Semantics
of Propositional Logic

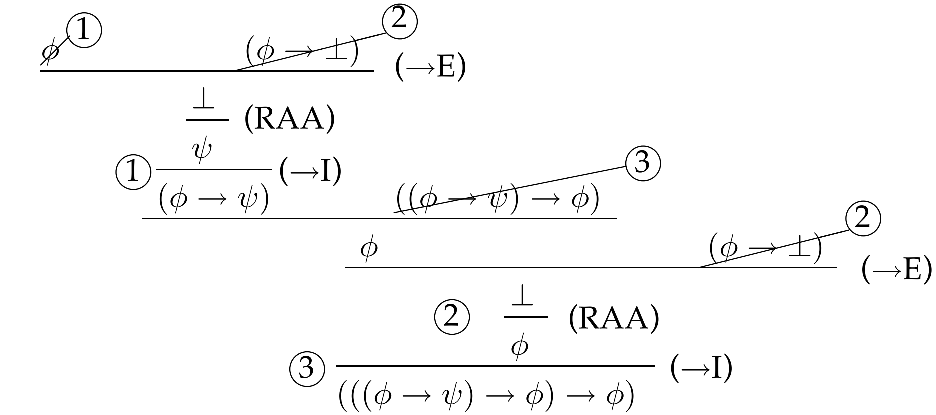
Problem

How do we know we have all the required rules for natural deduction?

Peirce gave an example of a theorem using only $\rightarrow$, whose proof needs $\perp$ as well.

Could we prove even more theorems of PROP by introducing more connectives,
or more rules for the given connectives?

Peirce's Law $(((\phi \rightarrow \psi) \rightarrow \phi) \rightarrow \phi)$



Charles S. Peirce
USA 1839–1914

One of the major inventors of semantics

David Hilbert
Germany 1862–1943

To show a sequent shouldn't be provable, give an interpretation of the formulas so that
the hypotheses are true
and the conclusion is false.

Example: To show that the sequent $(p_0 \to p_1) \vdash p_1$ shouldn't be provable.

Interpret both p_0 and p_1 as meaning:

$$2 = 3.$$

Then p_1 is false, but $(p_0 \to p_1)$ says

If 2=3 then 2=3,

which is true.

So we mustn't introduce a rule which would deduce p_1 from $(p_0 \to p_1)$.

In mathematics we count 'If ϕ then ψ' as true whenever ϕ is false.
For example we accept as true that:

If p is a prime > 2 then p is odd.

For example

If 3 is a prime > 2 then 3 is odd. (If TRUE then TRUE.)

But also

If 9 is a prime > 2 then 9 is odd. (If FALSE then TRUE.)

If 4 is a prime > 2 then 4 is odd. (If FALSE then FALSE.)

The one case we exclude is 'If TRUE then FALSE'.

Moral: To show that $(p_0 \to p_1) \vdash p_1$ ought not to be provable, we can interpret p_0 and p_1 as any two false statements.

The statements themselves don't matter;
only their truth values (T = True or F = False) matter.

The truth value of $(\phi \to \psi)$ is determined by those of ϕ and ψ by the *truth table*

ϕ	ψ	$(\phi \to \psi)$
T	T	T
T	F	F
F	T	T
F	F	T

Similarly we can give truth tables for all the connectives:

ϕ	ψ	$(\phi \wedge \psi)$	$(\phi \vee \psi)$	$(\phi \to \psi)$	$(\phi \leftrightarrow \psi)$	$(\neg\phi)$	$\bot$
T	T	T	T	T	T	F	F
T	F	F	T	F	F		
F	T	F	T	T	F	T	
F	F	F	F	T	T		

By a **truth valuation** we mean a function v that assigns a truth value
(T or F) to each proposition, in such a way that the truth tables hold.

For example if $v(p_1) = \text{T}$ and $v(p_2) = \text{F}$,
then $v((p_1 \wedge p_2)) = \text{F}$ and $v((p_1 \vee p_2)) = \text{T}$.

