Pi Formulas, Ramanujan, and the Baby Monster Group

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I. Introduction

In 1914, Ramanujan wrote a fascinating article in the Quarterly Journal of Pure and Applied Mathematics. The title was "*Modular equations and approximations to p*" and he discusses certain methods to derive exact and approximate evaluations. In fact, in his Notebooks, he gave several formulas, seventeen in all, for π (or to be more precise, $1/\pi$). For a complete list, see http://mathworld.wolfram.com/PiFormulas.html. He gave little explanation on how he came up with them, other than saying that there were "corresponding theories". One of the more famous formulas is given by,

$$1/(\pi\sqrt{8}) = 1/99^2\ \Sigma\ r\,(26390n+1103)/396^{4n}$$

where $r = (4n)!/(n!^4)$ and the sum Σ (*from this point of the article onwards*) is to go from $n = 0$ to ∞.

How did Ramanujan find such beautiful formulas? We can only speculate at the heuristics he used to find them, since they were only rigorously proven to be true in 1987 by the Borwein brothers. However, one thing we do know: Ramanujan wrote down in his Notebooks, among the others, the approximation[1],

$$e^{\pi\sqrt{58}} \approx 24591257751.9999998222\ldots$$

which can also be stated as,

$$e^{\pi\sqrt{58}} \approx 396^4 - 104$$

The appearance of 396^4 in both the exact formula and the approximation is no coincidence. The connection can be highlighted even further when you realize that 26390 = 5*7*13*58. What are these numbers 58 and 396^4? The former is a squarefree *discriminant d* of a *quadratic form* $ax^2+bxy+cy^2$ and 396 can be derived from the *class invariant*,

$$g_{58}^{2} = (5+\sqrt{29})/2$$

which in turn is related to the fundamental solution ($x_1 = 5$, $y_1 = 1$) of the *Pell equation* $x^2-29y^2 = -4$.

How can a formula for π have something to do with Pell equations? That, I believe, is what makes a mathematical journey so rewarding: the unexpected and fascinating connections we stumble on along the way. We will not be able to give an answer to that question in this article but we can do so for a more tractable one: How do we derive 396 or, in general, other algebraic numbers involved in pi formulas, from class invariants? It turns out the answer has to do with transforming roots of *Weber class polynomials* into roots of *Ramanujan class polynomials*, polynomials which we have mentioned in *"Ramanujan's Constant ($e^{p\ddot{O}163}$) and its Cousins"* [1]. In that article, we also discussed *Hilbert class polynomials* and pi formulas derived from them and that is where we will start.

II. Pi Formulas

A. The *j*-function and Hilbert Class Polynomials

Similar to $e^{\pi\sqrt{58}}$ but more famous, is $e^{\pi\sqrt{163}}$ which can be shown to be approximately,

$$e^{\pi\sqrt{163}} \approx 640320^3 + 743.99999999999925\ldots$$

The reason for such close approximation is because of the *j-function*, $j(t)$, which has the *q*-series expansion,

$$j(\tau) = 1/q + 744 + 196884q + 21493760q^2 + 864299970q^3 + \ldots$$

where $q = e^{2pit}$ and t is the *half-period ratio* defined as,

Case 1: For d = 4m,

$$\tau = \sqrt{(-m)}$$

Case 2: For d = 4m+3,

$$\tau = (1+\sqrt{(-d)})/2$$

where *m* is a positive integer, and *d* is the unsigned *discriminant*.

In [1], we were quite liberal with our use of what we called the j-function but we will be more strict here. First, while in some older texts the convention was to label the j-function as *j(q)*, it seems the modern one is to label it as $j(t)$, or "j-tau". Second, an obvious observation is that since *d* is positive, then τ is imaginary. We are evaluating $j(t)$ at imaginary arguments yet it will have a real value at those points. *However, it can be positive or negative*. For case 1, for all d > 0, then $j(t)$ is always positive. For case 2, for d > 0, there are three cases,

d < 3, $j(t)$ is positive

d = 3, $j(t)$ is zero
d > 3, $j(t)$ is negative

From this point, what we will call as *j(q)* will refer to the absolute value of the j-function, while $j(t)$ will refer to the signed value. This value is an algebraic integer of degree n where n is the *class number* of d. We have also observed that $j(t)$ was a good approximation for numbers of the form $e^{\pi\sqrt{d}}$.

