On Ramanujan's Other Pi Formulas

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I. Introduction

In "*Pi Formulas, Ramanujan, and the Baby Monster Group*" [1] we stated that Ramanujan came up with 17 formulas for $1/\pi$. However, in that article, we discussed only 13 of those formulas which depended on various modular functions and gave further examples per type. In this article, we will discuss two more formulas which depend on another modular function and give more examples, some *new*, of the same type of formula. The two formulas concerned are, let,

$$r_1 = (1/2)_n(1/3)_n(2/3)_n/(n!^3)$$

then,

$$1/\pi = 4/27\ \Sigma\ r_1\ (15n+2)(2/27)^n$$
$$1/\pi = 2/(15\sqrt{3})\ \Sigma\ r_1\ (33n+4)(4/125)^n$$

where the summation Σ is understood to go from n = 0 to ∞ (*from this point of the article onwards)* and $(a)_n$ is the *rising factorial*, aka *Pochhammer symbol*, such that $(a)_n = (a)(a+1)(a+2)\ldots(a+n-1)$. We will see that the denominators of this formulas, just like the others, closely approximate transcendental numbers of the form $e^{\pi\sqrt{d}}$, though it will take some manipulation to see these relations.

II. More Pi Formulas

To see the relationship of these formulas to certain transcendental numbers, we slightly re-define our rising factorial r_1. Let $\mathbf{r_2}$ be,

$$r_2 = 2^{2n}3^{3n}(1/2)_n(1/3)_n(2/3)_n/(n!^3)$$

where the superscript *n* is ordinary exponentiation and the subscript *n* is still the rising factorial. (The constant $2^2 3^3 = 108$ may not be necessary if r_2 is expressed as ordinary factorials, though it will appear again later.) Ramanujan's two formulas, plus another one he missed, are then given by,

$1/\pi = 1/(3\sqrt{3})\ \Sigma\ r_2\ (6n+1)\ /(6^3)^n$ (Chan and Liaw)
$1/\pi = 4/27\ \Sigma\ r_2\ (15n+2)\ /(2*9^3)^n$ (Ramanujan)
$1/\pi = 2/(15\sqrt{3})\ \Sigma\ r_2\ (33n+4)\ /(15^3)^n$ (Ramanujan)

Compare the denominators, we'll call them *h(q)*, to the approximations,

$e^{\pi\sqrt{(8/3)}} \approx 6^3 - 46.95$
$e^{\pi\sqrt{(16/3)}} \approx 2*9^3 - 42.55$
$e^{\pi\sqrt{(20/3)}} \approx 15^3 - 42.23$

and another by Chan and Liaw involving h(q) with quadratic irrationals,

$$1/\pi = 1/99\ \Sigma\ r_2\ (v_1n+v_2)\ /\ (6(4+3\sqrt{3}))^{3n}$$

where $v_1 = 15(27-\sqrt{3})$, $v_2 = 54-13\sqrt{3}$, so,

$$e^{\pi\sqrt{(44/3)}} \approx 6^3(4+3\sqrt{3})^3 - 42.004$$

These formulas then are somehow related to these approximations, and h(q) joins the club we have given for j(q), w(q), and r(q) discussed in [1]. The "excess" of the approximation is obviously approaching the integer *42*, especially for higher values of *m*, and we will see that there is also an established integer sequence that corresponds to this modular function h(q).

One can find the details for these formulas and others in Chan and Liaw's paper *"Cubic Modular Equations and New Ramanujan-Type Series for 1/p"* (2000) [2]. They give the formulas associated with discriminants *d* of form *d =12m* for m = 2, 4, 5 with class number 2 and for m = 7, 10, 11, 14, 19, 26, 31, 34, 59 with class number 4. (The above example involving quadratic irrationals uses m = 11). The largest *m* in the list, m = 59, yields,

$$e^{\pi\sqrt{(236/3)}} \approx 6^3(892+525\sqrt{3})^3 - 42.00000000062$$

since 236 = 4*59. The smallest *m*, m = 2, was missed by Ramanujan, giving the formula stated earlier,

$$1/\pi = 1/(3\sqrt{3})\ \Sigma\ r_2\ (6n+1)\ /\ 6^{3n}$$

though he had a similar-looking formula. Given $h = ((2n)!/(n!^2))^3$, then,

$$1/\pi = 1/4\ \Sigma\ h\ (6n+1)\ /2^{8n}$$

However, they are based on different discriminants as the former, with d = 12(2) = 24 has class number 2, while the latter, with d = 3 (see [1]) has class number 1.

