Bilattices, Intuitionism and Truth-knowledge Duality: Concepts and Foundations

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Abstract

We propose a family of intuitionistic bilattices with full truth-knowledge duality (D-bilattices) for a logic programming. The first family of perfect D-bilattices is composed by Boolean algebras with even number of atoms: the simplest of them, based on intuitionistic truth-functionally complete extension of Belnap's 4-valued bilattice, can be used in paraconsistent programming, that is, for knowledge bases with incomplete and inconsistent information. The other two families are useful for a probability theory where the uncertainty in the knowledge about a piece of information is in the form of belief types: as an interval (lower and upper boundary) probability or as a confidence level. Such programs can be parameterized by different kinds of probabilistic conjunctive/disjunctive strategies for their rules, based on intuitionistic implication. Such a framework offers a clear semantics for the satisfaction relation, and allows the extension of logic languages with intuitionistic implications also in the body of rules. From a theoretical point of view we introduce also a duality in higher-order bilatices, as in Temporal Probabilistic Logic, constructed as functional spaces over ordinary dual bilattices. Then we show the full truth-knowledge duality for a fixpoint semantics of logic programs based on dual bilattices. Finally we develop also an autoreferential version of Stone's Representation Theorem for the dual bilattices.

1 Introduction

In the past few decades, many classical as well as non-classical techniques for modeling the reasoning of intelligent systems have been proposed. Several frameworks, based on a many-valued logic with truth/knowledge partial orders (and lattices), for manipulating uncertainty, incompleteness, and inconsistency have been proposed in the form of particular extensions of classical logic programming and deductive databases.
Such truth/knowledge ordered lattices can be integrated into a unique structure called *bilattice* [26, 17, 18, 20, 23], and Ginsberg has shown that the same theorem prover can be used to simulate reasoning in first order logic, default logic, prioritized default logic and assumption truth maintenance system.
In this paper we will consider different belief truth structures (types) for knowledge systems: the simplest 4-valued bilattice, belief probability intervals [24, 32, 36, 1]

with the lower and upper probability boundary and confidence-level quantified data [34, 35] also, where the belief and doubt are considered separately for each fact.
Because of that we will present three levels of Heyting algebras over bilattices, built from the elements of the most simple algebras where belief is expressed by a single probability value (considered also as a truth value in this many-valued logics).
In our first contribution we will extend this research by introduction of a new *modal* operator for bilattices: for Belnap's 4-valued bilattice it corresponds to Fitting's conflation (knowledge negation) operator. It is well known that the modal operators are used in a variety of ways in knowledge systems, including reasoning about knowledge and belief, about time, with applications to nonmonotonic inference (in [33, 31, 15], and others). Ginsberg [25] has shown that Moore's and Kripke's ideas can be unified into a single approach if we view modal operators not in terms of possible worlds, but as mappings on the truth values assigned to various sentences: this is the way that we will use to define the semantics based on bilattices for our modal operator. This is a known fact about the algebraization of logics: a propositional logic becomes a Boolean algebra, the modal logics become the modal algebras by adding also an additional operator to Boolean algebra, in order to interpret an universal modal operator of these modal logics; in the case of S4 modal logics we obtain the closure algebras (known as topological Boolean algebras also) [11, 12], while in the case of the S5 modal logics used for auto-epistemic logic, we obtain the *monadic* algebras (or one-dimensional cylindric algebras) [16, 28].
This new modal operator plays a fundamental role in the truth/knowledge bilattice's *duality*, in logic programs and their least-fixpoint semantics: the existence of the truth/ knowledge duality is not a mathematical triviality, but such a perfect symmetry of two orderings is a very particular property for only strict subset of *distributive* bilattices, and the investigation of the general properties is one of the main issues of this paper.
Usually, the authors characterize their programs with a model theoretical semantics, where a minimum model is guaranteed to exist, and a corresponding monotonic fixpoint operator too. A lot of work was dedicated to different *many-valued* conjunctive/disjunctive probabilistic strategies used to define the different correlation types (independence, ignorance, mutual-exclusiveness, etc..) between simple facts (events). But a very little consideration is dedicated to the many-valued logical implication operator, used in logic programming rules. Often in the bilattice-based logic programs [18] are used only 2-valued implications, without any explanation why there is such asymmetry with all other logic connectives (usually, also the negation when used is many-valued).
There is a close relationship between logic programming and inductive definitions which are forms of *constructive* knowledge [7, 22]. Constructive information defines a collection of facts through a constructive process of iterating a recursive recipe. This recipe defines new instances of this collection in terms of presence (and sometimes the absence) of other facts of the collection. In the context of mathematics, constructive information appears by excellence in inductive definitions. Not a coincidence, inductive definitions have been studied in constructive mathematics and intuitionistic logic, in particular in the sub-areas of Inductive and Definition logics, Iterated Inductive Definition logics and Fixpoint logics.
We follow the *implication-based* approach to logic programming, and we adopt the