Case 1: If d = 4m, $\tau = \sqrt{(-m)}$, then $e^{\pi\sqrt{(4m)}} \approx j(\tau) - 744$, for d > 12.

Example 1. Let m = 5, d = 20 (class number 2),

$$j(\sqrt{-5}) = 2^3(25+13\sqrt{5})^3$$

so,

$$e^{\pi\sqrt{20}} \approx 2^3(25+13\sqrt{5})^3 - 744$$

Example 2. Let m = 10, d = 40 (class number 2),

$$j(\sqrt{-10}) = 6^3(65+27\sqrt{5})^3$$

so,

$$e^{\pi\sqrt{40}} \approx 6^3(65+27\sqrt{5})^3 - 744$$

Case 2: If d = 4m+3, $\tau = (1+\sqrt{-d})/2$, then $e^{\pi\sqrt{d}} \approx |\, j(\tau)| + 744$, for d > 12.

Example 1. Let d = 163 (class number 1),

$$j((1+\sqrt{-163})/2) = -640320^3$$

so,

$$e^{\pi\sqrt{163}} \approx 640320^3 + 744$$

Technically, case 2 is also defined for other d. For example, for d = 4m+2, say, d = 2, we have,

$$j((1+\sqrt{-2})/2) = 10^3(19-13\sqrt{2})^3$$

However, for the purposes of this article, we will limit case 2 only to form 4m+3. One should also take note that $j(\sqrt{-d})$ and $j((1+\sqrt{-d})/2)$ for d = 4m+3 are *different*, as illustrated by,

$$j(\sqrt{-7}) = 255^3, \quad \text{and} \quad j((1+\sqrt{-7})/2) = -15^3$$

which can easily be confirmed by an adequate computer algebra system. Anyway, now that we have the preliminaries out of the way, we can go to the pi formulas. To recall, we pointed out that there are exactly nine discriminants d with class number 1, the absolute values given as 3, 4, 7, 8, 11, 19, 43, 67, 163, with the larger values giving the better approximations:

$e^{\pi\sqrt{7}} \sim 15^3 + 697$
$e^{\pi\sqrt{11}} \sim 32^3 + 738$
$e^{\pi\sqrt{19}} \sim 96^3 + 744$
$e^{\pi\sqrt{43}} \sim 960^3 + 744$
$e^{\pi\sqrt{67}} \sim 5280^3 + 744$
$e^{\pi\sqrt{163}} \sim 640320^3 + 744$

We also have the formulas due to the Chudnovsky brothers (which had an their inspiration Ramanujan's formula cited in the opening paragraph):

Let, $c = (6n)!/((n!)^3(3n)!)$,

$1/(3\pi) = \Sigma\ c\ (-1)^n\ (63n+8)/(15^3)^{n+1/2}$
$1/(4\pi) = \Sigma\ c\ (-1)^n(154n+15)/(32^3)^{n+1/2}$
$1/(12\pi) = \Sigma\ c\ (-1)^n\ (342n+25)/(96^3)^{n+1/2}$
$1/(12\pi) = \Sigma\ c\ (-1)^n\ (16254n+789)/(960^3)^{n+1/2}$
$1/(12\pi) = \Sigma\ c\ (-1)^n\ (261702n+10177)/(5280^3)^{n+1/2}$
$1/(12\pi) = \Sigma\ c\ (-1)^n\ (545140134n+13591409)/(640320^3)^{n+1/2}$

The formulas obviously use the j(q)'s of the given approximations. The factorization of a term in the numerator also indicate what *d* is involved,

$63 = 3^2 * \mathbf{7}$
$154 = 2*7*\mathbf{11}$
$342 = 2*3^2 *\mathbf{19}$
$16254 = 2*3^3 *7*\mathbf{43}$
$261702 = 2*3^2 *7*31*\mathbf{67}$
$545140134 = 2*3^2 *7*11*19*127*\mathbf{163}$

Now why is this? It turns out the answer is in the general form of the formula for $d = 4m+3$, $d > 3$, (implying $j(\tau)$ as negative),

