Modern Physics

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Who can use this?

The lecture notes are tailor-made for my students at SBMJCE, Bangalore. It is the first of eight chapters in Engineering Physics [06PHY12] course prescribed by VTU for the first-semester (September 2008 - January 2009) BE students of all branches. Any student interested in exploring more about the course may visit the course homepage at *www.satheesh.bigbig.com/EnggPhy*. For those who are looking for the economy of studying this: this chapter is worth 20 marks in the final exam! Cheers ;-)

Syllabus as prescribed by VTU

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Introduction to blackbody radiation spectrum; Photoelectric effect;
Compton effect; Wave particle Dualism; de Broglie hypothesis:de Broglie
wavelength, Extension to electron particle; Davisson and Germer
Experiment, Matter waves and their Characteristic properties; Phase
velocity, group velocity and particle velocity; Relation between phase
velocity and group velocity; Relation between group velocity and
particle velocity; Expression for de Broglie wavelength using group
velocity.
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Reference

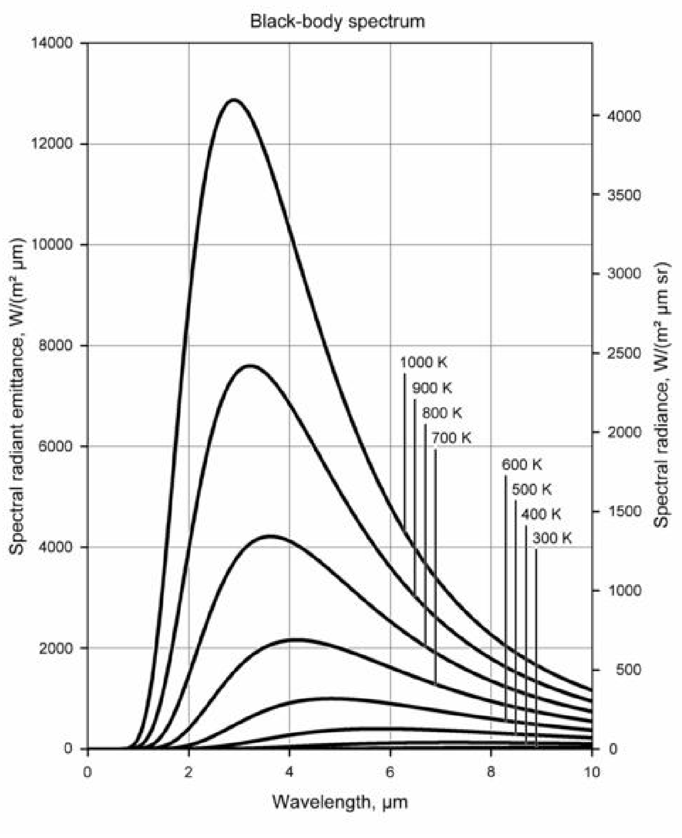
- **Arthur Beiser**, *Concepts of Modern Physics*, 6^{th} Edition, Tata McGraw-Hill Publishing Company Limited, ISBN- 0-07-049553-X.

1 Introduction

The turn of the 20^{th} century brought the start of a revolution in physics. In 1900, Max Planck published his explanation of blackbody radiation. This equation assumed that radiators are quantized, which proved to be the opening argument in the edifice that would become quantum mechanics. In this chapter, many of the developments which form the foundation of modern physics are discussed.

2 Blackbody radiation spectrum

A *blackbody* is an object that absorbs all light that falls on it. Since no light is reflected or transmitted, the object appears black when it is cold. The term *blackbody* was introduced by Gustav Kirchhoff in 1860. A perfect blackbody, in thermal equilibrium, will emit exactly as much as it absorbs at every wavelength. The light emitted by a blackbody is called *blackbody radiation*.



The plot of distribution of emitted energy as a function of wavelength and temperature of blackbody is know as *blackbody spectrum*. It has the following characteristics.

- The spectral distribution of energy in the radiation depends only on the temperature of the blackbody.
- The higher the temperature, the greater the amount of total radiation energy emitted and also energy emitted at individual wavelengths.
- The higher the temperature, the lower the wavelength at which maximum emission occurs.

Many theories were proposed to explain the nature of blackbody radiation based on classical physics arguments. But non of them could explain the complete blackbody spectrum satisfactorily. These theories are discussed in brief below.

