

# “Deriving The Bring-Jerrard Quintic Using A Quadratic Transformation”

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*Abstract:* It can be shown that by passing through the *Brioschi quintic form*, a *quadratic* transformation can suffice to transform the general quintic to the Bring-Jerrard form. This is in contrast to the *quartic* transformation found by Erland Bring and independently by George Jerrard, and the *cubic* one recently found by this author. A new one-parameter quintic form  $(z-5)(z^2+15)^2 + p = 0$  which the general quintic can be reduced to in radicals will also be discussed.

*Dedicated to Shelli Manuel-Tomacruz, for remembering.*

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## I. Introduction

Given a version of the principal quintic,

$$x^5 + 5rx^2 + 5sx + t = 0$$

we want to know *for what kth degree transformation* can one derive a Bring-Jerrard quintic form,

$$y^5 + J_1y + J_2 = 0$$

such that,

- a) the  $J_i$  are derived *in radicals* from the  $r, s, t$ .
- b) the roots  $x_i$  are related to  $y_i$  by the  $k$ th degree transformation for some *minimum*  $k$ .

To recall, a *Tschirnhausen transformation* of the equation  $f(x) = 0$  could be of form  $y = x+r$ ,  $y = x^2+ax+b$ ,  $y = 1/x$ , or in general  $y = g(x)/h(x)$  where  $g$  and  $h$  are polynomials such that  $h(x)$  vanishes for no root of  $f(x) = 0$ . (Dickson, p. 210)

It has been known since Tschirnhaus (1683) that a quadratic transformation  $y = g(x)$  can eliminate two terms from the general  $n$ th degree equation, or in the case of the general quintic, reduced it in radicals to principal quintic form. He also thought (Adamchik, Jeffrey, p.92) that a transformation  $y = g(x)$  where  $g$  was a cubic polynomial would suffice to eliminate, in radicals, three terms. However, as pointed out by Leibniz and others (Tignol), this entails that the  $J_1$  would be determined by roots of sextic equations which, in general, would not be solvable in radicals.

The first transformation known that satisfies the two conditions was found by Bring (1786) and later by Jerrard (c.1836) and was of form  $y = g(x)$  for  $k = 4$ . Recently, the author found that a *cubic* Tschirnhausen transformation could indeed derive in radicals the Bring-Jerrard form but it had to be the rational kind  $y = g(x)/h(x)$ . (And is only applicable to the quintic and sextic.) In this paper, we will discuss that even a *quadratic* transformation of form  $g(x,y) = h(x,y)$  will do and this in fact is just a *composition* of two Tschirnhausen transformations. Since it is highly doubtful that a linear relationship can exist between the  $x_i$  and  $y_i$ , then  $k = 2$  might be the minimum degree possible.

## II. The Brioschi Quintic

The *Brioschi quintic form* is important to this alternative way to derive the Bring-Jerrard form. As has already been pointed out, one of the uses of the Tschirnhausen transformation is that it enables one to find a formula to solve the general quintic, though one has to go beyond radicals. Felix Klein (1849-1925) found a hypergeometric formula for the principal quintic, though he also had to make use of Brioschi quintic. This form, named after Francesco Brioschi (1824-1897), is given by,

$$w^5 - 10cw^3 + 45c^2w - c^2 = 0$$

which has the nice discriminant  $D = 5^5 c^8 (-1 + 1728c)^2$ . In addition to the Bring and the Euler quintics this is the third *one-parameter* form to which the general quintic can be reduced to in radicals. This can be done by using the *rational* quadratic Tschirnhausen transformation,

$$x = (aw+b)/(c^{-1}w^2-3)$$

By eliminating  $w$  between the two using resultants we get a new quintic,

$$f_0 x^5 + f_3 x^2 + f_4 x + f_5 = 0$$

where the  $f_i$  explicitly are,

$$\begin{aligned} f_0 &= -1 + 1728c \\ f_3 &= 5c(8b^3 + ab^2 + a^3c + 72a^2bc) \\ f_4 &= 5c(-b^4 + a^3bc + 18a^2b^2c + 27a^4c^2) \end{aligned}$$