If $v((p_1 \leftrightarrow p_2)) = \text{F}$ then
either $v(p_1) = \text{T}$ and $v(p_2) = \text{F}$, or $v(p_1) = \text{F}$ and $v(p_2) = \text{T}$.

Let S be a set of propositional symbols.

By an *S-assignment* we mean a function a which assigns truth values to the propositional symbols in S.

We say that a truth valuation v *extends* the S-assignment a if for every propositional symbol p in S, $v(p) = a(p)$.
(This is the usual notion of one function extending another.)

Valuation Theorem (equivalent to Chiswell Proposition (2.8))
Let a be an S-assignment.
Then there is a truth valuation v that extends a.
Moreover if ϕ is a proposition whose propositional symbols come from S, then we can calculate the value $v(\phi)$ from ϕ and a; so if v' is another truth valuation extending a then $v'(\phi) = v(\phi)$.

Proof. First let ϕ be any proposition whose symbols come from S.
We show how to calculate $v(\phi)$, by induction on the length of ϕ.

If ϕ is a propositional symbol then $v(\phi) = a(\phi)$.

If ϕ is $\bot$ then $v(\bot) = \text{F}$.

If ϕ is $(\neg\chi)$ then ϕ determines χ uniquely,
by the unique readability lemma.
Also χ uses only propositional symbols in S.
By induction hypothesis we can calculate $v(\chi)$ from χ and a,
and hence from ϕ and a.
Then by the truth table for $\neg$,
$v(\phi)$ must be T if $v(\chi) = $ F, and F if $v(\chi) = $ T.

A similar argument applies if ϕ is $(\psi\Box\chi)$ where $\Box$ is
one of $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$.

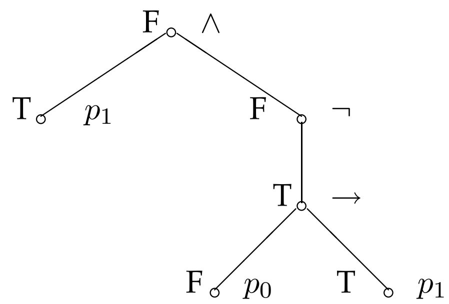
Now let b be an assignment which extends a
and assigns a truth value to each propositional symbol.
Then the argument above, with b in place of a, shows
how to calculate $v(\phi)$ for every proposition ϕ.
The calculation ensures that v is a truth valuation.

Since b extends a, v also extends a.
For each proposition ϕ whose propositional symbols come from S,
the calculation of $v(\phi)$ is exactly as before. $\Box$

The value $v(\phi)$ in the theorem depends only on ϕ and a,
so we write it as $a^\star(\phi)$.
We call $a^\star(\phi)$ the *truth value* of ϕ at a (or at v).

By the proof of the theorem, we can calculate the truth value of ϕ at a
by climbing step by step up the parsing tree of ϕ.
We can keep track of the calculation by writing the truth values
of the subformulas under appropriate symbols in ϕ.

Example: We calculate the truth value of $(p_1 \wedge (\neg(p_0 \rightarrow p_1)))$
under the assignment $a(p_0) = $ F, $a(p_1) = $ T:



p_0	p_1	$(p_1$	$\wedge$	$(\neg$	$(p_0$	$\rightarrow$	$p_1)))$
F	T	T	**F**	F	F	T	T

Let S be a finite set consisting of n propositional letters.
Then the number of S-assignment is 2^n (why?).
We can do the same calculation simultaneously for each assignment, in a table as follows.
Note how the S-assignments are listed at the left.

p_0	p_1	$(p_1$	$\wedge$	$(\neg$	$(p_0$	$\rightarrow$	$p_1)))$
T	T	T			T		T
T	F	F			T		F
F	T	T			F		T
F	F	F			F		F

p_0	p_1	$(p_1$	$\wedge$	$(\neg$	$(p_0$	$\rightarrow$	$p_1)))$
T	T	T	**F**	F	T	T	T
T	F	F	**F**	T	T	F	F
F	T	T	**F**	F	F	T	T
F	F	F	**F**	F	F	T	F

The bold column shows that this proposition is false at every $\{p_0, p_1\}$-assignment, and hence at every truth valuation.