2.1 Stefan-Boltzmann law

The Stefan-Boltzmann law, also known as Stefan's law, states that the total energy radiated per unit surface area of a blackbody in unit time (known variously as the blackbody irradiance, energy flux density, radiant flux, or the emissive power), $E_\star$, is directly proportional to the fourth power of the blackbody's thermodynamic temperature T (also called absolute temperature):

$$E_\star = \sigma T^4. \tag{1}$$

The constant of proportionality σ is called the Stefan-Boltzmann constant or Stefan's constant. It is not a fundamental constant, in the sense that it can be derived from other known constants of nature. The value of the constant is $5.6704 \times 10^{-8}\, J\, s^{-1}\, m^{-2}\, K^{-4}$. The Stefan-Boltzmann law is an example of a *power law.*

2.2 Wien's displacement law

Wien's displacement law states that there is an inverse relationship between the wavelength of the peak of the emission of a blackbody and its absolute temperature.

$$\lambda_{max} \propto \frac{1}{T}$$

$$T\lambda_{\text{max}} = b \tag{2}$$

where

λ_{max} is the peak wavelength in meters,

T is the temperature of the blackbody in kelvins (K), and

b is a constant of proportionality, called Wien's displacement constant and equals $2.8978 \times 10^{-3}\, mK$.

In other words, Wien's displacement law states that the hotter an object is, the shorter the wavelength at which it will emit most of its radiation.

2.3 Wien's distribution law

According to Wein, the energy density, E_λ, emitted by a blackbody in a wavelength interval λ and $\lambda + d\lambda$ is given by

$$E_\lambda\, d\lambda = \frac{c_1}{\lambda^5}\, e^{(-c_2/\lambda T)}\, d\lambda \tag{3}$$

where c_1 and c_2 are constants. This is known as *Wien's distribution law* or simply *Wein's law.* This law holds good for smaller values of λ but does not match the experimental results for larger values of λ. Wien received the 1911 Nobel Prize for his work on heat radiation.

2.4 Rayleigh-Jeans' law

According to Rayleigh and Jeans the energy density, E_λ, emitted by a blackbody in a wavelength interval λ and $\lambda + d\lambda$ is given by

$$E_\lambda\, d\lambda = \frac{8\pi kT}{\lambda^4}\, d\lambda, \tag{4}$$

where k is the Boltzmann's constant whose value is equal to $1.381 \times 10^{-23}\, JK^{-1}$.

It agrees well with experimental measurements for long wavelengths. However it predicts an energy output that diverges towards infinity as wavelengths grow smaller. This was not supported by experiments and the failure has become known as the **ultraviolet catastrophe** or *Rayleigh-Jeans catastrophe.* Here the word *ultraviolet* signifies shorter wavelength or higher frequencies and not the ultraviolet region of the spectrum. One more thing to note is that, it was not, as is sometimes asserted in physics textbooks, a motivation for quantum theory.

2.5 Planck's law of black-body radiation

Explaining the blackbody radiation curve was a major challenge in theoretical physics during the late nineteenth century. All the theories based on classical ideas failed in one or the other way. The wavelength at which the radiation is strongest is given by Wien's displacement law, and the overall power emitted per unit area is given by the Stefan-Boltzmann law. Wein's law could explain the blackbody radiation curve only for shorter wavelengths whereas Rayleigh-Jeans' law worked well only for larger wavelengths. The problem was finally solved in 1901 by Max Planck.

Planck came up with the following formula for the spectral energy density of blackbody radiation in a wavelength range λ and $\lambda + d\lambda$,

$$E_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \, d\lambda, \tag{5}$$

where h is the Planck's constant whose value is $6.626 \times 10^{-34}\, Js$. This formula could explain the entire blackbody spectrum and does not suffer from an ultraviolet catastrophe unlike the previous ones. But the problem was to justify it in terms of physical principles. Planck proposed a radically new idea that the oscillators in the blackbody do not have continuous distribution of energies but only in discrete amounts. An oscillator emits radiation of frequency ν when it drops from one energy state to the next lower one, and it jumps to the next higher state when it absorbs radiation of frequency ν. Each such discrete bundle of energy $h\nu$ is called *quantum*. Hence, the energy of an oscillator can be written as

$$E_n = nh\nu \qquad n = 0, 1, 2, 3, ... \tag{6}$$

2.5.1 Derivation Wien's law from Planck's law

The Planck's law of blackbody radiation expressed in terms of wavelength is given by

$$E_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \, d\lambda.$$

In the limit of shorter wavelengths, $hc/\lambda kT$ becomes very small resulting in

$$e^{hc/\lambda kT} \gg 1.$$

Therefore

$$e^{hc/\lambda kT} - 1 \approx e^{hc/\lambda kT}.$$

This reduces the Planck's law to

$$E_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT}} \, d\lambda$$

or

$$E_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5} e^{-hc/\lambda kT} \, d\lambda.$$

Now identifying $8\pi hc$ as c_1 and hc/k as c_2, the above equation takes the form

$$E_\lambda \, d\lambda = \frac{c_1}{\lambda^5} e^{(-c_2/\lambda T)} \, d\lambda.$$

This is the familiar Wien's law.