$1/\pi = \Sigma\ c\ (-1)^n\ (An+B)/(C)^{n+1/2}$

where,

$A = \sqrt{(d*(1728-j(\tau)))}$; $\qquad C = -j(\tau)$

Since, $1728 = 12^3$ (remember the taxicab anecdote?[2]) then,

$7(12^3 + 15^3) = 7^2 *27^2$
$11(12^3 + 32^3) = 4^2 *11^2 *14^2$
$19(12^3 + 96^3) = 12^2 *19^2 *18^2$
$43(12^3 + 960^3) = 12^2 *43^2 *378^2$
$67(12^3 + 5280^3) = 12^2 *67^2 *3906^2$
$163(12^3 + 640320^3) = 12^2 *163^2 *3344418^2$

For these *d*, the expression $d*(1728-j(t))$ is a perfect square! So there are beautiful Diophantine relationships behind these pi formulas as well, which explains why the factorization of *A* contains *d*. However, we are still missing the constant *B* and that is the fly in the ointment.

Technically, there is a closed-form expression for *B* in terms of Eisenstein series but it gets too complicated for this article. The interested reader is referred to "Ramanujan, Modular Equations, and Approximations to Pi" by Bailey, Borwein, and Borwein which contains a side article on the Chudnovskys' approach as well as their own which we shall discuss later. The other way to find *B* (since *A* and *C* are already known) is simply to solve the equation,

$$1/\pi = \Sigma\, c\, (-1)^n\, (An+B)/(C)^{n+1/2}$$

using enough *n* terms which should give *B* to a reasonable accuracy. For these six *d*, *B* conveniently is an integer, given by 3*8, 4*15, 12*25, 12*789, 12*10177, and 12*13591409, respectively. Since *A* and *B* have a common factor, then one can see why $1/\pi$ is divided by either 3, 4, or 12.

We have also pointed out that there are non-maximal or *non-fundamental discriminants* with class number 1 as shown by the approximations,

$$e^{\pi\sqrt{16}} \sim 66^3 - 744$$
$$e^{\pi\sqrt{28}} \sim 255^3 - 744$$

We can also use these two j(q)'s to find more pi formulas. The general form (with some small changes) for $d = 4m$ is given by,

$$1/\pi = \Sigma\, c\, (An+B)/(C)^{n+1/2}$$

where,

$$A = \sqrt{(-d*(1728-j(\tau)))}; \qquad C = j(\tau)$$

Note that there is no $(-1)^n$ for $d = 4m$. The presence of $(-1)^n$ in the formula for $d = 4m+3$ is simply the negative sign of $j(\tau)$. For $d = 4m$, since $j(\tau)$ is positive, then there is no need for it. The only difference really is in *A* since *d* is negated (though the product ends up positive). Solving for *B*, we then have two additional formulas,

$$1/\pi = 24\sqrt{2}\, \Sigma\, c\, (63n+5)/(66^3)^{n+1/2}$$
$$1/\pi = 162\, \Sigma\, c\, (133n+8)/(255^3)^{n+1/2}$$

We can, of course, use *d* with a higher class number *n*, though *B* would now be an algebraic number, usually of degree *n*. For the interested reader who wishes to find pi formulas using the method outlined above, one can find the defining polynomial of the real number *B* using the Integer Relations applet at http://www.cecm.sfu.ca/projects/IntegerRelations/.

B. Weber Class Polynomials

In [1], I remarked that I was not aware of pi formulas that used the roots of Weber class polynomials. In fact there are, and they were there all along in Ramanujan's 17 formulas! However, the roots were barely discernable and it took some time to recognize them. To recall, the *Weber modular function*, or *w(q)* had the *q*-series expansion,

$$w(q) = 1/q + 24 + 276q + 2048q^2 + 11202q^3 + \ldots$$

and $w(q)$, just like $j(q)$, are algebraic integers for suitable arguments involving imaginary quadratic irrationals. These algebraic integers are defined by the *Weber class polynomials*. For an extensive list of these polynomials, the reader is referred to Annegret Weng's website "Class Polynomials of CM-Fields", http://www.exp-math.uni-essen.de/zahlentheorie/classpol/class.html.