A lot of notions are defined in terms of truth valuations.

(1) We say that a proposition ϕ is a *tautology* if it is true at every truth valuation.

(2) We say that it is a *contradiction* if it is false at every truth valuation, and *satisfiable* if it is not a contradiction.

(3) We say that two propositions ϕ and ψ are *equivalent*, in symbols

$$\phi \text{ eq } \psi,$$

if $v(\phi) = v(\psi)$ for every truth valuation v.

We say that a truth valuation v is a *model* of the proposition ϕ if $v(\phi) = \mathrm{T}$.
We say that v is a *model* of the set of propositions Γ if v is a model of every proposition in Γ.
We say that Γ *semantically entails* ϕ, or that ϕ is a *semantic consequence* of Γ, in symbols

$$\Gamma \models \phi,$$

if every model of Γ is also a model of ϕ.

The symbol $\models$ is called *semantic turnstile*.

Then ϕ is a tautology if and only if $\models \phi$
(i.e. if the empty set semantically entails ϕ).

ϕ eq ψ if and only if both $\{\phi\} \models \psi$ and $\{\psi\} \models \phi$.

ϕ is a contradiction if and only if $\{\phi\} \models \bot$.

ϕ is satisfiable if and only if ϕ has a model.

Examples of Tautologies

(1) $((p_1 \to p_2) \leftrightarrow ((\neg p_2) \to (\neg p_1)))$.

(2) $((p_1 \to (\neg p_1)) \leftrightarrow (\neg p_1))$.

(3) $(p_1 \vee (\neg p_1))$.

(4) $(\bot \to p_1)$.

(5) $((p_1 \to (p_2 \to p_3)) \leftrightarrow ((p_1 \wedge p_2) \to p_3))$.

Examples of Equivalences

(1) Associative laws:
$p_1 \vee (p_2 \vee p_3)$ eq $(p_1 \vee p_2) \vee p_3$,
$p_1 \wedge (p_2 \wedge p_3)$ eq $(p_1 \wedge p_2) \wedge p_3$.

(2) Distributive laws:
$p_1 \vee (p_2 \wedge p_3)$ eq $(p_1 \vee p_2) \wedge (p_1 \vee p_3)$,
$p_1 \wedge (p_2 \vee p_3)$ eq $(p_1 \wedge p_2) \vee (p_1 \wedge p_3)$.

(3) Commutative laws:
$p_1 \vee p_2$ eq $p_2 \vee p_1$,
$p_1 \wedge p_2$ eq $p_2 \wedge p_1$.

(4) De Morgan laws:
$\neg(p_1 \vee p_2)$ eq $\neg p_1 \wedge \neg p_2$,
$\neg(p_1 \wedge p_2)$ eq $\neg p_1 \vee \neg p_2$.

(5) Idempotence laws:
$p_1 \vee p_1$ eq p_1,
$p_1 \wedge p_1$ eq p_1.

(6) Double negation: $\neg\neg p_1$ eq p_1.

p_1	p_2	p_3	p_1	$\vee$	$(p_2$	$\vee$	$p_3)$	$(p_1$	$\vee$	$p_2)$	$\vee$	$p_3,$
T	T	T	T	**T**	T	T	T	T	T	T	**T**	T
T	T	F	T	**T**	T	T	F	T	T	T	**T**	F
T	F	T	T	**T**	F	T	T	T	T	F	**T**	T
T	F	F	T	**T**	F	F	F	T	T	F	**T**	F
F	T	T	F	**T**	T	T	T	F	T	T	**T**	T
F	T	F	F	**T**	T	T	F	F	T	T	**T**	F
F	F	T	F	**T**	F	T	T	F	F	F	**T**	T
F	F	F	F	**F**	F	F	F	F	F	F	**F**	F

Some useful facts about equivalence

Equivalence is clearly an equivalence relation on the class of propositions. In other words:

Reflexive. For every proposition ϕ, ϕ eq ϕ.

Symmetric. If ϕ and ψ are propositions and ϕ eq ψ, then ψ eq ϕ.