$$f_5 = c(b^5 - 10a^2b^3c + a^5c^2 + 45a^4bc^2)$$

which, after dividing by the leading coefficient  $f_0$ , is already in principal quintic form,

$$x^5 + 5rx^2 + 5sx + t = 0$$

To transform the principal form to Briochi form, given  $r,s,t$  the objective is to equate coefficients to have a system of three equations in three unknowns  $a,b,c$ . For simplicity, let  $V = (-1+1728c)/c$ , hence,

$$Vr = 8b^3 + ab^2 + a^3c + 72a^2bc \quad (\text{eq.1})$$

$$Vs = -b^4 + a^3bc + 18a^2b^2c + 27a^4c^2 \quad (\text{eq.2})$$

$$Vt = b^5 - 10a^2b^3c + a^5c^2 + 45a^4bc^2 \quad (\text{eq.3})$$

By resolving this system to a single equation, the amazing thing is that the final resultant always has a quadratic factor! With modern computer algebra systems one in fact can do this mechanically and end up with a 24<sup>th</sup> degree equation which factors as a quadratic and a 22<sup>nd</sup> degree equation (not the expected  $3*4*5 = 60^{\text{th}}$  degree as it has trivial factors).

Dickson, by a series of ingenious algebraic manipulations involving polynomial invariants of the icosahedral group, managed to find explicitly this quadratic factor (in the variable  $b$ ) without elevation of degree. (The interested reader is referred to pp. 244-245 of *Modern Algebraic Theories*.) This is given by,

$$(r^4 - s^3 + rst)b^2 + (-11r^3s - 2s^2t + rt^2)b + (64r^2s^2 - 27r^3t - st^2) = 0$$

Incidentally, the discriminant  $D$  of this quadratic,

$$D = (-11r^3s - 2s^2t + rt^2)^2 - 4(r^4 - s^3 + rst)(64r^2s^2 - 27r^3t - st^2)$$

disregarding the factor  $r^2$ , is the same as the discriminant of  $x^5 + 5rx^2 + 5sx + t = 0$  which is given by  $5^5D$ , a result similar for the quadratic encountered in deriving the Bring-Jerrard form. To find  $a,c$ , the quadratics for them unfortunately are rather long and complicated expressions in terms of  $r,s,t$ . For convenience, they are usually defined also in terms of  $b$ . First define the variable  $V$  as,

$$r^2(-bs^2 + brt - st) V = (b^2r - 3bs - 3t)^3$$

Then  $a$  and  $c$  are,

$$a = (-8b^3r - 72b^2s - 72bt + Vr^2)/(b^2r + bs + t)$$

$$c = 1/(1728 - V)$$

(Duke, Tóth, p.10) and we have all the unknowns  $a,b,c$ ! Example, given,

$$x^5 + 5x^2 + 10x + 2 = 0$$

so  $r=1, s=2, t=2$ . Hence,

$$\begin{aligned} a &= (16/27)(5071-179\sqrt{871}) \\ b &= (1/3)(-17+\sqrt{871}) \\ c &= (1/1875929344)(615193+20894\sqrt{871}) \end{aligned}$$

and the roots  $x_i$  of  $x^5+5x^2+10x+2=0$  are related to the roots  $w_i$  of the Brioschi quintic  $w^5-10cw^3+45c^2w-c^2=0$ , by  $x=(aw+b)/(c^{-1}w^2-3)$ .

As one can see, even with small integral values for  $r,s,t$ , the numbers defining  $c$  are quite large. This can be explained since, interestingly, if the quadratics for  $a$  and  $c$  were given explicitly, they would be expressions of form,

$$\begin{aligned} (r^4-s^3+rst)^3a^2 + \text{Poly1}(r,s,t)a + \text{Poly2}(r,s,t) &= 0 \\ \text{Poly3}(r,s,t)^2c^2 + \text{Poly4}(r,s,t)c + (r^4-s^3+rst)^5 &= 0 \end{aligned}$$

