processing area as Haar transform. Graphs of Haar scaling function and mother wavelet are shown in Figure 3. This particular wavelet is ideal in situations with limited computational resources.

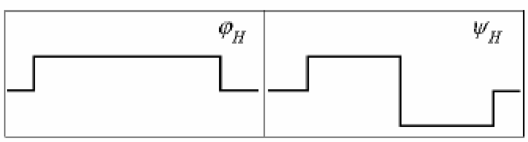


Figure 3- Left graph is the Haar scaling function and the right one is the Haar mother wavelet.

Due to simplicity and existence of fast computational algorithm, historically, Haar transform was a good choice for image processing. Advances in high speed VLSI not only provide the opportunity to utilize the Fourier transform in real-time processing of signals and images, but also provide opportunities to implement and explore new and more advanced signal and image processing algorithms.

Researchers in Applied Mathematics, Communications, and Signal/Image Processing areas have developed many different wavelet systems and some are still actively working in designing even newer wavelets with specialized characteristics. Wavelets can be divided in different classes in many different ways. For example, we can divide them based on their duration or support: infinite support wavelets and finite duration wavelets. There are several interesting wavelets with infinite support. Some of the infinite support wavelets are Gaussian wavelets, Mexican Hat, Morlet, and Meyer. Gaussian wavelets are obtained from derivatives of the Gaussian function. Several examples of the Gaussian wavelets along with other infinite duration wavelets are shown in Figure 4. Mexican Hat wavelet, referred to as ψ_{MH}, is similar to the Gaussian wavelet ψ_{G2}. Among these wavelets, only Meyer wavelet has a scaling function.

Orthogonal wavelets are very successful in numerical analysis like solving partial differential equations, speech coding and other similar applications, where symmetry is not a major requirement. Daubechies wavelets are very good in terms of their compact representation of signal details. They are, however, not efficient in representation of signal approximation at a given resolution. On the other hand, coiflets are similarly effective for both signal details and signal approximation. In image processing applications, biorthogonal wavelets, which are symmetric, are more desirable. Symmetric wavelets allow extension at the image boundaries and prevent image contents from shifting between subbands. In this case, due to human sensitivity to asymmetric errors, orthogonal wavelets usually are not used. For FBI digitized fingerprints compression[5], it is found that the biorthogonal wavelet system represented by $\varphi_{9,7}(t)$, $\tilde{\varphi}_{9,7}(t)$, $\psi_{9,7}(t)$, and $\tilde{\psi}_{9,7}(t)$, in Figure 7, is very successful.

37) **Why Haar Wavelets?**

Ans:There are a wide variety of popular wavelet algorithms, including Daubechies wavelets, Mexican Hat wavelets and Morlet wavelets. These wavelet algorithms have the advantage of better resolution for smoothly changing time series. But they have the disadvantage of being more expensive to calculate than the Haar wavelets.

38) What are the Advantages and Limitations of the Haar Wavelet Transform?

Ans: The Haar wavelet transform has a number of advantages:

it is conceptually simple.

it is fast.

it is memory efficient, since it can be calculated in place without a temporary array.

it is exactly reversible without the edge effects that are a problem with other wavelet trasforms.

The Haar transform also has limitations, which can be a problem for some applications.

In generating each set of averages for the next level and each set of coefficients, the Haar transform performs an average and difference on a pair of values. Then the algorithm shifts over by two values and calculates another average and difference on the next pair. The high frequency coefficient spectrum should reflect all high frequency changes. The Haar window is only two elements wide. If a big change takes place from an even value to an odd value, the change will not be reflected in the high frequency coefficients.

39) What are the investigations being carried out employing wavelets, in the field of numerical solutions of differential and integral equations?
Ans: