"A New Way To Derive The Bring-Jerrard Quintic In Radicals"

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Abstract: We derive, *in radicals*, the Bring-Jerrard quintic using a *cubic* Tschirnhausen transformation instead of the usual quartic Tschirnhausen transformation which was essentially the method employed by Erland Bring (1736-1798) and George Jerrard (1804-1863). Certain limitations of the new method as applied to higher degrees will also be discussed.

Dedicated to Cesar Piezas and Maribeth Piezas-Niere

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I. Introduction

Among the applications of the *Tschirnhausen transformation*, its more notable use is that it enables one to find a formula for the general quintic, though one has to go beyond radicals and use other functions, such as hypergeometric ones as was first done by Felix Klein (1849-1925) or elliptic functions as Charles Hermite (1822-1901) did. This transformation was named after Count Ehrenfried von Tschirnhaus (1651-1708) who in a short four-page paper "*On a method for removing all intermediate terms from a given equation*" proposed a method to solve the general *n*th degree equation.

It starts from the simple observation that given, say, a general cubic equation,

$$x^3+ax^2+bx+c = 0$$

by a change of variable $y = x+r$ for some indeterminate r, a new cubic is formed,

$$y^3 +(-3r+a)y^2+(3r^2-2ar+b)y+(-r^3+ar^2-br+c) = 0$$

One can then eliminate any of the intermediate terms (the y^2 or y term) by equating the coefficient to zero and solving for r involving an equation less than a cubic. This can obviously be applied to any *n*th degree equation to eliminate any of its intermediate terms using an equation of degree less than n. Tschirnhaus' insight was to allow more general substitutions,

$$y = x^m + r_{m-1}x^{m-1} + \ldots r_1 x + r_0$$

where m intermediate terms can be eliminated *simultaneously*, as the m parameters r_k enable us to fulfill m conditions. The procedure will be illustrated in the next two sections when we derive the principal and Bring-Jerrard quintic forms. However, this can be broadened even more by allowing *fractional* transformations. In its most general sense, the Tschirnhausen transformation then of a polynomial equation $f(x) = 0$ is of the form $y = g(x)/h(x)$ where g and h are polynomials and $h(x)$ does not vanish at a root of $f(x) = 0$, (Weisstein). We will make use of such a fractional form later.

II. The Principal Quintic

The principal quintic lacks two terms,

$$y^5 + P_1 y^2 + P_2 y + P_3 = 0$$

and the importance of this form is that it is solvable in terms of hypergeometric functions and a related icosahedral equation as was first demonstrated by Klein. Given the general quintic,

$$x^5+px^4+qx^3+rx^2+sx+t = 0$$

and the quadratic Tschirnhausen transformation,

$$y = x^2+ax+b$$

the variable x can be eliminated between the two using resultants to form a new quintic,

$$y^5+c_1y^4+c_2y^3+c_3y^2+c_4y+c_5 = 0$$

where the c_k are polynomials in terms of the coefficients p,q,r,s,t and *at most* of degree k in the unknowns a,b. This can easily be done in *Mathematica* by using the resultant function,

Resultant[*poly1*, *poly2*, *var*]

where *var* is the variable to be eliminated. (And similarly also for other computer algebra systems.) Explicitly c_1 and c_2 are then,

$$c_1 = -5b+ap-p^2+2q$$
$$c_2 = 10b^2+4b(p^2-2q)+a^2q+q^2-2pr+a(-4bp-pq+3r)+2s$$

The principal quintic form can then be acquired by setting $c_1=c_2=0$ and solving for the unknowns *a,b* which in general would only need a quadratic. The square root of the discriminant *D* of this quadratic was called by Klein as an *accessory radical*, as it does not diminish the Galois group of the quintic. (Doyle, McMullen, p. 20-21). Later we shall see that to derive the Bring-Jerrard form, we need to take the square root of the discriminant of the principal quintic.

Example:

Given the non-solvable quintic $x^5-x^4-x^3-x^2-x-1 = 0$ with discriminant $D = 2^4(599)$, the characteristic polynomial of the so-called *pentanacci numbers*, we use the resultant to eliminate *x* between it and the Tschirnhausen transformation $y = x^2+ax+b$. Collecting the variable *y*,

Collect[Resultant[x^5-x^4-x^3-x^2-x-1,y-(x^2+ax+b),x],y]

we get the new quintic $y^5+c_1y^4+c_2y^3+c_3y^2+c_4y+c_5 = 0$ where,

$$c_1 = -3-a-5b$$
$$c_2 = -3-4a-a^2+12b+4ab+10b^2$$

Solving $c_1 = c_2 = 0$ for this particular example conveniently involves only rational numbers and we find that $a = -3, b = 0$. So, using the transformation $y = x^2-3x$, the pentanacci equation has the principal quintic form,

$$y^5+2y^2+47y+122 = 0$$

III. The Bring-Jerrard Quintic via a Quartic Tschirnhausen Transformation

The Bring-Jerrard quintic on the other hand is important to Hermite's solution of the general quintic in terms of elliptic functions. It was named after Erland Bring (1736-1798) and George Jerrard (1804-1863) who worked independently of each other. This form lacks three terms,

$$z^5 + J_1z + J_2 = 0$$

and simple scaling can reduce it even further to the *Bring quintic form*,

$$z^5 + z + B_1 = 0$$

To eliminate *three* terms from the general quintic, it is reasonable to assume using a cubic Tschirnhausen transformation,

$$y = x^3+ax^2+bx+c$$

However, as was seen, the coefficients c_k would in general involve the kth powers of the unknowns a,b,c and by resolving this system of three equations one ends up with a sextic. This was noticed by the philosopher-mathematician Gottfried Liebniz (1646-1716), (Tignol), as a major difficulty of Tschirnhaus' method.

Around 1786, Bring (and later c. 1836, Jerrard in Hamilton's report) found a way around the problem by a method equivalent to using a *quartic* transformation, with the extra parameter used to prevent elevation of the degree of the final equation. In Adamchik's and Jeffrey's paper *"Polynomial Transformations of Tschirnhaus, Bring, and Jerrard"* [1] they give a very elegant derivation of this form and we'll reproduce this here with a small departure in the last step. Given the principal quintic form,

$$y^5 + ry^2 + sy + t = 0$$

and the quartic Tschirnhausen transformation

$$z = y^4 + ay^3 + by^2 + cy + d$$

eliminating y between the two again using resultants, we get the quintic,

$$z^5 + c_1 z^4 + c_2 z^3 + c_3 z^2 + c_4 z + c_5 = 0$$

where the c_k are in the coefficients r,s,t and in the unknowns a,b,c,d. Explicitly, c_1 and c_2 are,

$$c_1 = -5d + 3ar + 4s$$
$$c_2 = 10d^2 - 12adr + 3a^2r^2 - 3br^2 + 2b^2s - 16ds + 5ars + 6s^2 + 5abt - 4rt + c(3br + 4as + 5t)$$

Let $c_1 = 0$ and solving for d,

$$d = (3ar + 4s)/5$$

The next step is the ingenious algebraic trick of "freeing-up" the variable c for other tasks *by eliminating it in c_2*. Obviously this can be done by letting,

$$3br + 4as + 5t = 0$$

and solving for b,

$$b = -(4as + 5t)/(3r)$$

Setting $c_2 = 0$ and substituting into it these two expressions for b,d, we get a quadratic solely in the variable a and the coefficients r,s,t given by,

$$a^2(-27r^4 + 160s^3 - 300rst) + a(-27r^3s + 400s^2t - 375rt^2) + (-18r^2s^2 + 45r^3t + 250st^2) = 0 \qquad \text{(eq.1)}$$

and we find the first unknown! The discriminant D of this quadratic,

$$D = (-27r^3s+400s^2t-375rt^2)^2-4(-27r^4+160s^3-300rst)(-18r^2s^2+45r^3t+250st^2)$$

is in fact the discriminant of the principal quintic, up to the factor $45r^2$. Curiously, note that the analogous process applied to the "principal *sextic*" will yield a quadratic in the variable a whose discriminant is *not* the discriminant of the principal sextic (a result, in general, for the principal nth-ic for $n > 5$), hence making the quintic rather special.