Transitive. If ϕ, ψ and χ are propositions and ϕ eq ψ and ψ eq χ, then ϕ eq χ.

All three properties are immediate from the definition of eq.

Lemma ((2.9) in Chiswell)
Let ϕ, ψ be propositions whose propositional symbols come from a set S. The following are equivalent:

(i) $\{\phi\} \models \psi$ and $\{\psi\} \models \phi$.

(ii) $\models (\phi \leftrightarrow \psi)$.

(iii) ϕ eq ψ.

(iv) $a^\star(\phi) = a^\star(\psi)$ for every S-assignment a.

Proof (i) says that every model of ϕ is a model of ψ, and vice versa;
in other words, the truth valuations that are models of ϕ are exactly those that are models of ψ. This is (iii).
It is also equivalent to (ii) by the truth table for $\leftrightarrow$.
Finally (iv) is equivalent to (iii) by the Valuation Theorem. $\square$

Substitution theorems

These are a way of getting new tautologies, equivalences etc. out of old ones.

Let q be a propositional symbol and ϕ, ψ two propositions.
We write $\phi[\psi/q]$ for the proposition got from ϕ by replacing each occurrence of q by ψ.

Example:

$$(p_3 \land (\neg p_2))[(p_1 \to p_4)/p_2]$$

is

$$(p_3 \land (\neg(p_1 \to p_4))).$$

There are two Substitution Theorems ((2.14) in Chiswell notes)). They say:

Let q be a propositional symbol, ϕ, ψ_1, ψ_2 propositions and Γ a set of propositions.

(1) If ψ_1 eq ψ_2 then $\phi[\psi_1/q]$ eq $\phi[\psi_2/q]$.

(2) If $\Gamma \models \psi_2$, then $\{\psi[\phi/q] : \psi \in \Gamma\} \models \psi_2[\phi/q]$.

Part (1) is otherwise known as *Compositionality*.

Part (2) is otherwise known as the *Replacement Theorem*.

Example of Compositionality:

$$(p_1 \wedge p_2) \;\text{eq}\; \neg(\neg p_1 \vee \neg p_2)$$

so

$$(p_1 \wedge p_2) \to p_3 \;\text{eq}\; \neg(\neg p_1 \vee \neg p_2) \to p_3.$$

Example of Replacement Theorem:

$$p_1 \wedge \neg p_1 \text{ is a contradiction (i.e. } \{p_1 \wedge \neg p_1\} \models \bot)$$

so for every proposition ϕ

$$\phi \wedge \neg\phi \text{ is a contradiction.}$$

We can **prove** the Substitution Theorem by the following observations.

Compositionality says that at any truth valuation, the truth value of a proposition won't change if we replace the parsing tree from some node n downwards, as long as the truth value at n is not changed.

The Replacement Theorem says: suppose that for every truth valuation v,

$$v \text{ a model of } \Gamma \;\Rightarrow\; v(\psi_2) = \mathrm{T},$$

then the same is true for every truth valuation w defined by

$$w(p_i) = \begin{cases} v(\phi) & \text{if } p_i \text{ is } q, \\ v(p_i) & \text{otherwise .} \end{cases}$$

But this must be true. If something holds for *all* truth valuations, then it holds for all truth valuations of a certain form.

Warning from experience

These proofs of the parts of the Substitution Theorem are correct. But for more complicated languages one must be more careful. Two famous and well-respected textbooks

> Hilbert and Ackermann, *Foundations of Mathematical Logic*, 1928;
> Lloyd, *Foundations of Logic Programming*, 1984.

contained false theorems about substitution in their first editions. So for more complicated languages one should be prepared to define $\phi[\psi/q]$ carefully by induction on the length of ϕ, and then prove theorems about substitution by induction on the length of formulas.

Disjunctive and conjunctive normal forms

Let S be a set of propositional symbols and ϕ a proposition whose propositional symbols come from S.

Consider the truth table for ϕ.
The rows on the left list all the S-assignments,
and for each row the corresponding truth value of ϕ is given on the right.