Hence, the leading coefficient of the quadratic in  $a$  is a cube, for  $c$  is a square, plus the constant term of the latter is a perfect fifth power! Considering that the Brioschi quintic is expressed in terms of  $c$  and this form enables one to solve the *general* quintic in terms of hypergeometric functions, it is quite reminiscent of solving *solvable* quintics (non-binomial) in terms of radicals since it can be shown that its quartic Lagrange resolvent also has a constant term that is a perfect fifth power. For example, given the solvable,

$$y^5+10y^3+15y^2-140y+2897=0$$

then its solution in radicals is given by the quartic with a fifth-power constant term,

$$z^4+597z^3+26919z^2-2614747z-19^5=0$$

such that  $y_1 = z_1^{1/5} + z_2^{1/5} + z_3^{1/5} + z_4^{1/5}$  where the  $z_i$  are the quartic roots.

### III. The Quadratic Transformation

To get to the Bring-Jerrard form we simply do a *second* transformation on the Brioschi quintic. This is given by  $y=(w+d)/(c^{-1}w^2-3)$ . Again eliminating  $w$ , we get,

$$y^5 + g_3y^2 + g_4y + g_5 = 0$$

where,

$$\begin{aligned} g_3 &= 5c(8d^3 + d^2 + 72cd + c)/(-1+1728c) \\ g_4 &= 5c(-d^4 + 18cd^2 + cd + 27c^2)/(-1+1728c) \\ g_5 &= c(d^5 - 10cd^3 + 45c^2d + c^2)/(-1+1728c) \end{aligned}$$

Since  $c$  is already known, one can just set  $g_3 = 0$  and solve for  $d$  to get the Bring-Jerrard quintic (or  $g_4 = 0$  for the Euler-Jerrard one). The principal quintic (in  $x$ ) and the Bring-Jerrard (in  $y$ ) are then connected by the “bridge” which is the Brioschi quintic (in  $w$ ). By eliminating  $w$  between the two transformations,

$$x = (aw+b)/(c^{-1}w^2-3), \quad y = (w+d)/(c^{-1}w^2-3)$$

we then get a quadratic relation between the roots  $x_i$  and  $y_i$ .

*Theorem 1.*

*The general quintic can be transformed in radicals to Bring-Jerrard form using only **quadratic** transformations.*

Given the principal quintic,

$$x^5 + 5rx^2 + 5sx + t = 0$$

one can derive a particular Bring-Jerrard form,

$$y^5 + g_4y + g_5 = 0$$

such that only a *quadratic* transformation (or mapping) exists between the roots  $x_i$  and  $y_i$  and which is given by,

$$(-3c+d^2)x^2 + 2(3ac-bd)xy - c(b-ad)(x-ay) + (b^2-3a^2c)y^2 = 0$$

where  $a, b, c, d$  and  $g_4, g_5$  are radicals in terms of  $r, s, t$ . Explicitly, these are,

$$a = (-8b^3r - 72b^2s - 72bt + Vr^2)/(b^2r + bs + t)$$

$$c = 1/(1728 - V)$$

$$g_4 = 5c(-d^4 + 18cd^2 + cd + 27c^2)/(-1 + 1728c)$$

$$g_5 = c(d^5 - 10cd^3 + 45c^2d + c^2)/(-1 + 1728c)$$

where  $V$  is,

$$r^2(-bs^2 + brt - st) V = (b^2r - 3bs - 3t)^3$$

and  $b, d$  is any root of,

$$(r^4 - s^3 + rst)b^2 + (-11r^3s - 2s^2t + rt^2)b + (64r^2s^2 - 27r^3t - st^2) = 0$$

$$8d^3 + d^2 + 72cd + c = 0$$

Example. Using the same principal quintic,

$$x^5 + 5x^2 + 10x + 2 = 0$$

where  $a, b, c$  have been solved for earlier what remains is to solve the cubic in  $d$  and choose its real root  $d = -0.02306458\dots$ , giving the Bring-Jerrard form (with approximate coefficients),

$$y^5 + (6.0977 \times 10^{-8})y + (2.8128 \times 10^{-10}) = 0$$

which has the sole real root  $y_1 = -0.004579\dots$  Using this value for  $y$  in the quadratic relation and solving for  $x$ ,

$$x_1 = -0.225328\dots, \quad x_2 = 0.604459\dots$$

with the first as the correct root of the principal quintic.