With a,b,d now determined, the role of the variable c appears. In [1], the authors suggested using the *power sums* of the coefficients to eliminate the z^2 term (as well as for the z^4 and z^3). However, since as was pointed out the coefficients c_k (easily given by *Mathematica*) are expressions in the unknowns at most of degree k and collecting the variable c for these,

$$c_3 = v_0c^3 + v_1c^2 + v_2c + v_3$$
$$c_4 = w_0c^4 + w_1c^3 + w_2c^2 + w_3c + w_4$$

(where both the v_i and w_i are expressions in r,s,t, and a,b,d) one can ask why not simply set $c_3 = 0$ and solve the cubic in c? That is what we'll do to derive the *Bring-Jerrard quintic*,

$$x^5 + J_1x + J_2 = 0$$

as was first done by Bring, or set $c_4 = 0$, solve the quartic in c to get the *Euler-Jerrard quintic*,

$$z^5 + E_1z^2 + E_2 = 0$$

as was first achieved by Euler (Weisstein, "*Quintic Equation*"). In general, this kind of Tschirnhausen transformation can simultaneously eliminate in radicals the x^{n-1}, x^{n-2}, and x^{n-3} (*or* x^{n-4}) terms of the general nth degree equation for $n > 3$, though it is pointless to apply it to the case $n = 4$ as one ends up solving the same quartic (Hamilton, p.10). This is *not* to say though that the quartic cannot be transformed into binomial form in radicals in a non-trivial manner. One can use a cubic transformation to reduce it so. While indeed it would involve a final equation that is a sextic, it is quite easy to show that this would be solvable in radicals. In general, we can prove that irreducible solvable equations of the nth degree can be reduced to *binomial* form in radicals using a Tschirnhausen transformation of n-1 degree. See "*Solving Solvable Quintics Using One Fifth Root Extraction*" by this author.

Example:

Using the principal form given earlier,

$$y^5+2y^2+47y+122 = 0$$

and the quartic Tschirnhausen transformation $z = y^4+ay^3+by^2+cy+d$, by eliminating *y* between the two we get,

$$z^5+c_1z^4+c_2z^3+c_3z^2+c_4z+c_5 = 0$$

where,

$$c_1 = 188+6a-5d$$
$$c_2 = 2(6139+235a+6a^2-6b+305ab+47b^2)-8(94+3a)d+10d^2+2(305+94a+3b)c$$

Following the procedure outlined earlier, we find,

$$a = (-1509782+3243\sqrt{2995})/411589$$
$$b = (5461621-101614\sqrt{2995})/411589$$
$$d = 2(34160020+9729\sqrt{2995})/2057945$$

Note that the discriminant of the pentanacci equation is $2^4(599)$ and that 5*599 = 2995. These values are enough to set $c_1=c_2=0$. To set $c_3 = 0$, we need to solve a rather complicated cubic in *c*, which for this case is a one-real root cubic. Approximately this is *c* = 1010.29006103… and with all *a,b,c,d* known the Bring-Jerrard quintic form, with approximate coefficients, is then,

$$z^5 + (4.840918\text{x}10^{13})z + (1.258842\text{x}10^{17}) = 0$$

with the unique real root $z_1 = -1976.819519\ldots$ Reversing the transformation,

$$z_1 = y^4+ay^3+by^2+cy+d$$

by solving this quartic, three of the roots will be extraneous but one,

$$y_1 = -2.032892\ldots$$

is precisely the real root of the given principal quintic $y^5+2y^2+47y+122 = 0$.