idea of using relative pseudo complements as implications, which has been proposed in the literature several times and is nowadays well understood, for distributive bilattices also. In fact, the use of relative pseudo-complements is adopted in the fuzzy, multi-adjoint and residuated logic programming and allows also the assignment of weights to rules. Consequently, as a second contribution, here one particular family of *intuitionistc* bilattices, that is, distributive bilattices extended by logic implication $\Rightarrow$, defined as a relative pseudo complement $x \Rightarrow y = \bigvee\{z|z \wedge x \leq y\}$ (here $\wedge$ is the meet operation, $\leq$ is the ordering of each of two lattices of a bilattice and $\bigvee$ is the l.u.b). It has nice mathematical properties. Semantically, the basic notion is that of relative pseudo complement from the algebraic semantics of intuitionistic logic [9]. More over, such an implication can be used for *nested* implications in bodies of rules, in order to extend the expressive power of logic programming (see, for example, in [4]), but such possibility needs further investigation and is not the subject of this work.
Another contribution is given by a family of parameterized bilattice operators, for different probabilistic strategies in a belief and confidence-level based logic programming. The particular attention is given to the 4-valued Belnap's bilattice, which is the *smallest* non-trivial billatice which satisfies the perfect truth/knowledge symmetry. It can be used as a 'minimal' many-valued structure for logic programming and databases with *incomplete and inconsistent* information, thus with a relevant application impact in web data integration. Consequently, a number of new concepts developed in this paper will be exemplified for the case of Belnap's bilattice, and the last contribution of this paper is given by a new representation theorem for it.

The plan of this paper is the following: after a brief introduction to Heyting algebras and bilattices, in Section 2 we present the family of intuitionistic D-bilatices with full truth-knowledge duality: Boolean algebras with even number of atoms (the simplest case is the 4-valued Belnap's bilattice), and the belief and confidence-level bilattices parameterized with a number of possible probabilistic conjunctive strategies. In Section 3 we extend the D-bilattices with functional higher-order bilattices and show how they can be applied for Temporal Probabilistic Logic Programs. In Section 4 we show the full duality for the least fixpoint semantics of logic programs based on D-bilattices. Finally, in Section 4, we develop a version of Autoreferential Representation Theorem for D-bilattices.

1.1 Introduction to modal Heyting algebras

Heyting algebras are algebraic structures that play in relation to intuitionistic logic a role analogous to that played by Boolean algebras in relation to classical logic. They are most simple defined as a certain type of lattice. A *lattice* is a partially ordered set $(L, \leq)$, in which every pair of elements $x, y \in L$ has a least upper bound, or *join*, denoted by $x \vee y$, and a greatest lower bound, or *meet*, denoted by $x \wedge y$. A top (bottom) element of a lattice L is an element, denoted by 1 (0), such that $x \leq 1$ ($0 \leq x$) for all $x \in L$.
Now a Heyting algebra is defined to be a lattice $(L, \leq)$, possessing distinct top and bottom elements, such that, for any pair of elements $x, y \in L$, the set of $z \in L$ satisfying $z \wedge x \leq y$ has a largest element. This element, which is uniquely determined by x

and y, is denoted by $x \Rightarrow y$: thus, $x \Rightarrow y$ is characterized by the following condition (Galois connection): for all $z \in L$, $\quad z \leq x \Rightarrow y \quad$ iff (if and only if) $\quad z \wedge x \leq y$. This binary operation $\Rightarrow$ is the relative pseudo-complement for x and y and is called *intuitionistic implication*, that is, the well known Godel's t-norm (relative pseudo-complement), such that $x \Rightarrow y = \bigvee\{z | z \wedge x \leq y\} = 1$, if $x \leq y$, y otherwise. The operation which sends each element x to the element $\neg x = x \Rightarrow 0$ is called *intuitionistic negation* (pseudo-complement): we note that the latter operation satisfies
$z \leq \neg x \quad$ iff $\quad z \wedge x = 0 \quad$ iff $\quad x \leq \neg z$.
It is not difficult to show that any Heyting algebra is a distributive lattice, so that hold: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, $\quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. In such complete distributive lattices we have that for any x, $x \wedge \neg x = 0$, but generally $x \vee \neg x \neq 1$.
A function $l : X \rightarrow Y$ between posets X, Y is *monotone* if $x \leq x'$ implies $l(x) \leq l(x')$ for all $x, x' \in X$.
Such a function $l : X \rightarrow Y$ is said to have right (or upper) adjoint if there is a function $r : Y \rightarrow X$ in the reverse direction such that $l(x) \leq y$ iff $x \leq r(y)$ for all $x \in X, y \in Y$. Such a situation forms a Galois connection and will often be denoted by $l \dashv r$. Then l is called left (or lover) adjoint of r. If X, Y are complete lattices (posets) then $l : X \rightarrow Y$ has a right adjoint iff l preserves all joins (it is *additive*, i.e., $l(x \vee y) = l(x) \vee l(y)$ and $l(0_X) = 0_Y$ where $0_X, 0_Y$ are bottom elements in complete lattices X and Y respectively). The right adjoint is then $r(y) = \bigvee\{z \in X \mid l(z) \leq y\}$. Similarly, a monotone function $r : Y \rightarrow X$ is a right adjoint (it is *multiplicative*, i.e., has a left adjoint) iff r preserves all meets; the left adjoint is then $l(x) = \bigwedge\{z \in Y \mid x \leq r(z)\}$. The *selfadjoint* modal operator $l \dashv l$ is both multiplicative and additive.
Each monotone function $l : X \rightarrow Y$ on a complete lattice (poset) X has both a *least* fixed point $\mu l \in X$ and *greatest* fixed point $\nu l \in X$. these can be described explicitly as: $\quad \mu l = \bigwedge\{x \in X \mid l(x) \leq x\} \quad$ and $\quad \nu l = \bigvee\{x \in X \mid x \leq l(x)\}$.
In what follows we denote by $y < x$ iff $y \leq x$ and not $x \leq y$, and we denote by $x \bowtie y$ two unrelated elements in X (so that not ($x \leq y$ or $y \leq x$)). An element in a lattice $x \neq 0$ is *join-irreducible* element iff $x = a \vee b$ implies $x = a$ or $x = b$ for any $a, b \in X$. An element in a lattice $x \in X$ is an *atom* iff $x > 0$ and $\nexists y.(x > y > 0)$.
The *modal* Heyting algebras are standard Heyting algebras extended by unary adjoint operators.

1.2 Introduction to Bilattices

Bilattice theory is a ramification of multi-valued logic by considering both truth $\leq_t$ and knowledge $\leq_k$ partial orderings. Given two truth values x and y, if $x \leq_t y$ then y is at least as true as x, i.e., $x \leq_t y$ iff $x <_t y$ or $x = y$. The two operations corresponding to this ordering (t-lattice) are the meet (greatest lower bound) $\wedge$ and the join (least upper bound) $\vee$.
For the knowledge (or precision in probabilistic theory) ordering $x \leq_k y$ means that y is more precise than x, i.e., $x \leq_k y$ iff $x <_k y$ or $x = y$. The operations $\otimes$ and $\oplus$ correspond to the greatest lower bound and least upper bound respectively in the knowledge ordering (k-lattice). The negation operation for these two orderings are defined as the involution operators which satisfy De Morgan law between the join and

meet operations.

Definition 1 *(Ginsberg [23]) A bilattice $\mathcal{B}$ is defined as a sextuple* $(\mathcal{B}, \wedge, \vee, \otimes, \oplus, \neg)$, *such that:*
1. The t-lattice $(\mathcal{B}, \leq_t, \wedge, \vee)$ *and the k-lattice* $(\mathcal{B}, \leq_k, \otimes, \oplus)$ *are both complete lattices.*
2. $\neg : \mathcal{B} \to \mathcal{B}$ *is an involution* ($\neg\neg$ *is the identity) mapping such that:*
$\neg$ *is lattice homomorphism from* $(\mathcal{B}, \wedge, \vee)$ *to* $(\mathcal{B}, \vee, \wedge)$ *and* $(\mathcal{B}, \otimes, \oplus)$ *to itself.*

Notice that from this definition, the negation $\neg$ is an antitonic operator w.r.t. the $\leq_t$, with $\neg 1_t = 0_t$, $\neg 0_t = 1_t$ (where $0_t, 1_t$ are the bottom and the top elements w.r.t the $\leq_t$ respectively), but *monotonic* w.r.t. the $\leq_k$. This homomorphism $\neg$ inverts the truth partial ordering, that is, if $x \leq_t y$ then $\neg x \geq_t \neg y$, while it preserves the knowledge preordering, that is, if $x \leq_k y$ then $\neg x \leq_k \neg y$. The last property is very important for the fixpoint semantics of logic programs based on a bilattice of algebraic truth values, because we can use the monotonicity w.r.t. the $\leq_k$ ordering. That is the reason that in the case of the distributive lattices we will not use directly the pseudo-complement as negation operator $\neg$, because it generally does not satisfy this property. As a result, the distributive lattices which are Boolean algebras w.r.t. the pseudo-complement negation operator, will not be Boolean algebras w.r.t this bilattice negation, and in such distributive lattices will not be satisfied condition $x \wedge \neg x = 0$. Because of that generally the bilattice negation $\neg$, differently from the pseudo-complement, is a *paraconsistent* negation.
We can see such (bi)lattice structures as two algebras, denoted $(\mathcal{B}, \leq_t, \alpha_t)$ and $(\mathcal{B}, \leq_k, \alpha_k)$, with a carrier set $\mathcal{B}$ of algebraic truth values and the set $\alpha_t = \{\wedge, \vee\}$, $\alpha_k = \{\otimes, \oplus\}$ of n-ary operations, $f : \prod \mathcal{B} \to \mathcal{B}$, $g : \prod \mathcal{B} \to \mathcal{B}$, where $f \in \alpha_t$, $g \in \alpha_k$ and $\prod \mathcal{B} = \mathcal{B} \times \mathcal{B}$ is the cartesian product.
Any homomorphism $h : (\mathcal{B}, \alpha) \to (\mathcal{B}, \beta)$ of algebras is a mapping such that for any term $f(a_1, .., a_n)$, where $f \in \alpha_t$ is a n-ary operation of the first algebra and $a_i \in \mathcal{B}$, $h(f(a_1, .., a_n)) = h(f)(h(a_1), .., h(a_n))$, where $h(f) \in \alpha_k$ is the correspondent n-ary operation of the second algebra.
For example for $h = \neg : (\mathcal{B}, \alpha_t) \to (\mathcal{B}, \beta)$, where $\beta = \{\vee, \wedge\}$, we have that, from the antitonic property of $\neg$ w.r.t. the $\leq_t$, $\neg(\wedge(a, b))$ (that is, $\neg(a \wedge b)$) is equal to $\neg(\wedge)(\neg(a), \neg(b)) = \neg a \vee \neg b$, where $\neg(\wedge) = \vee$, that is, this homomorphism corresponds to De Morgan law.
The smallest bilattice is the trivial bilattice, in which $\mathcal{B}$ consists of a single value, so that it is useless.
In any bilattice with the top and bottom truth values 1_t and 0_t w.r.t. the $\leq_t$ respectively, we have that $\neg 1_t = 0_t$, $\neg 0_t = 1_t$, while in any nontrivial bilattice (where $\leq_t \neq \leq_k$) they are k-incomparable, i.e., $1_t \bowtie_t 0_t$.

1.2.1 Belnap's bilattice

The smallest *nontrivial* bilattice is Belnap's 4-valued bilattice [26] $\mathcal{B}_4 = \{t, f, \perp, \top\}$ where $t = 1_t$ is *true*, $f = 0_t$ is *false*, $\top = 1_k$ is inconsistent (both true and false) or *possible* , and $\perp = 0_k$ is *unknown*. As Belnap observed, these values can be given two natural orders: *truth* order, $\leq_t$, and *knowledge* order, $\leq_k$, such that $f \leq_t \top \leq_t t$,

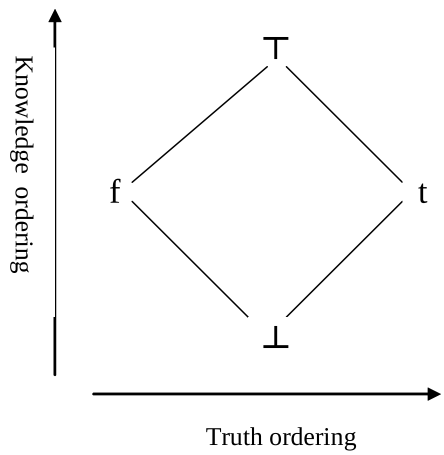

Figure 1: Belnap's bilattice

$f \leq_t \perp \leq_t t$, and $\perp \leq_k f \leq_k \top$, $\perp \leq_k t \leq_k \top$. and $b \leq_k a$). Thus any two members α, β in a bilattice are equal, $\alpha = \beta$, if and only if (shortly 'iff') $\alpha =_t \beta$ and $\alpha =_k \beta$. Meet and join operators under $\leq_t$ are denoted $\wedge$ and $\vee$; they are natural generalizations of the usual conjunction and disjunction notions. Meet and join under $\leq_k$ are denoted $\otimes$ (*consensus*, because it produces the most information that two truth values can agree on) and $\oplus$ (*gullibility*, it accepts anything it's told), such that hold: $f \otimes t = \perp$, $f \oplus t = \top$, $\top \wedge \perp = f$ and $\top \vee \perp = t$.
There is a natural notion of truth negation, denoted $\neg$, (reverses the $\leq_t$ ordering, while preserving the $\leq_k$ ordering): switching f and t, leaving $\perp$ and $\top$, and corresponding knowledge negation (*conflation*) [18], denoted $\sim$, (reverses the $\leq_k$ ordering, while preserving the $\leq_t$ ordering), switching $\perp$ and $\top$, leaving f and t. These two kinds of negation commute: $\sim \neg x = \neg \sim x$ for every member x of a bilattice.
It turns out that the operations $\wedge, \vee$ and $\neg$, restricted to $\{f, t, \perp\}$ are exactly those of Kleene's strong 3-valued logic. A more general information about this smallest bilattice may be found in [18].

1.3 Introduction to probabilistic conjunctive/disjunctive strategies

The aim of this introduction is to give an overview of the most known conjunctive/disjunctive strategies. The underling lattices and the algebraic properties of these strategies will be investigated in what follows (Propositions 9,10) , where we will demonstrate that defined parameterized bilattices addopt these strategies.
In temporal-probabilistic databases the precise probability of compound event (conjunction or disjunction of events) depends upon the known relationships (eg. independence, mutual exclusion, ignorance of any relationship, etc..) between the primitive events that constitute the compound event. The query answering in such temporal-probabilistic databases [3, 35] can be parameterized by user in order to take in consideration such different interdependencies of events of a database.

In the case when the probability of a compound event belongs to an interval $[x, y]$, $x, y \in [0, 1]$, rather than a point, such strategies for conjunctions are defined as follows (mutually exclusive strategy is slightly different from that in [35]):
1. *Positive correlation* strategy, $[x, y] \wedge_{pc} [x_1, y_1] = [min\{x, x_1\}, min\{y, y_1\}]$,
2. *Ignorance* strategy, $[x, y] \wedge_{ig} [x_1, y_1] = [max\{0, x + x_1 - 1\}, min\{y, y_1\}]$,
3. *Negative correlation* strategy,
$[x, y] \wedge_{nc} [x_1, y_1] = [max\{0, x + x_1 - 1\}, max\{0, y + y_1 - 1\}]$,
4. *Independence* strategy, $[x, y] \wedge_{in} [x_1, y_1] = [x \cdot x_1, y \cdot y_1]$.
5. *Mutually-exclusive* strategy, $[x, y] \wedge_{ex} [x_1, y_1] = [0, 0]$.
For disjunctions we have that
1. $[x, y] \vee_{pc} [x_1, y_1] = [max\{x, x_1\}, max\{y, y_1\}]$,
2. $[x, y] \vee_{ig} [x_1, y_1] = [max\{x, x_1\}, min\{1, y + y_1\}]$,
3. $[x, y] \vee_{nc} [x_1, y_1] = [min\{1, x + x_1\}, min\{1, y + y_1\}]$,
4. $[x, y] \vee_{in} [x_1, y_1] = [x + x_1 - x \cdot x_1, y + y_1 - y \cdot y_1]$,
5. $[x, y] \vee_{ex} [x_1, y_1] = [1, 1]$.
Notice that only in the case of ignorance strategy such compositions correspond to the possible-worlds based semantics of probabilistic programs; other strategies are rather *approximations* for such pure probabilistic logic.
In the case when the probability of a compound event is a confidence level, that is a couple of a belief $[x, y]$ and a doubt $[z, v]$, which we associate to the facts of our uncertain and incomplete knowledge base, rather than a point, such probabilistic strategies are defined as follows [34, 35]: Let $\alpha = ([x, y], [z, v])$, $\beta = ([x_1, y_1], [z_1, v_1])$, then, conjunction strategises are defined by: $\alpha \wedge \beta =$
1. *Positive correlation* strategy,
$= ([min\{x, x_1\}, min\{y, y_1\}],\ [max\{z, z_1\}, max\{v, v_1\}])$,
2. *Ignorance* strategy conjunction,
$= ([max\{0, x + x_1 - 1\}, min\{y, y_1\}],\ [max\{z, z_1\}, min\{1, v + v_1\}])$,
3. *Negative correlation* strategy conjunction,
$= ([max\{0, x + x_1 - 1\}, max\{0, y + y_1 - 1\}],\ [min\{1, z + z_1\}, min\{1, v + v_1\}])$,
4. *Independence* strategy conjunction,
$= ([x \cdot x_1, y \cdot y_1],\ [z + z_1 - z \cdot z_1, v + v_1 - v \cdot v_1])$,
5. *Mutually-exclusive* strategy conjunction, $= ([0, 0], [1, 1])$.
The disjunction strategises are defined by: $\alpha \vee \beta =$
1. *Positive correlation* strategy (join operation),
$= ([max\{x, x_1\}, max\{y, y_1\}],\ [min\{z, z_1\}, min\{v, v_1\}])$,
2. *Ignorance* strategy conjunction,
$= ([max\{x, x_1\}, min\{1, y + y_1\}],\ [max\{0, z + z_1 - 1\}, min\{v, v_1\}])$,
3. *Negative correlation* strategy conjunction,
$= ([min\{1, x + x_1\}, min\{1, y + y_1\}],\ [max\{0, z + z_1 - 1\}, max\{0, v + v_1 - 1\}])$,
4. *Independence* strategy conjunction,
$= ([x + x_1 - x \cdot x_1, y + y_1 - y \cdot y_1],\ [z \cdot z_1, v \cdot v_1])$,
5. *Mutually-exclusive* strategy conjunction, $= ([1, 1], [0, 0])$.
In [3] are introduced hybrid probabilistic logic programs which define general properties that conjunctive and disjunctive strategies should obey. The ones described in this paper are some important examples which we will use in order to parameterize D-bilatices for belief and confidence level logic programs, as will be shown in the