The w(q) also provide excellent approximations to the transcendental numbers $e^{\pi\sqrt{d}}$ for d > 0. However, we will limit our discussion to **odd** *d*. Since case 1 was for d = 4m, this restricts us to case 2 for d = 4m+3. Case 2 can be divided into *d* of form 8n+3 and 8n+7, a distinction made in [1]. However, we can further refine this as to whether *d* is divisible by 3, hence giving us four cases. Let *x be the appropriate real root* of the Weber class polynomial:

A. ***For 8n+3***: Degree of poly is *3n*, where *n* is the class number of *d.* (With the exception of *d* = 3 which has a Weber class polynomial of degree 1.)

Case 1. If 3 doesn't divide *d*, then $e^{\pi\sqrt{d}} \approx x^{24} - 24$.

Example: Let *d* =163, *n* = 1,

$$e^{\pi\sqrt{163}} \approx x^{24} - 24; \ (x^3-6x^2+4x-2=0)$$

Case 2. If 3 divides *d*, then $e^{\pi\sqrt{d}} \approx 2^8x^8 - 24$.

Example: Let *d* = 51, *n* = 2,

$$e^{\pi\sqrt{51}} \approx 2^8x^8 - 24; \ (x^6-8x^5-3x^4+6x^3+9x^2+2x+1=0)$$

B. ***For 8n+7***: Degree of poly is *n*, where *n* is the class number of *d.*

Case 3. If 3 doesn't divide *d*, then $e^{\pi\sqrt{d}} \approx 2^{12}x^{24} - 24$.

Example: Let *d* = 23, *n* = 3,

$$e^{\pi\sqrt{23}} \approx 2^{12}x^{24} - 24; \ (x^3-x-1=0)$$

Case 4. If 3 divides *d*, then $e^{\pi\sqrt{d}} \approx 2^{12}x^8 - 24$.

Example: Let *d* =39, *n* = 4,

$$e^{\pi\sqrt{39}} \approx 2^{12}x^8 - 24; \ (x^4-3x^3-4x^2-2x-1=0)$$

Again, having the preliminaries out of the way, we can go to the pi formulas. In the list given by Mathworld, the first three of Ramanujan's formulas are,

$$4/\pi = \Sigma\ (6n+1)(1/2)_n^{\ 3}\ /(4^n(n!)^3)$$
$$16/\pi = \Sigma\ (42n+5)(1/2)_n^{\ 3}/(64^n(n!)^3)$$
$$32/\pi = \Sigma\ (42\sqrt{5}n+5\sqrt{5}+30n-1)(1/2)_n^{\ 3}\ ((\sqrt{5}-1)/2)^{8n}\ /(64^n(n!)^3)$$

where $(a)_n$ is the *rising factorial*, aka *Pochhammer symbol*, such that $(a)_n = (a)(a+1)(a+2)\dots(a+n-1)$. In this guise, it is hard to see the Weber class polynomial root, though the presence of $(\sqrt{5}-1)/2$ is a clue. One way then is to reformulate them in terms of ordinary factorials.

Let, $h = ((2n)!/(n!^2))^3$. Then,

$$1/\pi = 1/4\ \Sigma\ h\ (6n+1)\ /2^{8n}$$
$$1/\pi = 1/16\ \Sigma\ h\ (42n+5)\ /2^{12n}$$
$$1/\pi = 1/32\ \Sigma\ h\ (42\sqrt{5}n+5\sqrt{5}+30n-1)\ /(2^{12}((1+\sqrt{5})/2)^8)^n$$

Compare to,

$$e^{\pi\sqrt{3}} \approx 2^8(1)^8 - 25.23 \qquad \text{(Case 2)}$$
$$e^{\pi\sqrt{7}} \approx 2^{12}(1)^{24} - 24 \qquad \text{(Case 3)}$$
$$e^{\pi\sqrt{15}} \approx 2^{12}((1+\sqrt{5})/2)^8 - 24 \qquad \text{(Case 4)}$$

Since $w(q)_3 = 1$, $w(q)_7 = 1$, $w(q)_{15} = ((1+\sqrt{5})/2)^8$ then the denominators of these formulas use the *w(q)'s*, just like the *j(q)'s* found in the Chudnovskys' formulas! (Though the *w(q)'s* are multiplied by appropriate powers of 2.) The general form of the formula is then,

$$1/\pi = \Sigma\ h\ (An+B)\ /(2^y w(q))^n$$

In Weng's website one find the polynomials that define *w(q)* for *d* up to 422500, so one can find analogous formulas, though Ramanujan found the smaller and easier cases. (Since $w(q) = x^k$ for some power *k*, one has to find *k* first.) However, unlike the formulas derived from the Hilbert class polynomials, I haven't come across any explicit references how to derive A and B from *w(q)*, other than in a passing mention in a paper (see paper by Chan, Gee, and Tan) specifically for odd *d*. It should be interesting to find other formulas like these, especially one that uses $d = 163$.