2.5.2 Derivation of Rayleigh-Jeans' law from Planck's law

The Planck's law of blackbody radiation expressed in terms of wavelength is given by

$$E_\lambda\, d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}\, d\lambda.$$

In the limit of long wavelengths, the term in the exponential becomes small. Now expressing it in the form of power series ($e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$),

$$e^{hc/\lambda kT} = 1 + \left(\frac{hc}{\lambda kT}\right) + \frac{\left(\frac{hc}{\lambda kT}\right)^2}{2!} + ...$$

Since $\frac{hc}{\lambda kT}$ is small, any higher order of the same will be much smaller, so we truncate the series beyond the first order term,

$$e^{hc/\lambda kT} \approx 1 + \left(\frac{hc}{\lambda kT}\right).$$

Therefore, the Planck's law takes the form

$$E_\lambda\, d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{(1+hc/\lambda kT)-1}\, d\lambda,$$

that is,

$$E_\lambda\, d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{(hc/\lambda kT)}\, d\lambda,$$

$$E_\lambda\, d\lambda = \frac{8\pi hc}{\lambda^5}\frac{\lambda kT}{hc}\, d\lambda.$$

This gives back the Rayleigh-Jeans Law

$$E_\lambda\, d\lambda = \frac{8\pi kT}{\lambda^4}\, d\lambda.$$

3 Photo-electric effect

The phenomenon of electrons being emitted from a metal when struck by incident electromagnetic radiation of certain frequency is called *photoelectric effect*. The emitted electrons can be referred to as *photoelectrons*. The effect is also termed the *Hertz Effect* in the honor of its discoverer, although the term has generally fallen out of use.

3.1 Experimental results of the photoelectric emission

1. The time lag between the incidence of radiation and the emission of a photoelectron is very small, less than 10^{-9} second.

2. For a given metal, there exists a certain minimum frequency of incident radiation below which no photoelectrons can be emitted. This frequency is called the *threshold frequency* or *critical frequency*, denoted by ν_0. The energy corresponding to this threshold frequency is the minimum energy required to eject a photoelectron from the surface. This minimum energy is the characteristic of the material which is called *work function* (ϕ).

3. For a given metal and frequency of incident radiation, the number of photoelectrons ejected is directly proportional to the intensity of the incident light.

4. Above the threshold frequency, the maximum kinetic energy of the emitted photoelectron is independent of the intensity of the incident light but depends on the frequency of the incident light.

5. The photoelectron emission can be stopped by applying the voltage in a reverse way. This reverse voltage required to stop the photoelectron emission is called the *stopping potential.* This is independent of the intensity but increases with increase in the frequency of incident radiation.

3.2 Einstein's explanation of the photoelectric effect

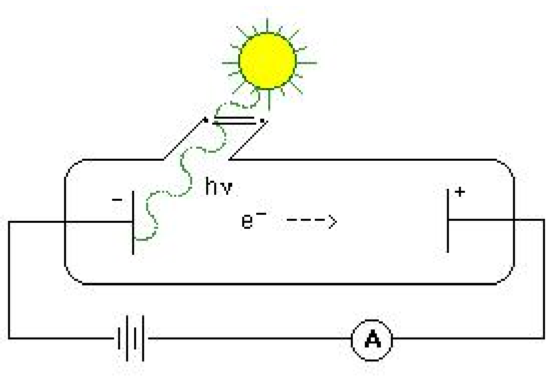
The above experimental results were at odds with Maxwell's wave theory of light, which predicted that the energy would be proportional to the intensity of the radiation. In 1905, Einstein solved this paradox by describing light as composed of discrete quanta, now called *photons*, rather than continuous waves. Based upon Planck's theory of blackbody radiation, Einstein theorized that the energy in each quantum of light was equal to the frequency multiplied by a constant, called Planck's constant. A photon above a threshold frequency has the required energy to eject a single electron, creating the observed effect. Einstein came up the following explanation

Energy of incident photon = Energy needed to remove an electron + Kinetic energy of the emitted electron

Algebraically,

$$h\nu = \phi + KE_{max} \tag{7}$$

where
h is Planck's constant,
ν is the frequency of the incident photon,
$\phi = h\nu_0$ is the work function where ν_0 is the threshold frequency,
$KE_{max} = \frac{1}{2}mv^2$ is the maximum kinetic energy of ejected electrons,
m is the rest mass of the ejected electron, and
v is the speed of the ejected electron.
Since an emitted electron cannot have negative kinetic energy, the equation implies that if the photon's energy ($h\nu$) is less than the work function (ϕ), no electron will be emitted.