subsections 2.1 and 2.2.

2 Family of bilattices with truth-knowledge duality

In this section we will introduce a particular family of bilattices, denominated D-bilattices, with intuitionistic implication and with a perfect duality between its truth and knowledge lattices, denominated t-lattice and k-lattice respectively.

Lemma 1 *In each bilattice $(\mathcal{B}, \wedge, \vee, \otimes, \oplus, \neg)$, given by Definition 1, the bilattice negation involution operator $\neg$ is selfadjoint modal operator w.r.t. the $\leq_k$.*

Proof: The operator $\neg$ is surjective: suppose that there exists $y \in \mathcal{B}$ such that $\nexists x \in \mathcal{B}.y = \neg x$; but for $x = \neg y$ we have that $\neg x = \neg\neg y = y$ (from the involution of $\neg$) which is a contradiction.
From the homomorphism $\neg : (\mathcal{B}, \leq_k, \otimes, \oplus) \rightarrow (\mathcal{B}, \leq_k, \otimes, \oplus)$ we have that if $x \leq_k y$, i.e., $x = x \otimes y$, then $\neg x = \neg(x \otimes y) = \neg x \otimes \neg y$, i.e., $\neg x \leq_k \neg y$, thus $\neg$ is monotone w.r.t. $\leq_k$. From the fact that for every $x \in \mathcal{B}$, $\neg x = \neg(x \otimes 1_k) = \neg x \otimes \neg 1_k$, i.e. $\forall x \in \mathcal{B}.\neg x \leq_k \neg 1_k$, thus, from the surjective property of $\neg$ must hold that $\neg 1_k$ is the top element in the k-lattice, i.e., $\neg 1_k = 1_k$, and from homomorphic property $\neg(x \otimes y) = \neg x \otimes \neg y$, we conclude that $\neg$ is multiplicative modal operator for the k-lattice.
Analogously, from from the fact that for every $x \in B$, $\neg x = \neg(x \oplus 0_k) = \neg x \oplus \neg 0_k$, i.e. $\forall x \in \mathcal{B}.\neg x \geq_k \neg 0_k$, thus, from the surjective property of $\neg$ must hold that $\neg 0_k$ is the bottom element in the k-lattice, i.e., $\neg 0_k = 0_k$, and from homomorphic property $\neg(x \oplus y) = \neg x \oplus \neg y$, we conclude that $\neg$ is additive modal operator for the k-lattice. Consequently, $\neg$ is a selfadjoint modal operator for the k-lattice.
□

The intuitive meaning for the D-bilattices is that any property defined over the truth ordering (t-lattice) can be equivalently defined over the knowledge ordering (k-lattice), and viceversa. That is, they are perfect dual images.

Definition 2 *A D-bilattice $\mathcal{B}$ is a distributive bilattice $(\mathcal{B}, \wedge, \vee, \otimes, \oplus, \neg)$ with the isomorphism of truth-knowledge lattices $\partial : (\mathcal{B}, \leq_t) \simeq (\mathcal{B}, \leq_k)$, which is an involution. Let us define the unary operator $\sim =_{def} \partial\neg\partial : \mathcal{B} \rightarrow \mathcal{B}$. Then we say that a D-lattice is* `perfect` *if two truth negations, the intuitionistic negation $\neg_t$ (pseudo complement), such that $\neg_t x = \bigvee\{z | z \wedge x = 0_t\}$, and the bilattice negation $\neg$, are correlated by $\neg = \neg_t \sim$.*