C. Ramanujan Class Polynomials

The last class polynomials we discussed in [1] were the Ramanujan class polynomials. To recall, the *r*-function *r(q)* has the q-series expansion,

$$r(q) = 1/q + 104 + 4372q + 96256q^2 + 1240002q^3 + \dots$$

and the appropriate root of the class polynomial gives the *r(q)'s*. These will useful in understanding Ramanujan's last ten pi formulas in Mathworld's list. This time, the formulas involve **even** discriminants $d = 4m$, where *d* has class number 2 (both fundamental and non-fundamental). Analogous to the Hilbert class polynomials, the approximation is,

$$e^{\pi\sqrt{m}} \sim r(q) -/+ 104$$

for even and odd *m*, respectively. For even *m*, we have,

$$e^{\pi\sqrt{6}} \sim (4\sqrt{3})^4 - 106$$
$$e^{\pi\sqrt{10}} \sim 12^4 - 104$$
$$e^{\pi\sqrt{18}} \sim 28^4 - 104$$
$$e^{\pi\sqrt{22}} \sim (12\sqrt{11})^4 - 104$$

$$e^{\pi\sqrt{58}} \sim 396^4 - 104$$

where d = 4*18 is non-fundamental. In the list, the pi formulas were in terms of rising factorials. To make the connection more clear, one can just express it in terms of ordinary factorials. Let, r = $(4n)!/(n!^4)$ and we have,

$$1/(8\pi) = \sqrt{3}\,\Sigma\, r\,(8n+1)/(4\sqrt{3})^{4n+2}$$
$$1/(16\pi) = \sqrt{8}\,\Sigma\, r\,(10n+1)/12^{4n+2}$$
$$1/(16\pi) = 3\sqrt{3}\,\Sigma\, r\,(40n+3)/28^{4n+2}$$
$$1/(8\pi) = \sqrt{11}\,\Sigma\, r\,(280n+19)/(12\sqrt{11})^{4n+2}$$
$$1/(16\pi) = \sqrt{8}\,\Sigma\, r\,(26390n+1103)/396^{4n+2}$$

One can easily see the *r(q)'s* in the denominators. For odd m, we have (with the last two d = 4*9, 4*25 as non-fundamental),

$$e^{\pi\sqrt{5}} \sim (4\sqrt{2})^4 + 100$$
$$e^{\pi\sqrt{13}} \sim (12\sqrt{2})^4 + 104$$
$$e^{\pi\sqrt{37}} \sim (84\sqrt{2})^4 + 104$$

$$e^{\pi\sqrt{9}} \sim 3*8^4 + 104$$
$$e^{\pi\sqrt{25}} \sim 20*24^4 + 104$$

giving us,

$$1/(4\pi) = \Sigma\, r\,(-1)^n(20n+3)/(4\sqrt{2})^{4n+2}$$
$$1/(4\pi) = \Sigma\, r\,(-1)^n(260n+23)/(12\sqrt{2})^{4n+2}$$
$$1/(4\pi) = \Sigma\, r\,(-1)^n(21460n+1123)/(84\sqrt{2})^{4n+2}$$

$$1/(4\pi) = 3\,\Sigma\, r\,(-1)^n(28n+3)/(3*8^4)^{n+1/2}$$
$$1/(4\pi) = 5\,\Sigma\, r\,(-1)^n(644n+41)/(20*24^4)^{n+1/2}$$

and the *r(q)'s* are apparent in the denominators as well. So how do we find formulas like these? It turns out the answer lies in *transforming Weber polynomials into Ramanujan class polynomials*, or $w(q) \to r(q)$. Thus, the transformation not only gives us pi formulas but allows us to verify the polynomials we derived for *r(q)*. The transformations are given by,