IV. The Bring-Jerrard Quintic via a Cubic Tschirnhausen Transformation

It turns out we *can* use a cubic Tschirnhausen transformation, *though it has to be of the fractional sort*. Again, given the principal quintic,

$$y^5 +ry^2+sy+t = 0$$

and the cubic Tschirnhausen transformation

$$z = (y^3+ay^2+by+c)/(y+d)$$

we eliminate the variable y still using resultants to get the quintic,

$$c_0z^5+c_1z^4+c_2z^3+c_3z^2+c_4z+c_5 = 0$$

where the c_k are slightly messier expressions in the coefficients r,s,t and in the unknowns a,b,c,d. Explicitly, c_1 is given by,

$$c_1 = -ad(3d^2r-4ds+5t)+b(3d^2r-4ds+5t)-c(5d^4-2dr+s)+d^2(3d^2r-4ds+5t)$$

Set $c_1 = 0$ and among the linear variables, solve for a. (Solving b or c needs more work later.) Substitute this to $c_2 = 0$ and we get the equation,

$$P(c,d)-b(5c-3r)(3d^2r-4ds+5t)^2 = 0$$

where $P(c,d)$ is a polynomial in r,s,t and c,d and which is complicated to write down. The more important point is that obviously the above equation is susceptible to the same algebraic trick used earlier, namely "freeing-up" a variable, this time b by letting,

$$5c-3r = 0$$

or,

$$c = 3r/5.$$

Substituting this into $P(c,d) = 0$ (which makes $c_2 = 0$), we get the quadratic solely in the unknown d,

$$d^2(-27r^4+160s^3-300rst)+d(27r^3s-400s^2t+375rt^2)+(-18r^2s^2+45r^3t+250st^2) = 0 \qquad \text{(eq.2)}$$

which if one notices is, up to sign, essentially the same equation as (eq.1)! Solving for d, and substituting a,c,d into $c_3 = 0$, one has to solve a cubic in b. Alternatively, if the Euler-Jerrard quintic form is desired, into $c_4 = 0$ and solve a quartic in b. These known values of b,c,d will then define the numerical value of a.

Example: Using the same principal quintic for comparison,

$$y^5+2y^2+47y+122 = 0$$

but this time the cubic Tschirnhausen transformation $z = (y^3+ay^2+by+c)/(y+d)$, by eliminating y between the two we get,

$$c_0z^5+c_1z^4+c_2z^3+c_3z^2+c_4z+c_5 = 0$$

where the explicit expressions are still a bit messy. Using the procedure and starting with the known result that $c = 3r/5$, we find that,

$$c = 6/5$$

$$ad(305-94d+3d^2) = b(305-94d+3d^2)-(1/5)(141-12d-1525d^2+470d^3)$$
$$d = (1509782+3243\sqrt{2995})/411589$$

Substituting these values for *a*,*c*,*d* into $c_3 = 0$, we have to solve a cubic in *b*, which this time has three real roots. Arbitrarily choosing one, $b = -435.63831050\ldots$, which together with *d* should define the value of *a*. The Bring-Jerrard form is then,

$$(3.73359\text{x}10^6)z^5 + (1.49365\text{x}10^{16})z - (6.17569\text{x}10^{18}) = 0$$

(one can just divide by the leading coefficient) which has the unique real root $z_1 = 234.85708481\ldots$. such that by solving the cubic

$$z_1(y+d) = (y^3+ay^2+by+c)$$

which has three real roots, with two extraneous but one,

$$y_1 = -2.032892\ldots$$

is again the real root of the given principal quintic $y^5+2y^2+47y+122 = 0$.