From this isomorphism we have that the involution duality operator ∂, with the identity $\partial\partial = id_{\mathcal{B}}$ (the bijective mapping ∂ is equal to its inverse), preserves the ordering, that is, for any two $x, y \in \mathcal{B}$, we have that $x \leq_t y$ iff $\partial x \leq_k \partial y$, i.e., $x \leq_k y$ iff $\partial x \leq_t \partial y$, and $\partial(x \wedge y) = \partial x \otimes \partial y$. Thus, $\partial 1_t = 1_k$ and $\partial 0_t = 0_k$, where 1_k and 0_k are the top and the bottom values w.r.t. $\leq_k$.
Both truth negations are antitonic w.r.t. $\leq_t$, thus hold the De Morgan laws $\neg(x \wedge y) = \neg x \vee \neg y$ and $\neg_t(x \wedge y) = \neg_t x \vee \neg_t y$, with $\neg 1_t = \neg_t 1_t = 0_t$, and $\neg 0_t = \neg_t 0_t = 1_t$. Notice that $\neg$ is monotonic w.r.t. $\leq_k$, while for $\neg_t$ it is not generally valid.

Proposition 1 *In each D-bilattice $(\mathcal{B}, \wedge, \vee, \otimes, \oplus, \neg)$, the operator $\sim$ is selfadjoint modal operator w.r.t. the $\leq_t$, and the bilattice negation operator for k-lattice satisfy $\sim 1_k = 0_k$, $\sim 0_k = 1_k$, while $\sim 1_t = 1_t$, $\sim 0_t = 0_t$ with $1_k \bowtie_t 0_k$.*

Proof: The operator $\sim$ is an involution $(\sim\sim = (\partial\neg\partial)(\partial\neg\partial) = (\partial(\neg(\partial\partial)\neg)\partial) = id_{\mathcal{B}})$ thus, surjective: suppose that there exists $y \in \mathcal{B}$ such that $\nexists x \in \mathcal{B}.y = \sim x$; but for $x = \sim y$ we have that $\sim x = \sim\sim y = y$ (from the involution of $\sim$) which is a contradiction.
If $x \leq_t y$ then $\partial x \leq_k \partial y$ and $\neg\partial x \leq_k \neg\partial y$, thus $\partial\neg\partial x \leq_t \partial\neg\partial y$, i.e., $\sim x \leq_t \sim y$, that is, $\sim$ is monotonic for the t-lattice. We have, based on Lemma 1, that $\sim 1_t = \partial\neg\partial 1_t = \partial\neg 1_k = \partial 1_k = 1_t$ and $\sim (x \wedge y) = \partial\neg\partial(x \wedge y) = \partial\neg(\partial x \otimes \partial y) = \partial(\neg\partial x \otimes \neg\partial y) = (\partial\neg\partial x) \wedge (\partial\neg\partial y) = \sim x \wedge \sim y$, so that $\sim$ is a multiplicative modal operator.
Also, $\sim 0_t = \partial\neg\partial 0_t = \partial\neg 0_k = \partial 0_k = 0_t$ and $\sim (x \vee y) = \partial\neg\partial(x \vee y) = \partial\neg(\partial x \oplus \partial y) = \partial(\neg\partial x \oplus \neg\partial y) = (\partial\neg\partial x) \vee (\partial\neg\partial y) = \sim x \vee \sim y$, so that $\sim$ is also additive modal operator, and, consequently, it is a selfadjoint involution modal operator for the t-lattice.
If $x \leq_k y$ then $\partial x \leq_t \partial y$ and $\neg\partial x \geq_t \neg\partial y$. Thus, $\partial\neg\partial x \geq_k \partial\neg\partial y$, i.e., $\sim x \geq_k \sim y$, that is, $\sim$ is antitonic for the k-lattice. From the fact that for every x, $\sim x = \sim (x \otimes 1_k) = \partial\neg\partial(x \otimes 1_k) = \partial\neg(\partial x \wedge \partial 1_k) = \partial(\neg\partial x \vee \neg\partial 1_k) = (\partial\neg\partial x) \oplus (\partial\neg\partial 1_k) = \sim x \oplus \sim 1_k$, i.e. $\forall x \in \mathcal{B}. \sim x \geq_k \sim 1_k$. Thus, from the surjective property of $\sim$ must hold that $\sim 1_k$ is the bottom element in the k-lattice, i.e., $\sim 1_k = 0_k$.
Analogously, from from the fact that for every x, $\sim x = \sim (x \oplus 0_k) = \partial\neg\partial(x \oplus 0_k) = \partial\neg(\partial x \vee \partial 0_k) = \partial(\neg\partial x \wedge \neg\partial 0_k) = (\partial\neg\partial x) \otimes (\partial\neg\partial 0_k) = \sim x \otimes \sim 0_k$, i.e. $\forall x \in \mathcal{B}. \sim x \leq_k \sim 0_k$. Thus, from the surjective property of $\sim$ must hold that $\sim 0_k$ is the top element in the k-lattice, i.e., $\sim 0_k = 1_k$. Consequently, $\sim$ is a bilattice negation operator for the k-lattice.
In any bilattice [23] we have that $1_t \bowtie_k 0_t$. Let us show that in D-bilattices holds also that $1_k \bowtie_t 0_k$: suppose that $1_k \leq_t 0_k$ then $\partial 1_k = 1_t \leq_k \partial 0_k = 0_t$ which is a contradiction (the same holds if we suppose that $1_k \geq_t 0_k$).
$\square$
From this point of view, the D-lattices are Heyting algebras w.r.t. $\leq_t$ ordering, enriched by the modal selfadjoint operator $\sim$.