For even m,

$$r(q)_{4m} = (4\sqrt{C})^4, \quad \text{where } C = (g_m^{12} + g_m^{-12})/2$$

For odd m,

$$r(q)_{4m} = (4\sqrt{C})^4, \quad \text{where } C = (G_m^{12} - G_m^{-12})/2$$

where g_m and G_m are the *Ramanujan g- and G-functions*. See http://mathworld.wolfram.com/Ramanujang-andG-Functions.html. These are basically identical to the w(q) as given by the relationships,

$$w(q)_{4m} = g_m^{24}, \qquad w(q)_{4m} = G_m^{24}$$

Thus, let $d = 4m$ and the transformation is,

For even m,

$$r(q)_d = (4\sqrt{C})^4, \quad \text{where } C = (w(q)_d^{1/2} + w(q)_d^{-1/2})/2$$

and for odd m,

$$r(q)_d = (4\sqrt{C})^4, \quad \text{where } C = (w(q)_d^{1/2} - w(q)_d^{-1/2})/2$$

Before we go into the pi formulas, we can give an example. Since we have,

$$e^{\pi\sqrt{58}} \sim 2^6((5+\sqrt{29})/2)^{12} + 24$$

Then,

$$w(q)_{232} = ((5+\sqrt{29})/2)^{12}$$

So,

$$C = (((5+\sqrt{29})/2)^6 + ((5+\sqrt{29})/2)^{-6}) /2 = 99^2$$

Since,

$$r(q)_{232} = (4\sqrt{(99^2)})^4 = 396^4$$

Thus,

$$e^{\pi\sqrt{58}} \sim 396^4 - 104$$

as expected. In [1], we pointed out that Ramanujan gave r(q) for d with class number 2 and 4. However, we managed to find some for $d = 4m$ with class number 6, for m = 26, 106, 202, 298 using the Integer Relations applet, the first one being,

$$e^{\pi\sqrt{26}} \sim (4y)^4 - 104; \; (y^3-13y^2-9y-11=0)$$

With this transformation, we can verify if indeed these are correct. From Weng's website, we find that the polynomial for d = 4*26 = 104 is given by,

$$x^6-2x^5-2x^4+2x^2-2x-1 = 0 \qquad \text{(eq.1)}$$

such that,

$$e^{\pi\sqrt{26}} \sim 2^6x^{12} + 24$$

with x the appropriate real root. So $w(q)_{104} = x^{12}$. Or,

$$C = (x^6 + x^{-6}) /2$$

We will assume C to be a perfect square, $C = y^2$,

$$y^2 = (x^6 + x^{-6})/2 \qquad \text{(eq.2)}$$

By eliminating the variable *x* between eq.1 and eq.2, we get,

$$(y^3+13y^2-9y+11)^2(y^3-13y^2-9y-11)^2 = 0$$

since,

$$r(q)_d = (4\sqrt{C})^4$$

then $r(q)_{104} = (4\sqrt{C})^4 = (4y)^4$. Or,

$$e^{\pi\sqrt{26}} \sim (4y)^4 - 104; \ (y^3-13y^2-9y-11=0)$$

for the real root of *y*, which is precisely the same relation as the original one! So satisfying to know that the applet was correct. The same way can be used to prove the three relations for the other *d*. And not only proving, but finding other r(q) as well. In [1], we asked for the explicit equation for the r(q) of *d* with class number 8, with $d = 4m$, and *m* = 178, 226, 466, 562 (which are form *2q* for prime *q*). Using the same approach as the one above, we finally find the first one as,

$$e^{\pi\sqrt{178}} \sim (12y)^4 - 104; \ (y^4-2962y^3+72y^2-382y-71=0)$$

with *y* the appropriate real root. The same method can be used for the other *d* as well as for those with even higher class numbers.

However, two things should be pointed out. First, it will not always be the case that $w(q)_{4m} = x^{12}$, where *x* is the appropriate root of the listed polynomial. In general, it is $w(q)_{4m} = x^k$, where,

$$e^{\pi\sqrt{m}} \sim 2^6x^k +/- 24$$

for some integer *k*. Second, it will not always be that *C* will be a perfect square. Our examples use $m = 2q$ where *q* is prime. If *q* is composite, *C* may not be a square.