#### IV. Other One-Parameter Forms

It was mentioned that there are three one-parameter forms that the general quintic can be reduced to in radicals, not counting equivalent forms that can be obtained by a multiplicative inverse. However, one in fact can find a distinct *fourth*. These four are given by,

$$\begin{aligned} w^5 + w + B_1 &= 0 \\ w^5 + w^2 + E_1 &= 0 \\ w^5 - 10cw^3 + 45c^2w - c^2 &= 0 \\ w^5 - 5gw^3 + 10g^2w - g^2 &= 0 \end{aligned}$$

with the last one already tantalizingly close to the solvable *de Moivre's quintic*,

$$x^5 - 5mx^3 + 5m^2x - m^2 = 0$$

The fourth was found by analogy to the Brioschi quintic. Defining,

$$w^5 + pcw^3 + qc^2w - c^2 = 0$$

and the transformation  $y = (aw+b)/(c^{-1}w^2+d)$  with unknowns  $p, q$  and  $a, b, c, d$ , we get,

$$h_0y^5 + h_1y^4 + h_2y^3 + h_3y^2 + h_4y + h_5 = 0$$

with  $h_1 = -c(5d^2 - 3dp + q)(a + bd^2 - bdp + bq)$ . Setting  $h_1 = 0$  and using the first factor, then,

$$q = -5d^2 + 3dp$$

While the expression for  $h_2$  is not given, by substituting this into  $h_2 = 0$ , we get,

$$h_2 = c(10d-3p)(ab-2b^2d^2+2a^2cd^3+b^2dp-a^2cd^2p) = 0$$

Solving for  $p$  using the first factor yields the Brioschi form while the second factor yields the new form. Sparing the reader the rest of the algebra, in summary, given,

$$w^5 - 5gw^3 + 10g^2w - g^2 = 0$$

with discriminant  $D = 5^5g^8(-1+32g)(-1+36g)$  and the transformation,

$$y = (aw+b)/(g^1w^2-1)$$

where  $g = -(ab+3b^2)/(3a^2)$  gives a quintic already in Bring-Jerrard form,

$$y^5 - (5/27)(a+3b)b^3y + (1/27)(a+3b)(a+4b)b^3 = 0$$

Equating coefficients with a generic Bring-Jerrard  $y^5 + sy + t = 0$  and solving for the unknown  $a, b$  involves only a quartic and in the variable  $b$  is given by,

$$5b^4s + 25b^3t - 27s^2 = 0$$

As a last point, the last two forms can also be put into a more aesthetic form. Using the transformations  $w = 1/(z^2+20)$  and  $w = 4/(z^2+15)$  on the Brioschi and the fourth, respectively, yields,

$$c(z-5)(z^2+20)^2 + 1 = 0$$

$$g(z-5)(z^2+15)^2 + 32 = 0$$

Simple scaling on the variables  $c$  and  $g$  implies the *general* quintic can be reduced, in radicals, to any of the beautiful and simple forms,

$$(z-5)(z^2+20)^2 + p_1 = 0$$

$$(z-5)(z^2+15)^2 + p_2 = 0$$

## V. Beyond The Quintic

### A. Sextics

By extrapolating Jerrard's quartic transformation to the  $n$ th degree equation, the sextic, after scaling variables, can be reduced to a form depending only on *two* parameters,

$$\begin{aligned}x^6 + x^2 + a_1x + a_2 &= 0 \\x^6 + x^3 + a_1x + a_2 &= 0\end{aligned}$$

The general method utilizes what Klein calls an “*accessory irrationality*”, a radical whose usual function allows the elimination of the  $x^{n-1}$  and  $x^{n-2}$  terms. In 1861, P. Joubert derived what is called the *Joubert sextic form*,

$$x^6 + a_1x^4 + a_2x^2 + a_3x + a_4 = 0$$

which can be attained “...without any *accessory irrationalities*”!<sup>1</sup> Another form, the *Maschke sextic form* named after Heinrich Maschke (1853-1908), can be transformed into the Joubert form using a cubic Tschirnhausen transformation. (See <http://www.math.rutgers.edu/courses/535/535-f02/cubics.pdf>). Finally, the general 6<sup>th</sup> degree equation can also be reduced to the *Valentiner sextic form* (after Siegfried Valentiner(?)), which can be solved in terms of *generalized* hypergeometric functions, in a manner analogous to Klein's solution of the general quintic by reducing it to the Brioschi form.