Corollary 1 *For any D-bilattice $\mathcal{B}$ the duality operator ∂ can be extended to the following isomorphism of modal Heyting algebras $\partial : (\mathcal{B}, \leq_t, \alpha_t) \simeq (\mathcal{B}, \leq_k, \alpha_k)$, with $\alpha_t = \{\wedge, \rightharpoonup, \sim\}$, $\alpha_k = \{\otimes, \rightharpoondown, \neg\}$, where $\rightharpoonup$ and $\rightharpoondown$ are the intuitionistic implications (the relative pseudo-complements) w.r.t. the $\leq_t$ and $\leq_k$ respectively.*

Proof: Let us prove that ∂ is a homomorphism also for relative pseudo-complements, that is, $\partial(x \rightharpoonup y) = \partial(\bigvee\{z | z \wedge x \leq_t y\}) = \oplus\{\partial z | z \wedge x \leq_t y\}$ =(from homomorphic property of ∂ w.r.t. $\vee$ operator) $= \oplus\{\partial z | z \wedge x \leq_t y\}$ =(from homomorphic property of ∂ w.r.t. $\wedge$ operator) $= \oplus\{\partial z | \partial z \otimes \partial x \leq_k \partial y)\} = \partial x \rightharpoondown \partial y$. That is, $\partial \rightharpoonup = \rightharpoondown \partial$.
Thus, we have that $\partial\neg_t = \neg_k\partial$ (from $\neg_t x = x \rightharpoonup 0_t$, $\neg_k x = x \rightharpoondown 0_k$, and $0_k = \partial 0_t$).
The operator $\sim$ is a selfadjoint modal operator for the t-lattice, while $\neg$ is a selfadjoint modal operator for the k-lattice. Consequently ∂ is an isomorphism between modal

truth/knowledge Heyting algebras.
□
Notice that important point of the Definition 2 above is that it introduces two modal selfadjoint operators $\sim$ and $\neg$. In the case of perfect D-lattices the modal operator $\sim$ is the *belief* operator which models the S5 autoepistemic modal logic. They are very important in the case when, from the fact the intuitionistic negation operators $\neg_t, \neg_k$ are non monotonic for *both* orderings, they can not be used in order to define the fixpoint semantics for logic programs (as, for example in Belnap's bilattice): in that case, for example $\neg = \neg_t \sim$ is not monotonic w.r.t. the truth ordering, but is monotonic w.r.t. the knowledge ordering, and the normal 3-valued logic programs can use it as default (or epistemic) negation with fixpoint semantics computed w.r.t. knowledge ordering. For example, consider the clause $A \leftarrow B_1, \neg B_2$, where $\neg$ is the epistemic (default) negation in logic programs, such that $\neg = \neg_t \sim$, where $\sim$ is a modal belief operator and $\neg_t$ 'classical' (i.e., intuitionistic) negation operator in α_t (i.e., the standard way to represent normal (3-valued) logic programs as autoepistemic logic programs).
We will show that such a family of D-bilattices is very important for applications in logic, and that the simplest non trivial bilattice is also a member of this family. Moreover, every Boolean algebra (distributive lattice) with even number of atoms is a D-bilattice. The fact that such an isomorphism preserves the ordering is very important, so that if we define a fixpoint semantics for logic programs w.r.t. some truth-ordering monotonic operator, we are able to define also the semantics w.r.t. the correspondent knowledge-ordering monotonic operator, and viceversa.

Lemma 2 *Let **2** $= (\{0,1\}, \leq, \cdot, max, ^{-})$ denote the classic 2-valued Boolean algebra and $(\mathbf{2}^n, \leq_t)$ denote the the Boolean algebra, obtained as the n-times cartesian product of **2**, such that $\langle a_1, ..., a_n\rangle \leq_t \langle b_1, ..., b_n\rangle$ iff $a_j \leq b_j$, $a_j, b_j \in \mathbf{2}$ for all $1 \leq j \leq 2n$. Then, given a set S of $|S| = n, n \geq 1$ elements with an enumeration $m_N : \{1, 2, ..., n\} \rightarrow S$, the mapping $i_B : \mathcal{P}(S) \rightarrow \mathbf{2}^n$, (where $\mathcal{P}(S)$ is the powerset of S, such that for any $X \in \mathcal{P}(S)$, $i_B(X) = \langle a_1, ..., a_n\rangle$ with $a_j = 1$ if $m_N(j) \in X$; 0 otherwise), is also the following isomorphism of the Powerset Boolean algebra and the tuple-based Boolean algebra $i_B : (\mathcal{P}(S), \subseteq, \bigcap, \bigcup, -) \simeq (\mathbf{2}^n, \leq_t, \wedge, \vee, ^{-})$.*

Proof: It is easy to show that the inverse mapping i_B^{-1} if defined by $i_B^{-1}(\langle a_1, ..., a_n\rangle) = \{m_N(j) \mid$ if $1 \leq j \leq n$ and $a_j = 1\}$. The top element in the tuple based lattice is $1_t = \langle 1, ..., 1\rangle$, while the bottom element is $0_t = \langle 0, ..., 0\rangle$. In the tuple-based Boolean algebra the operators are defined by, for any two $X = i_B^{-1}(\langle a_1, ..., a_n\rangle)$, $Y = i_B^{-1}(\langle b_1, ..., b_n\rangle)$, by
$i_B(-X) = \overline{\langle a_1, ..., a_n\rangle} = \langle \overline{a_1}, ..., \overline{a_n}\rangle$, where $\overline{0} = 1$, and $\overline{1} = 0$,
$i_B(X \bigcap Y) = \langle a_1, ..., a_n\rangle \wedge \langle b_1, ..., b_n\rangle = \langle a_1 \cdot b_1, ..., a_n \cdot b_n\rangle$, where $\cdot$ is a multiplication, $i_B(X \bigcup Y) = \langle a_1, ..., a_n\rangle \vee \langle b_1, ..., b_n\rangle = \langle max(a_1, b_1), ..., max(a_n, b_n)\rangle$.
□
We will use these tuple-based Boolean algebras for the proof of the following proposition (the D-bilattices which are not Boolean will be presented in last two subsections):