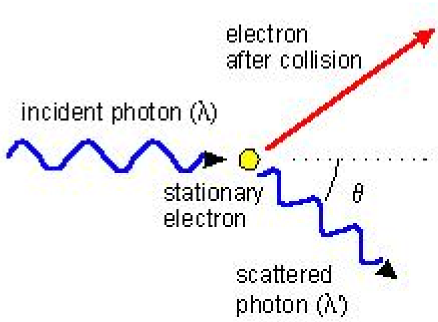


The photoelectric effect helped propel the then-emerging concept of the dualistic nature of light, that light exhibits characteristics of waves and particles at different times. The effect was impossible to understand in terms of the classical wave description of light, as the energy of the emitted electrons did not depend on the intensity of the incident radiation. In his famous paper of 1905, Einstein extended Planck's quantum hypothesis by postulating that quantization was not a property of the emission mechanism, but rather an intrinsic property of the electromagnetic field. Using this hypothesis, Einstein was able to explain the observed phenomenon. Explanation of the photoelectric effect was one of the first triumphs of quantum mechanics and earned Einstein the Nobel Prize in 1921.

4 Compton effect

Compton scattering or the *Compton effect* is the decrease in energy (increase in wavelength) of an X-ray or gamma ray photon, when it interacts with matter. The amount the wavelength increases by is called the *Compton shift.* The Compton effect was observed in 1923 by Arthur Compton who got the 1927 Nobel Prize in Physics for the discovery.

The interaction between electrons and high energy photons results in the electron being given part of the energy and a photon containing the remaining energy being emitted in a different direction from the original, so that the overall momentum of the system is conserved. In this scenario, the electron is treated as free or loosely bound. If the photon is of sufficient energy, it can eject an electron from its host atom entirely resulting in the Photoelectric effect instead of undergoing Compton scattering.



The Compton scattering equation is given by,

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta) \tag{8}$$

where
λ is the wavelength of the photon before scattering,
λ' is the wavelength of the photon after scattering,
m_e is the mass of the electron,
θ is the angle by which the photon's heading changes,
h is Planck's constant, and
c is the speed of light.
$\frac{h}{m_e c} = 2.43 \times 10^{-12}\, m$ is known as the *Compton wavelength.*

The effect is important because it demonstrates that light cannot be explained purely as a wave phenomenon. Light must behave as if it consists of particles in order to explain the Compton scattering. Compton's experiment convinced physicists that light can behave as a stream of particles whose energy is proportional to the frequency.

5 Wave-particle dualism

Albert Einstein's analysis of the photoelectric effect in 1905 demonstrated that light possessed particle-like properties, and this was further confirmed with the discovery of the Compton scattering in 1923. Later on, the diffraction of electrons would be predicted and experimentally confirmed, thus showing that electrons must have wave-like properties in addition to particle properties. The *wave-particle duality* is the concept that all matter and energy exhibits both wave-like and particle-like properties. This duality addresses the inadequacy of classical concepts like 'particle' and 'wave' in fully describing the behavior of small-scale objects. This confusion over particle versus wave properties was eventually resolved with the advent and establishment of quantum mechanics in the first half of the 20th century.

In 1924, Louis de Broglie formulated the *de Broglie hypothesis*, claiming that all matter has a wave-like nature and the wavelength (denoted as λ) of a moving particle of momentum (denoted as p) is given by:

$$\lambda = \frac{h}{p} \tag{9}$$

where h is Planck's constant. de Broglie's formula was confirmed three years later for electrons with the observation of electron diffraction and he was awarded the Nobel Prize for Physics in 1929 for his hypothesis.

The above formula holds true for all particles. In most of the laboratory experiments for measuring the de Broglie wavelength, we accelerate a charged particle using an electric field. When an electron at rest is accelerated by applying a potential difference of V, it will have a kinetic energy given by

$$\frac{1}{2}mv^2 = eV.$$

Expressing the kinetic energy in terms of linear momentum $p(= mv)$, we rewrite the above equation as

$$\frac{p^2}{2m} = eV,$$

that is

$$p = \sqrt{2meV}.$$

Now plugging this equation into the expression for de Broglie wavelength, we get

$$\lambda = \frac{h}{\sqrt{2meV}}.$$

Substituting the numerical values of the natural constants ($h = 6.626 \times 10^{-34}\,Js$, $m = 9.11 \times 10^{-31}\,kg$ and $e = 1.602 \times 10^{-19}\,C$), we get