For the third time, now that we have the preliminaries out of the way, we can discuss the pi formulas. The general form is given by, let $r = (4n)!/(n!^4)$,

1) **Even** *m.*

$$1/(16\pi) = \Sigma\ r\,(An + B)/(4\sqrt{C})^{4n+2}$$

where,

$$A = ((g_m^{12} - g_m^{-12})/2)\sqrt{m}; \qquad C = (g_m^{12} + g_m^{-12})/2$$

2) **Odd** *m.*

$$1/(16\pi) = \Sigma\ r\ (-1)^n(An + B)/(4\sqrt{C})^{4n+2}$$

where,

$$A = ((G_m^{12} + G_m^{-12})/2)\sqrt{m}; \qquad C = (G_m^{12} - G_m^{-12})/2$$

Note that there is an exchange of arithmetic operations for the expressions for A and C. There is an explicit expression for *B* but again it is too complicated for this article. The curious reader is referred to the paper by Bailey and the Borweins for the details. The alternative method that we will use is just to solve for *B* using enough terms in the summation and to use the Integer Relations applet to determine its defining polynomial.

As an example, since we know that $w(q)_{232} = x^k = ((5+\sqrt{29})/2)^{12}$, then,

$$A = ((x^{k/2} - x^{-k/2})/2)\sqrt{58} = 52780\sqrt{2}$$

$$C = (x^{k/2} + x^{-k/2})/2 = 99^2$$

and solving for *B* up to a sufficient accuracy, the applet gives us the quadratic polynomial which has a root $B = 2206\sqrt{2}$. Thus,

$$1/(16\pi) = \Sigma\ r\ (52780\sqrt{2}n + 2206\sqrt{2})/396^{4n+2}$$

or, moving the common factor $2\sqrt{2}$,

$$1/(16\pi\sqrt{8}) = \Sigma\ r\ (26390n+1103)/396^{4n+2}$$

and we have Ramanujan's formula! In [1] we wondered if we could use higher class numbers and we certainly can. Using $d = 4m$ with class number 4 for m = 34, 82, we have the formulas,

$$1/(8\pi\sqrt{2}) = (4+\sqrt{17})\ \Sigma\ r\ (v_1n+v_2)/(4v_3)^{4n+2}$$
$$1/(16\pi\sqrt{2}) = (32+5\sqrt{41})\ \Sigma\ r\ (w_1n+w_2)/(4w_3)^{4n+2}$$

where,

$$v_1 = 8(165+41\sqrt{17})^{1/2}\sqrt{17};\ \ v_2 = (411+163\sqrt{17})^{1/2};\ \ v_3 = 3(4+\sqrt{17});$$

and,

$$w_1 = 20(1347+211\sqrt{41})^{1/2}\sqrt{82};\ \ w_2 = (51099+9097\sqrt{41})^{1/2};\ \ w_3 = 3(51+8\sqrt{41})$$

Note that $(4+\sqrt{17})$ and $(32+5\sqrt{41})$ are *fundamental units*, involved in the solutions to the Pell equations $x^2 - 17y^2 = -1$ and $x^2 - 41y^2 = -1$. They were factored from the original expressions of the v_i and w_i for i = 1, 2 which were unwieldy. For class number 6 for m = 26, we have,

$$1/(16\pi\sqrt{8}) = \Sigma\ r\ (x_1n+x_2)/(4x_3)^{4n+2}$$

where,

$x_1 = (x^3-338x^2-364x-1352)_1$; $x_2 = (8x^3-168x^2-27x-27)_1$; $x_3 = (x^3-13x^2-9x-11)_1$

or x_1, x_2, x_3 are the real roots of the cubics above, respectively, which anyway are just one-real root cubics. These are just examples and there are certainly other *d* we can use with class number 4, 6, 8, *ad infinitum.*

III. Baby Monster Group

What is the *Baby Monster*? Richard Borcherds, who proved the *Monstrous Moonshine Conjecture*, had a cute answer to this question. In a Notice of the AMS, he wrote a short article, "What is the Monster?" As to what the Baby Monster is, he jokingly wrote that when he was a grad student, his supervisor John Conway[3] would bring his one-year old son to the department, who was soon known as the baby monster.[4] For the full article, see http://math.berkeley.edu/~reb/papers/whatismonster/whatismonster.pdf.

Joking aside, the Baby Monster, also known as Fischer's baby monster group, is the second largest of the sporadic finite groups. Many of its properties were described by B. Fischer, who also contributed to the discovery of the largest sporadic finite group, namely the *Monster Group*, also known as Fischer-Griess Monster.