We are familiar that the formulas based on j(q) and r(q) come in two types: plain sums and alternating sums. *For h(q), there is also an alternating sum version*. (We only had three examples based on w(q) and they were plain sums but I wouldn't be surprised if there is also an alternating sum type.) The first few are,

$$1/\pi = (1/4)\sqrt{3}\ \Sigma\ r_2\ (-1)^n (5n+1)\ /(3*4^3)^n$$

$$1/\pi = (1/12)\sqrt{15}\,\Sigma\ r_2\ (-1)^n\ (9n+1)\ /(5*12^3)^n$$
$$1/\pi = (1/108)\sqrt{7}\,\Sigma\ r_2\ (-1)^n\ (165n+13)\ /(7*36^3)^n$$

corresponding to *non-fundamental discriminants* d = 3m for m = 9, 25, 49 with class number 2 and related to the approximations,

$$e^{\pi\sqrt{(9/3)}} \approx 3*4^3 + 38.8$$
$$e^{\pi\sqrt{(25/3)}} \approx 5*12^3 + 41.91$$
$$e^{\pi\sqrt{(49/3)}} \approx 7*36^3 + 41.998$$

Since the *m* are squares of the first few odd primes, the h(q) look especially nice, being the product of that prime and a cube. For *fundamental* discriminants d = 3m also with class number 2, we have m = 17, 41, 89. So,

$$1/\pi = 1/(12\sqrt{3})\,\Sigma\ r_2\ (-1)^n\ (51n+7)\ /(12^3)^n$$
$$1/\pi = 1/(96\sqrt{3})\,\Sigma\ r_2\ (-1)^n\ (615n+53)\ /(48^3)^n$$
$$1/\pi = 1/(1500\sqrt{3})\,\Sigma\ r_2\ (-1)^n\ (14151n+827)\ /(300^3)^n$$

related to,

$$e^{\pi\sqrt{(17/3)}} \approx 12^3 + 41.6$$
$$e^{\pi\sqrt{(41/3)}} \approx 48^3 + 41.99$$
$$e^{\pi\sqrt{(89/3)}} \approx 300^3 + 41.99997$$

One can easily see the h(q) in the denominators of the formulas. For the fundamental discriminants, the factorization of a term in the numerator also indicates what *d* is involved,

51 = 3***17**
615 = 3*5***41**
14151 = 3*53***89**

Only the formulas for m = 9, 17 are from *"Cubic Singular Moduli, Ramanujan's Class Invariants l_n, and the Explicit Shimura Reciprocity Law"* (2003) [3] by Chan, Gee, and Tan, while the other four are new to the literature and have been derived by this author.

III. Dedekind Eta Function

So how do we find these formulas? In [2], the authors gave an algorithm for finding them. We can use an alternative method, based on theirs, that may be relatively more straightforward. There are three parameters in our formulas; given h(q), one can immediately find a second parameter. The third can be derived from a rather complicated expression or one can use a shortcut and use *integer relations*. Thus, the crucial value is h(q). Fortunately, it can be expressed in terms of another modular function called the *Dedekind eta function*, $h(t)$. This is defined as,

$$\eta(\tau) = q^{1/24}\,\Pi\ (1-q^k)$$

where the product is to go from $k = 1$ *to* ¥, $q = e^{2pit}$, and argument t is the half-period ratio. See http://mathworld.wolfram.com/DedekindEtaFunction.html. Just like the j-function, in the context of our article we can distinguish two cases, *depending on whether the discriminant d is even or odd*,

Case 1 (Even): For d = 12m, *m* any positive integer. Let,

$eta_{1a} = \eta(\sqrt{(-m/3)})$; $eta_{1b} = \eta(\sqrt{(-3m)})$;

and the eta quotient,

$y_m = (eta_{1a}/eta_{1b})^{12}$

then our *h(q)* for even *d* is defined as,

$h(q)_m = (y_m+27)^2/y_m$

Case 2 (Odd): For d = 3k, *k* an **odd** integer. Let,

$eta_{2a} = \eta((1+\sqrt{(-k/3)})/2)$; $eta_{2b} = \eta((1+\sqrt{(-3k)})/2)$;

and,

$y_k = (eta_{2a}/eta_{2b})^{12}$

then *h(q)* for odd *d* is,

$h(q)_k = (y_k+27)^2/y_k - 108$

(Notice the appearance of the constant $108 = 2^2 3^3$.)