$$\lambda = \frac{1.226 \times 10^{-9}}{\sqrt{V}}\,m. \tag{10}$$

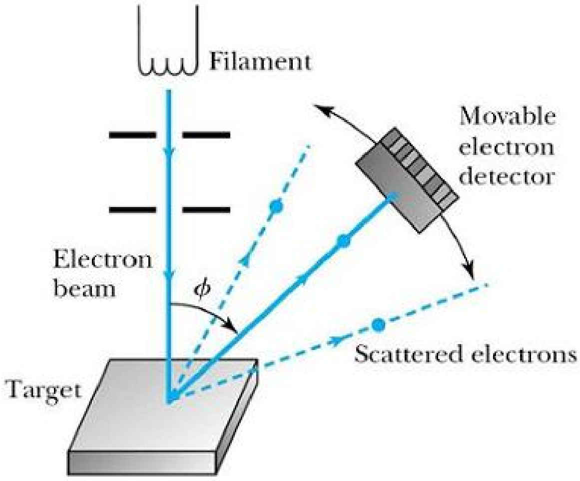
5.1 Davisson and Germer Experiment

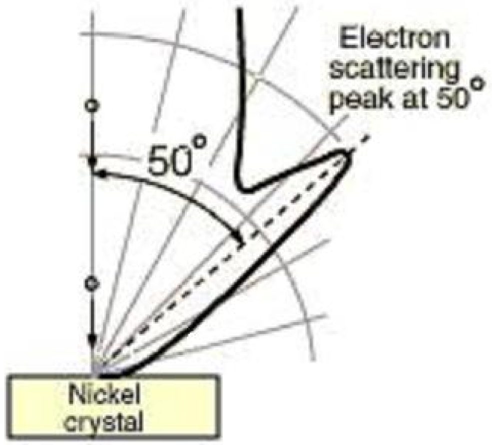
In 1927, while working for Bell Labs, Clinton Davisson and Lester Germer performed an experiment showing that electrons were diffracted at the surface of a crystal of nickel. The basic idea is that the planar nature of crystal structure provides scattering surfaces at regular intervals, thus waves that scatter from one surface can constructively or destructively interfere from waves that scatter from the next crystal plane deeper into the crystal. This celebrated Davisson-Germer experiment confirmed the de Broglie hypothesis that particles of matter have a wave-like nature, which is a central tenet of quantum mechanics. In particular, their observation of diffraction allowed the first measurement of a wavelength for electrons. The measured wavelength agreed well with de Broglie's equation.

The Davisson-Germer consisted of firing an electron beam from an electron gun on a nickel crystal at normal incidence i.e. perpendicular to the surface of the crystal. The electron gun consisted of a heated filament that released thermally excited electrons, which were then accelerated through a potential difference V, giving them a kinetic energy of eV where e is the charge of an electron. The angular dependence of the reflected electron intensity was measured, and was determined to have the same diffraction pattern as those predicted by Bragg for X-rays. An electron detector was placed at an angle $\theta = 50^\circ$ and measured the number of electrons that were scattered at that particular angle.

According to the de Broglie relation, a beam of $54\,eV$ had a wavelength of $0.165\,nm$. This matched the predictions of Bragg's law

$$n\lambda = 2d\sin\left(90^\circ - \frac{\theta}{2}\right),$$





for $n = 1$, $\theta = 50°$, and for the spacing of the crystalline planes of nickel ($d = 0.091\,nm$) obtained from previous X-ray scattering experiments on crystalline nickel.

This was also replicated by George Thomson. Thomson and Davisson shared the Nobel Prize for Physics in 1937 for their experimental work. This, in combination with Arthur Compton's experiment, established the wave-particle duality hypothesis, which was a fundamental step in quantum theory.

5.2 Properties of Matter-waves

1. Matter-waves are associated with any moving body and their wavelength is given by $\lambda = \frac{h}{mv}$.
2. The wavelength of matter-waves is inversely proportional to the velocity of the body. Hence, a body at rest has an infinite wavelength whereas the one traveling with a high velocity has a lower wavelength.
3. Wavelength of matter-waves depends on the mass of the body and decreases with increase in mass. Because of this, the wave-like behavior of heavier objects is not very evident whereas the wave nature of subatomic particles can be observed experimentally.
4. Amplitude of the matter-waves at a particular space and time depends on the probability of finding the particle at that space and time.
5. Unlike other waves, there is no physical quantity that varies periodically in the case of matter-waves.
6. Matter waves are represented by a wave packet made up of a group of waves of slightly differing wavelengths. Hence, we talk of group velocity of matter waves rather than the phase velocity.
7. Matter-waves show similar properties as other waves such as interference and diffraction.