So the table describes a function f_ϕ from the set of S-assignments to the set of truth values, and

$$f_\phi(a) = a^\star(\phi) \text{ for each } S\text{-assignment } a.$$

We can write f_ϕ as f_ϕ^S when we need to show what S is.

Post's Theorem ((2.12) in Chiswell)
Let S be a set of m propositional symbols $q_1, \ldots, q_m$ ($m > 0$), and let g be a function from the set of S-assignments to the set $\{\mathrm{T}, \mathrm{F}\}$.
Then there is a proposition ψ using at most the propositional symbols in S, such that $g = f_\psi$.

Proof We split into three cases.

Case One: $g(a) = \mathrm{F}$ for all S-assignments a.
Then we take ψ to be $q_1 \wedge \neg q_1$, which is always false.

Case Two: There is exactly one S-assignment a such that $g(a) = \mathrm{T}$.

Then take ψ to be $q'_1 \wedge \ldots \wedge q'_m$ where

$$q'_i = \begin{cases} q_i & \text{if } a(q_i) = \mathrm{T}, \\ \neg q_i & \text{if } a(q_i) = \mathrm{F}. \end{cases}$$

We write ψ_a for this formula ψ.

Then for every S-assignment c,

$$\begin{aligned} f_{\psi_a}(c) = \mathrm{T} &\Leftrightarrow c^\star(\psi_a) = \mathrm{T} \\ &\Leftrightarrow c^\star(q_i') = \mathrm{T} \text{ for all } i\ (1 \leqslant i \leqslant m) \\ &\Leftrightarrow c(q_i) = a(q_i) \text{ for all } i\ (1 \leqslant i \leqslant m) \\ &\Leftrightarrow c = a. \end{aligned}$$

So $f_{\psi_a} = g$.

Case Three: $g(a) = \mathrm{T}$ exactly when a is one of $a_1, \ldots, a_k$ with $k > 1$.

In this case let ψ be $\psi_{a_1} \vee \ldots \vee \psi_{a_k}$.

Then for every S-assignment c,

$$\begin{aligned} f_\psi(c) = \mathrm{T} &\Leftrightarrow c^\star(\psi) = \mathrm{T} \\ &\Leftrightarrow c^\star(\psi_{a_j}) = \mathrm{T} \text{ for some } j\ (1 \leqslant j \leqslant k) \\ &\Leftrightarrow c = a_j \text{ for some } j\ (1 \leqslant j \leqslant k). \end{aligned}$$

So again $f_\psi = g$. □

Example

We find a formula to complete the truth table

p_1	p_2	p_3	?
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

There are three rows with value T:

p_1	p_2	p_3	?	
T	T	T	F	
T	T	F	T	$\Leftarrow a_1$
T	F	T	T	$\Leftarrow a_2$
T	F	F	F	
F	T	T	T	$\Leftarrow a_3$
F	T	F	F	
F	F	T	F	
F	F	F	F	

The proposition ψ_{a_1} is $p_1 \wedge p_2 \wedge \neg p_3$.
The proposition ψ_{a_2} is $p_1 \wedge \neg p_2 \wedge p_3$.
The proposition ψ_{a_3} is $\neg p_1 \wedge p_2 \wedge p_3$.

So the required proposition is

$$(p_1 \wedge p_2 \wedge \neg p_3) \vee (p_1 \wedge \neg p_2 \wedge p_3) \vee (\neg p_1 \wedge p_2 \wedge p_3).$$

The formula

$$(\neg \phi)$$

is called the *negation* of the formula ϕ.

A *literal* is a formula which is either atomic
or the negation of an atomic formula
(but not $\bot$ or $\neg\bot$).

The formula

$$\phi_1 \wedge \ldots \wedge \phi_n$$

is called a *conjunction* and the formulas ϕ_i are called its *conjuncts*.

The formula

$$\phi_1 \vee \ldots \vee \phi_n$$

is called a *disjunction* and the formulas ϕ_i are called its *disjuncts*.

A *basic conjunction* is a conjunction of one or more literals,
and a *basic disjunction* is a disjunction of one or more literals.
A single literal counts as a basic conjunction and a basic disjunction.