## B. Septics

Likewise, the general septic<sup>2</sup> (unfortunate name!) can be reduced to a form depending now on *three* parameters,

$$\begin{aligned}x^7 + x^3 + a_1x^2 + a_2x + a_3 &= 0 \\x^7 + x^4 + a_1x^2 + a_2x + a_3 &= 0 \\(x^2 + \tfrac{1}{2})^2 x^3 + p_1x^2 + p_2x + p_3 &= 0\end{aligned}$$

with the last also found by Jerrard. One of Hilbert's famous problems (the 13<sup>th</sup>) asks if the general seventh degree equation could be reduced even further and be expressed “...by means of functions of only two arguments”. (D. Joyce). The question was finally settled in 1957 by Andrei Kolmogorov and Vladimir Arnold.

There is a *different* kind of transformation, still in the radicals, for the septic (and above). Jerrard proposed that the general  $n$ th degree equation can be reduced to,

$$x^n + a_1x^{n-2} + ka_1^2x^{n-4} + a_2x^{n-5} + \dots + a_{n-3} = 0$$

for some arbitrary  $k$ , with the hope that as applied to  $n = 5$ , it would transform it to the solvable de Moivre quintic. However, Hamilton (p.3-4) stated that the method works *only for  $n = 7$  and higher*. So by choosing  $k = \frac{1}{4}$  and scaling the variable  $x$ , it then reduces the  $n$ th degree equation, for  $n > 6$ , to the form,

$$(x^2 + \frac{1}{2})^2 x^{n-4} + p_1 x^{n-5} + \dots + p_{n-4} = 0$$

## VII. Conclusion

We can conclude this paper by asking some questions:

1. *Are there any other one-parameter reduced quintic forms?*
2. *Can there be a quadratic transformation (maybe a composition of two transformations) to eliminate three terms from the general sextic?*
3. *Is there a cubic transformation to eliminate three terms from the general septic?*

For [1], duplicates resulting from elementary transformations can be weeded out simply by looking at the discriminant. For the four, these are given by,

$$\begin{aligned} D_1 &= 4^4 + 5^5 B_1, & D_2 &= 2^2 3^3 + 5^5 E_1 \\ D_3 &= 5^5 c^8 (-1 + 12^3 c)^2, & D_4 &= 5^5 g^8 (-1 + 2^5 g)(-1 + 6^2 g) \end{aligned}$$

hence those of additional reduced forms should be distinct. Furthermore, while we now know of transformations of degree  $k = 2, 3, 4$  between the principal and Bring-Jerrard quintics, it seems unlikely there will be one for  $k = 1$ . As was pointed out by other authors, inverting these transformations involve getting spurious numerical solutions to the principal quintic, the one for  $k = 4$  naturally generating the most, with the correct solutions ascertained only by testing. Minimum  $k$  would imply less extraneous solutions.

For [2] and [3], in a paper prior to this one, it was mentioned that the rational *cubic* transformation to eliminate three terms in radicals from the general  $n$ th degree equation would suffice for  $n = 5, 6$  but no higher, since the coefficients of these transformations were dependent on  $n$ , and for  $n > 6$  already involve higher than quartic roots. Is  $k = 3$  the least degree then of a transformation to eliminate three terms from the sextic? Similarly, is it  $k = 4$  for the septic?

--End--

### Footnotes:

1. A quadratic Tschirnhausen transformation can eliminate the  $x^{n-1}$  and  $x^{n-3}$  terms, though normally its coefficients are determined by a root of a cubic. *If* Joubert managed to avoid any accessory irrationality, he must have used something else. Whether this is indeed the case this author has yet to find out.
2. Klein, in addition to his work on the quintic, also investigated solving the general sextic and septic. For the latter, see his seminal article, “*On the order-seven transformation of elliptic functions*” in the book, “*The Eightfold Way: The Beauty of Klein’s Quartic Curve*”.

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