6 Phase velocity, group velocity and particle velocity

The ***phase velocity*** of a wave is the rate at which the phase of the wave propagates in space. This is the speed at which the phase of any one frequency component of the wave travels. For such a component, any given phase of the wave (for example, the crest) will appear to travel at the phase velocity.

The phase speed is given in terms of the wavelength λ and period T as

$$v_{\text{phase}} = \frac{\lambda}{T}. \tag{11}$$

Or, equivalently, in terms of the wave's angular frequency ω and wavenumber k by

$$v_{\text{phase}} = \frac{\omega}{k}. \tag{12}$$

In quantum mechanics, particles also behave as waves with complex phases. By the de Broglie hypothesis, we see that

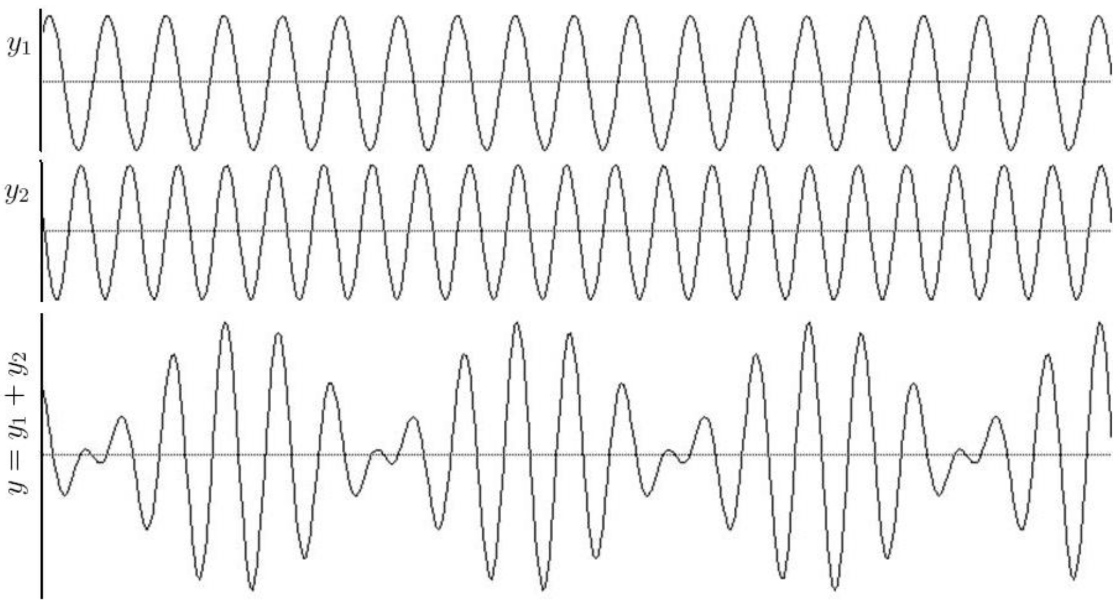
$$v_{\text{phase}} = \frac{\omega}{k} = \frac{E/\hbar}{p/\hbar},$$

$$v_{\text{phase}} = \frac{E}{p}. \tag{13}$$

The phase velocity of electromagnetic radiation may under certain circumstances (e.g. in the case of anomalous dispersion) exceed the speed of light in a vacuum, but this does not indicate any superluminal information or energy transfer. It was theoretically described by physicists such as Arnold Sommerfeld and Leon Brillouin.

The ***group velocity*** of a wave is the velocity with which the variations in the shape of the wave's amplitude (known as the modulation or envelope of the wave) propagate through space. For example, imagine what happens if you throw a stone into the middle of a very still pond. When the stone hits the surface of the water, a circular pattern of waves appears. It soon turns into a circular ring of waves with a quiescent center. The ever expanding ring of waves is the group, within which one can discern individual wavelets of differing wavelengths traveling at different speeds. The longer waves travel faster than the group as a whole, but they die out as they approach the leading edge. The shorter waves travel slower and they die out as they emerge from the trailing boundary of the group.

Now, we shall arrive at the expression for the group velocity using the concept of superposition of two almost similar waves.