There are exactly *four* values of *m* and *eight* values of *k* such that h(q) is an integer, namely m = 1, 2, 4, 5 and k = 1, 5, 9, 17, 25, 41, 49, 89 (though for *m = 1* and *k = 1, 5* yields h(q) < 108). We can use the remaining nine values to find pi formulas, and the complete list is in this article. For other *m* or *k*, h(q) is an algebraic integer of higher degree. Since for such h(q) we already have an example for *m* (m = 11), we can also give an example for *k*. Let k = 13, d = 3(13) = 39, with class number 4 and we have,

$1/\pi = 1/9 \, \Sigma \, r_2 \, (-1)^n (w_1 n+w_2)/(w_3)^n$

where $w_1 = (-19+14\sqrt{13})^{1/2}\sqrt{13}$; $w_2 = 2/3(-35+13\sqrt{13})^{1/2}\sqrt{2}$; $w_3 = (27/2)(23+7\sqrt{13})$.

The general form of these formulas and approximations are given by:

Let, $r_2 = 2^{2n}3^{3n}(1/2)_n(1/3)_n(2/3)_n/(n!^3)$ for both cases.

Case 1: For *d = 12m*,

$1/\pi = \Sigma \, r_2 \, (An+B)/C^n$

where,

$$A = 1/3\sqrt{(d(1-2^2 3^3 h(q)^{-1}))}, \qquad C = h(q)$$

and,

$$e^{\pi\sqrt{(4m/3)}} \approx h(q) - 42$$

Case 2: For $d = 3k$, for odd k,

$$1/\pi = \Sigma\, r_2\, (-1)^n(An+B)/C^n$$

where,

$$A = 1/3\sqrt{(d(1+2^2 3^3 h(q)^{-1}))}, \qquad C = h(q)$$

and,

$$e^{\pi\sqrt{(k/3)}} \approx h(q) + 42$$

We are still missing B. Since we already have two of the three unknowns, we can use the same method in [1], i.e., by using a sufficient number of n terms and solving the general form to get a close numerical approximation for B. One can then use the Integer Relations applet at http://www.cecm.sfu.ca/projects/IntegerRelations/ to get its defining polynomial. Or one can use the explicit expression for B found in [2]. For h(q), most computer algebra systems can compute Dedekind eta functions though not necessarily its defining polynomial if its value happened to be algebraic. One can just use the applet though at some other time we will discuss a more efficient way of deriving it from known polynomials.

IV. Monster Group (Again) and Conclusion

We mentioned earlier that h(q) has an integer sequence associated with it. Its *q-series* expansion is given by,

$$h(q) = 1/q + 42 + 783q + 8672q^2 + 65367q^3 + \ldots$$

(A030197, Sloane's Encyclopedia of Integer Sequences) and the integer *42* explains the disparity, positive or negative, of the approximations.

And what is this sequence also known as? It's again a *McKay-Thompson series*, this time for class 3A of Monster! So no wonder there were all those 3's, whether as 3^3, $\sqrt{3}$, or $e^{\pi\sqrt{(n/3)}}$. To recall, j(q) was associated with a McKay-Thompson series of class 1, w(q) and r(q) for class 2 (or if signed, class 4). And now we have h(q), of class 3. But these series *go on* for higher degrees. *So will there be pi formulas using modular functions associated with higher order series?* That should be an interesting question to answer.

As if to tantalize us with a possible answer, Ramanujan had *two* more formulas up his sleeve. The last two in the list are given by,

$$1/\pi = \sqrt{(12/125)}\, \Sigma\, r_3\, (11n+1)(4/125)^n$$
$$1/\pi = (18\sqrt{3})/(85\sqrt{85})\, \Sigma\, r_3\, (133n+8)(4/85)^n$$

where $r_3 = (1/2)_n(1/6)_n(5/6)_n/(n!^3)$ and $(a)_n$ still is the rising factorial. If these turn out to be associated with a McKay-Thompson series of class 6, then it should not be a surprise.

In conclusion, there is this wonderful relationship between two seemingly different mathematical objects, the transcendental numbers π and *e* on one hand, and finite groups on the other. Ramanujan could not have known about the Monster Group (it was constructed by Griess only in 1982), but he knew π very well, and we are just peering over his shoulder to see what was so fascinating to him.

While we have pointed out before that Ramanujan didn't really come up with the number eponymously known after him, namely $e^{\pi\sqrt{163}}$ (a better candidate is $e^{\pi\sqrt{58}}$), I recently learned that he was in fact working on the numbers 11, 19, 43, 67, and 163, discriminants *d* with class number 1. In his lost Notebook, he jotted down some of the results of his investigation on the properties of these particular numbers. See "Eisenstein Series and Approximations to Pi" by Berndt and Chan. I have no doubt that if only he lived long enough, he would have found the formulas which the Chudnovsky brothers found in 1987.

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www.geocities.com/titus_piezas/ramanujan.html← Click here for an index of articles.

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