A formula is in *disjunctive normal form* (DNF) if it is a disjunction of one or more basic conjunctions.
A formula is in *conjunctive normal form* (CNF) if it is a conjunction of basic disjunctions.

Examples

(1)

$$p_1 \wedge \neg p_1$$

is a basic conjunction, so it is in DNF.
But also p_1 and $\neg p_1$ are basic disjunctions,
so the proposition is in CNF too.

(2)

$$(p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge p_2 \wedge p_3)$$

is in DNF.

(3) Negating the proposition in (2), applying the De Morgan laws and removing double negations gives

$$\begin{array}{ll} & \neg((p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge p_2 \wedge p_3)) \\ \text{eq} & \neg(p_1 \wedge \neg p_2) \wedge \neg(\neg p_1 \wedge p_2 \wedge p_3) \\ \text{eq} & (\neg p_1 \vee \neg\neg p_2) \wedge (\neg\neg p_1 \vee \neg p_2 \vee \neg p_3) \\ \text{eq} & (\neg p_1 \vee p_2) \wedge (p_1 \vee \neg p_2 \vee \neg p_3) \end{array}$$

which is in CNF.

Theorem ((2.13) in Chiswell)
Every proposition ϕ in PROP is equivalent
to a proposition ϕ^{DNF} in disjunctive normal form,
and to a proposition ϕ^{CNF} in conjunctive normal form.
If S is a nonempty set of propositional symbols,
and every propositional symbol in ϕ is in S,
then ϕ^{DNF} and ϕ^{CNF} can be chosen so that they use only
propositional symbols from S.

Proof The proof of Post's Theorem constructs a proposition ψ using only propositional symbols from S, such that $f_\psi = f_\phi$.

By inspection, the proposition ψ is in disjunctive normal form.

Since $f_\phi = f_\psi$, we have for every S-assignment a

$$a^\star(\phi) = f_\phi(a) = f_\psi(a) = a^\star(\psi),$$

so ϕ eq ψ. Hence we can take ϕ^{DNF} to be ψ.

To find ϕ^{CNF}, first use the argument above to find $(\neg\phi)^{DNF}$, call it θ.
Then $\neg\theta$ uses only propositional symbols in S, and is equivalent to ϕ.

Then use the method of Example (3) above,
pushing the negation sign $\neg$ inwards by the De Morgan rules
and then cancelling double negations,
to get an equivalent proposition in CNF. □

Satisfiability of propositions in DNF and CNF

First consider a basic conjunction

$$\phi_1 \wedge \ldots \wedge \phi_m.$$

This proposition is satisfiable if and only if there is a valuation v such that

$$v(\phi_1) = \ldots = v(\phi_m) = \mathrm{T}.$$

Since the ϕ_i are literals, we can find such a v unless there are two literals among $\phi_1, \ldots, \phi_n$ which are respectively p and $\neg p$ for the same propositional symbol p.
We can easily check this condition by inspecting the proposition.

So checking the satisfiability of propositions in DNF, and finding a model if there is one, are trivial.

But a lot of significant mathematical problems can be written as the problem of finding a model for a proposition in CNF.

The general problem of determining whether a proposition in CNF is satisfiable is known as SAT.
Many people think that the question of finding a fast algorithm for solving SAT, or proving that there isn't one, is one of the major unsolved problems of 21st century mathematics.
(It is the "P = NP" problem.)

Example A *proper m-colouring* of a map is a function assigning one of m colours to each country in the map, so that no two countries with a common border have the same colour as each other.
A map is *m-colourable* if it has a proper m-colouring.

Suppose a map has countries $c_1, \ldots, c_n$.
Write p_{ij} for 'Country c_i has the j-th colour'.
Then finding a proper m-colouring of the map is equivalent to finding a model of this proposition in CND:

$$\begin{array}{ll} & (p_{11} \vee p_{12} \vee \ldots \vee p_{im}) \wedge \ldots \wedge (p_{n1} \vee \ldots \vee p_{nm}) \\ \wedge & (\neg p_{ik} \vee \neg p_{jk}) \wedge \ldots \end{array}$$

(for all k and all countries c_i, c_j with a common border)