Let the two waves be given by

$$y_1 = A\,\cos(\omega t - kx),$$

$$y_2 = A\,\cos[(\omega + \Delta\omega)t - (k + \Delta k)x].$$

When these two waves superimpose, we get

$$y = y_1 + y_2,$$

$$y = A\,\cos(\omega t - kx) + A\,\cos[(\omega + \Delta\omega)t - (k + \Delta k)x].$$

Using the trigonometric relation

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right),$$

we get

$$y = 2A\cos\left(\frac{[\omega t - kx] + [(\omega + \Delta\omega)t - (k + \Delta k)x]}{2}\right)\cos\left(\frac{[\omega t - kx] - [(\omega + \Delta\omega)t - (k + \Delta k)x]}{2}\right),$$

$$y = 2A\cos\left(\frac{(2\omega + \Delta\omega)t - (2k + \Delta k)x}{2}\right)\cos\left(\frac{\Delta\omega t - \Delta kx}{2}\right).$$

Since $\Delta\omega$ is too small compared to 2ω, we can write

$$2\omega + \Delta\omega \approx 2\omega$$

Now using this in the above equation and rearranging the terms, we end up with

$$y = 2A\cos\left(\frac{\Delta\omega t - \Delta kx}{2}\right)\cos\left(\frac{(2\omega)t - (2k)x}{2}\right).$$

Further simplifying it, gives us

$$y = 2A\cos\left(\frac{\Delta\omega t - \Delta kx}{2}\right)\cos(\omega t - kx).$$

Identifying $2\cos\left(\frac{\Delta\omega t - \Delta kx}{2}\right)$ as the constant amplitude of the superposed wave, we can write

$$2A\cos\left(\frac{\Delta\omega t - \Delta kx}{2}\right) = constant$$

i.e.,

$$\left(\frac{\Delta\omega t - \Delta kx}{2}\right) = constant$$

$$(\Delta\omega t - \Delta kx) = constant$$

$$x = \left(\frac{\Delta\omega t}{\Delta k}\right) + constant$$

Differentiating the above equation with respect to t, we get the group velocity,

$$v_{group} = \frac{dx}{dt} = \frac{\Delta\omega}{\Delta k}$$

under the limiting condition, we get

$$v_{group} = \frac{d\omega}{dk} \tag{14}$$

This is the defining equation of group velocity. In a dispersive medium, the phase velocity varies with frequency and is not necessarily the same as the group velocity of the wave.

The ***particle velocity*** is the velocity v of a particle in a medium as it transmits a wave. For a particle of mass m possessing a linear momentum p, the particle velocity is given by

$$v_{particle} = \frac{p}{m}. \tag{15}$$

In many cases this is a longitudinal wave of pressure as with sound, but it can also be a transverse wave as with the vibration of a taut string. When applied to a sound wave through a medium of air, particle velocity would be the physical speed of an air molecule as it moves back and forth in the direction the sound wave is traveling as it passes. Particle velocity should not be confused with the speed of the wave as it passes through the medium, i.e. in the case of a sound wave, particle velocity is not the same as the speed of sound.

6.1 Relation between group velocity and phase velocity

The group velocity of a matter wave is given by

$$v_{group} = \frac{d\omega}{d\kappa},$$

whereas phase velocity is given by

$$v_{phase} = \frac{\omega}{\kappa}.$$

From the definition of phase velocity, we can write

$$\omega = v_{phase}\,\kappa.$$

Substituting this in the expression for group velocity, we get

$$v_{group} = \frac{d\left(v_{phase}\,\kappa\right)}{d\kappa}.$$

Differentiating using the *product rule*, we get

$$v_{group} = v_{phase} + \kappa\frac{dv_{phase}}{d\kappa}.$$

We rewrite this in the following form

$$v_{group} = v_{phase} + \kappa\frac{dv_{phase}}{d\lambda}\,\frac{d\lambda}{d\kappa}.$$

Since

$$\kappa = \frac{2\pi}{\lambda},$$

we have

$$\frac{d\kappa}{d\lambda} = \frac{-2\pi}{\lambda^2}.$$

Plugging these two equations into the v_{group} expression ,we get

$$v_{group} = v_{phase} + \left(\frac{2\pi}{\lambda}\right)\frac{dv_{phase}}{d\lambda}\left(\frac{-\lambda^2}{2\pi}\right).$$

Further simplifying,

$$v_{group} = v_{phase} - \lambda\frac{dv_{phase}}{d\lambda}. \tag{16}$$

6.2 Relation between group velocity and particle velocity

The group velocity of a matter wave is given by

$$v_{group} = \frac{d\omega}{d\kappa},$$

where

$$\omega = 2\pi\nu$$

and

$$\kappa = \frac{2\pi}{\lambda}.$$

From Planck's equation $E = h\nu$, we can write

$$\nu = \frac{E}{h};$$

and from de Broglie wavelength, we can write

$$\lambda = \frac{h}{p}.$$

Using the above equations, we rewrite the expressions for ω and κ,

$$\omega = 2\pi\frac{E}{h}$$

and

$$\kappa = 2\pi\frac{p}{h}.$$

Now, differentiating the expressions for ω and κ, we get

$$d\omega = \frac{2\pi}{h}dE$$

and

$$d\kappa = \frac{2\pi}{h}dp.$$

Substituting the expressions for $d\omega$ and $d\kappa$ into the v_{group} equation,