What is the connection between these groups and the modular functions *j(q)*, *w(q)*, and *r(q)* that we have discussed? In [1], we have pointed out that the coefficients of their expansions are in fact *McKay-Thompson series* for the Monster. Another way to illustrate the connection is to express the coefficients of the *normalized* j-function *j(t)* in terms of the irreducible representations of the Monster. To recall,

$$j(\tau) - 744 = 1/q + 196884q + 21493760q^2 + 864299970q^3 + \dots$$

Since the degrees of the irreducible representations of Monster Group *"M"* are given by 1, 196883, 21296876, 842609326, etc…(A001379, Sloane's Online Encyclopedia of Integer Sequences), we observe that,

1 = 1
196884 = 1 + 196883
21493760 = 1 + 196883 + 21296876
864299970 = 1(2) + 196883(2) + 21296876 + 842609326

or the coefficients of *j(t)* are simple linear sums of the representations of the Monster. (The second identity, which led to the Monstrous Moonshine Conjecture, was observed by John McKay in the late 1970's. J.G. Thompson carried it further to the other coefficients.)

What I noticed was that there seemed to be an analogous relationship between Ramanujan's function *r(q)*, (which is also defined by the transformation rule $w(q) \rightarrow r(q)$ that we have given) and the Baby Monster. The normalized r(q) would be,

$$r(q) - 104 = 1/q + 4372q + 96256q^2 + 1240002q^3 + \dots$$

(See A007267.) Given the degrees of irreducible representations of Baby Monster Group "*B*": 1, 4371, 96255, 1139374, etc… (A001378), we can observe that,

1 = 1
4372 = 1 + 4371
96256 = 1 + 4371(0) + 96255
1240002 = 1(2) + 4371 + 96255 + 1139374

Is this coincidence? Are the coefficients of the *q*-series of r(q) simple linear sums of the representations of the Baby Monster, or will the relationship exist only for the first few terms? The relationship between $j(\tau)$ and the Monster has already been proven so does this automatically imply this relationship? After all, the coefficient list of r(q) is also the McKay-Thompson series of class 2A for Monster, and since the *double cover* of the Baby Monster is a subgroup of the Monster, then r(q) and the Baby Monster should be connected via the Monster group, though that is just a guess.

IV. Conclusion

This article is meant to be some sort of a sequel to *Ramanujan's Constant and Its Cousins* [1], as there were some loose ends left hanging there. I hope we have clarified some points though I believe there are still others that can be explored further. The conclusions we have stated here are to supersede those of the earlier article, especially with regards to pi formulas, since this is the expanded version of that topic after all.

Because of space constraints, there were a *lot* of intriguing avenues that I didn't go into in this article. For example, while in [1] we discussed the modular functions j(q), w(q), and r(q) and here we discussed the transformation $w(q) \to r(q)$, there is in fact also a *second* transformation $j(q) \to r(q)$. However, for the purposes of this article, since Weng's website has w(q) polynomials galore and there is no corresponding website for j(q) (and understandably, as these have LARGE coefficients), the first transformation was more useful. However, there is still a *third* transformation $w(q) \to j(q)$ and that might be interesting to explore, though perhaps we can do so for another time.

To sum up the main theme of this article, which was on pi formulas, we have discussed three formulas that use the modular function w(q) and ten formulas that use r(q). That makes thirteen. But Ramanujan had *seventeen* formulas. What about the other four? It turns out that they use still *another* modular function.

But that is another story.

Footnotes:

1. Makes you wonder how people calculated numbers like $e^{\pi\sqrt{58}}$ before computers.
2. The taxicab anecdote, as if you didn't know, was Ramanujan's instant recognition of 1729 as the smallest integer that is the sum of two cubes in two ways, namely, $1729 = 1^3 + 12^3 = 9^3 + 10^3$.

3. John Conway also discovered the sporadic simple group that described the symmetries of the Leech lattice, a 24^{th}-dimensional grid. And, of course, his "Game Of Life".
4. My little nephew and godson, who is also French, can sometimes be quite the baby monster (*mon Dieu!*), so I understand the joke perfectly.

-- End --

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www.geocities.com/titus_piezas/ramanujan.html← Click here for an index of articles.

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