As was pointed out in [1], by inverting the Tschirnhausen transformation one has to deal with extraneous solutions. To quote, "…*It is interesting to note that if one used Tschirnhaus' cubic transformation to solve a quintic (using something other than radicals), then one would obtain 15 solution candidates. By using a quartic transformation, Bring and Jerrard simplified the intermediate expressions at the price of now generating 20 solution candidates.*" (p.93)

It turns out that one in fact *can* use a cubic Tschirnhausen transformation, though of the fractional sort, and the extra variable in the denominator is enough to enable us to derive the Bring-Jerrard quintic form still in the radicals. On the downside, while the quartic transformation suggested by Bring-Jerrard can be non-trivially applied to equations of degree $n = 5$ and above, this fractional cubic transformation can be non-trivial in radicals *only for n = 5 and 6*. For $n = 6$, one can use the same steps to have a final equation in the unknown *d*, but now it is a quartic! For $n = 7$ and above, it steadily gets higher hence giving a limit to the applicability of the method.

V. Conclusion: Beyond three terms and eliminating *four* terms

While Jerrard is noted for the transformation that can eliminate three terms from the general quintic and higher degrees, he in fact proposed *four* kinds of transformations in radicals. In Hamilton's report [3], these are described, namely:

1) Eliminating x^{n-1}, x^{n-2}, x^{n-3}.
2) Eliminating x^{n-1}, x^{n-2}, x^{n-4}.
3) Eliminating x^{n-1}, x^{n-3} and setting $c_2^{\,2} = mc_4$ for an arbitrary *m*.

4) Eliminating x^{n-1}, x^{n-2}, x^{n-3}, x^{n-4}.

The first two are familiar and we know how to attain them. The third, a variant of which is to eliminate x^{n-1}, x^{n-3}, x^{n-5}, would reduce solving sextics to solving a quintic. Finally, with the fourth, Jerrard hoped it could solve the general quintic by reducing it to binomial form. Unfortunately, in Hamilton's analysis while these ingenious transformations of an *n*th degree equation were indeed valid, each had an effective lower limit, being $n = 5$ for (1) and (2), $n = 7$ for (3), and $n = 10$ for (4). It then implies that starting with the *decic*, as much as four terms can be eliminated in radicals from the general *n*th degree equation!

With the advent of computer algebra systems, it might be feasible and interesting to find the explicit Tschirnhausen transformation for the fourth Jerrard transformation and present it in a relatively concise manner. For those interested, the paper is: "*Inquiry into the validity of a method recently proposed by George B., Jerrard, Esq., for transforming and resolving equations of elevated degrees*", http://www.maths.tcd.ie/pub/HistMath/People/Hamilton/Jerrard/

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References:

1. Adamchik, V., and Jeffrey, D., "Polynomial Transformations of Tschirnhaus, Bring, and Jerrard", ACM SIGSAM Bulletin, Vol 37, No. 3, Sept 2003, http://www.apmaths.uwo,ca/~djeffrey/Offprints/Adamchik.pdf
2. Doyle, P., and McMullen, C., "Solving The Quintic By Iteration", http://abel.math.harvard.edu/_ctm/papers/home/text/papers/icos/icos.pdf
3. Hamilton, W.R., "Inquiry into the validity of a method recently proposed by George B., Jerrard, Esq., for transforming and resolving equations of elevated degrees", http://www.maths.tcd.ie/pub/HistMath/People/Hamilton/Jerrard/
4. MacTutor History of Mathematics Archive, http://www-gap.dcs.st-and.ac.uk/~history/index.html
5. Piezas, T., "Solving Solvable Quintics Using One Fifth Root Extraction", http://www.geocities.com/titus_piezas/quintics.html.
6. Tignol, J., Galois' Theory of Algebraic Equations, World Scientifc, 2001
7. Weisstein, E., "Quintic Equation", CRC Concise Encyclopedia of Mathematics, Chapman & Hall/CRC, 1999, or at http://mathworld.wolfram.com.