$$v_{group} = \frac{\frac{2\pi}{h}dE}{\frac{2\pi}{h}dp},$$

that is

$$v_{group} = \frac{dE}{dp}.$$

Since we are dealing with the matter-waves E can be the kinetic energy of particle in wave motion. Using the relation

$$E = \frac{p^2}{2m}$$

and differtiating it with respect to p, we get

$$v_{group} = \frac{dE}{dp} = \frac{d}{dp}\left(\frac{p^2}{2m}\right) = \frac{2p}{2m},$$

so

$$v_{group} = \frac{p}{m}.$$

Right hand is nothing but the particle velocity $v_{particle}$. Therefore

$$v_{group} = v_{particle}. \tag{17}$$

6.3 Relation between phase velocity and particle velocity

The phase velocity of a matter-wave is given by

$$v_{phase} = \frac{\omega}{\kappa},$$

where

$$\omega = 2\pi\nu$$

and

$$\kappa = \frac{2\pi}{\lambda}.$$

From Planck's equation $E = h\nu$, we can write

$$\nu = \frac{E}{h};$$

and from de Broglie wavelength, we can write

$$\lambda = \frac{h}{p}.$$

Using the above equations, we rewrite the expressions for ω and κ,

$$\omega = 2\pi\frac{E}{h}$$

and

$$\kappa = 2\pi\frac{p}{h}.$$

Now, substituting these into the expression for v_{phase}, we get

$$v_{phase} = \frac{2\pi\frac{E}{h}}{2\pi\frac{p}{h}},$$

that is,

$$v_{phase} = \frac{E}{p}.$$

From Einstein's mass-energy equivalence relation, we have

$$E = mc^2$$

and from the definition of linear momentum of a particle, we have

$$p = mv_{particle}.$$

Using E and p expressions in equation of v_{phase}, we get

$$v_{phase} = \frac{mc^2}{mv_{particle}},$$

or

$$v_{phase} = \frac{c^2}{v_{particle}},$$

which gives us

$$v_{phase} \cdot v_{particle} = c^2. \tag{18}$$

Since $v_{group} = v_{particle}$, we can also write

$$v_{phase} \cdot v_{group} = c^2. \tag{19}$$

6.4 Expression for de Broglie wavelength using group velocity

Consider particle moving with kinetic energy $mv^2/2$. This can be associated with energy $h\nu$. Therefore

$$h\nu = \frac{mv^2}{2},$$

$$\nu = \frac{m}{2h}v^2.$$

Differentiating the above expression with respect to λ,

$$\frac{d\nu}{d\lambda} = \frac{m}{2h}2v\frac{dv}{d\lambda},$$

or

$$\frac{d\nu}{d\lambda} = \frac{mv}{h}\frac{dv}{d\lambda}.$$

The group velocity of a matter wave is given by

$$v_{group} = \frac{d\omega}{d\kappa},$$

where

$$\omega = 2\pi\nu$$

and

$$\kappa = \frac{2\pi}{\lambda}.$$

Differentiating ω and κ, we get

$$d\omega = 2\pi d\nu$$

and

$$d\kappa = \frac{-2\pi}{\lambda^2}d\lambda.$$

$$v_{group} = \frac{2\pi d\nu}{\frac{-2\pi}{\lambda^2}d\lambda},$$

$$v_{group} = -\lambda^2\frac{d\nu}{d\lambda},$$

We can express this in the following way

$$\frac{d\nu}{d\lambda} = \frac{-v_{group}}{\lambda^2},$$

Equating the two expressions for $\frac{d\nu}{d\lambda}$, we get

$$\frac{mv}{h}\frac{dv}{d\lambda} = \frac{-v_{group}}{\lambda^2}.$$

Simplifying this equation leaves us with

$$\frac{dv}{d\lambda} = \frac{-h}{m\lambda^2}.$$

Rewriting this in the following form

$$dv = -\frac{h}{m\lambda^2}d\lambda$$

Integrating the above equation

$$\int dv = -\int \frac{h}{m\lambda^2}d\lambda$$

we get

$$v = \frac{h}{m\lambda} + constant.$$

To fix the constant we use the condition: as $\lambda \to \infty$, $v \to 0$. This makes $constant = 0$

$$v = \frac{h}{m\lambda}$$

That is

$$\lambda = \frac{h}{mv} \tag{20}$$

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