Proposition 2 *Boolean algebras with even number of atoms are perfect D-bilattices. Other Boolean algebras are not bilattices, thus are not D-bilattices.*

Proof: Each Boolean algebra with even number of atoms is isomorphic to the powerset Boolean algebra $(\mathcal{P}(S), \subseteq, \bigcap, \bigcup, -)$ with $|S| = 2n, n \geq 1$ of atoms, and, consequently, to the tuple-based Boolean algebra $(\mathbf{2}^{2n}, \leq_t, \wedge, \vee, ^{-})$. Let us define the knowledge ordering in this Boolean lattice by:
$\langle a_1, ..., a_n\rangle \leq_k \langle b_1, ..., b_n\rangle$ iff $a_j \leq b_j,\ 1 \leq j \leq n$ and $a_j \geq b_j,\ n+1 \leq j \leq 2n$, with the top $1_k = \langle a_1, ..., a_{2n}\rangle$ and the bottom element $0_k = \langle \overline{a_1}, ..., \overline{a_{2n}}\rangle$ respectively, such that $a_j = 1$, for $1 \leq j \leq n$ and $a_j = 0$, for $n+1 \leq j \leq 2n$. Then this tuple-based Boolean algebra is also the bilattice $(\mathbf{2}^{2n}, \wedge, \vee, \otimes, \otimes, \neg)$ with the meet and join operators defined by $\langle a_1, ..., a_{2n}\rangle \oplus \langle b_1, ..., b_{2n}\rangle = \langle max(a_1, b_1), ..., max(a_n, b_n), a_{n+1} \cdot b_{n+1}, ..., a_{2n} \cdot b_{2n}\rangle$,
$\langle a_1, ..., a_{2n}\rangle \otimes \langle b_1, ..., b_{2n}\rangle = \langle a_1 \cdot b_1, ..., a_n \cdot b_n, max(a_{n+1}, b_{n+1}), ..., max(a_{2n}, b_{2n})\rangle$.
Let us define the bilattice negation $\neg$ and the duality operator ∂ as follows:
$\neg\langle a_1, ..., a_{2n}\rangle =_{def} \langle \overline{a_{n+1}}, ..., \overline{a_{2n}}, \overline{a_1}, \overline{a_n}\rangle$,
so that $\neg\neg$ is an identity, and $\neg$ is an antitonic negation operator for the $\leq_t$, with $\neg 1_t = 0_t$ and $\neg 0_t = 1_t$. It is monotonic w.r.t. $\leq_k$, and
$\partial\langle a_1, ..., a_{2n}\rangle =_{def} \langle a_1, ..., a_n, \overline{a_{n+1}}, \overline{a_{2n}}\rangle$.
It is easy to show that $\partial\partial$ is an identity and that is valid the following isomorphism:
$\partial : (\mathbf{2}^{2n}, \leq_t, \wedge, \vee, \sim) \simeq (\mathbf{2}^{2n}, \leq_k, \otimes, \oplus, \neg)$, where $\sim =_{def} \partial\neg\partial$ is a selfadjoint modal operator w.r.t. $\leq_t$, while $\neg$ is a selfadjoint modal operator w.r.t. $\leq_k$.
In any Boolean algebra (a distributive lattice with complements) The pseudo-complement $\neg_t$ coincides with Boolean negation, thus in this tuple based Boolean algebra we have that $\neg_t$ is equal to $^{-}$, as follows:
$\neg_t\langle a_1, ..., a_{2n}\rangle = \bigvee\{\langle c_1, ..., c_{2n}\rangle \mid \langle c_1, ..., c_{2n}\rangle \wedge \langle a_1, ..., a_{2n}\rangle = \langle 0, ..., 0\rangle\} =$
$= \bigvee\{\langle c_1, ..., c_{2n}\rangle \mid c_i \cdot a_i = 0, 1 \leq i \leq 2n\} = \langle \overline{a_1}, ..., \overline{a_{2n}}\rangle = \overline{\langle a_1, ..., a_{2n}\rangle}$.
Thus, we have that $\neg_t \sim \langle a_1, ..., a_{2n}\rangle = \neg_t\langle a_{n+1}, ..., a_{2n}, a_1, ..., a_n\rangle =$
$= \langle \overline{a_{n+1}}, ..., \overline{a_{2n}}, \overline{a_1}, ..., \overline{a_n}\rangle = \neg\langle a_1, ..., a_{2n}\rangle$.
Consequently, the tuple-based Boolean algebra $(\mathbf{2}^{2n}, \leq_t, \wedge, \vee, ^{-})$ and, based on the isomorphism i_B of the Lemma 2, the original Boolean algebra with even number of atoms, are the perfect D-lattices.
Let us show that any Boolean algebra $\mathcal{P}(S), \subseteq, \bigcap, \bigcup, -)$ with $|S| = 2n+1, n \geq 1$ atoms can not be a D-lattice. It is enough to consider the, isomorphic to it, tuple-based Boolean algebra: in such algebra every maximal chain $0_t \leq_t ..., \leq_t 1_k \leq_t ... \leq_t 1_t$ has $2n + 2$ elements. Let us consider the subset C of such chains which contain the top knowledge element 1_k. For every chain in C, the number of elements strictly lower than 1_k is greater (in the following we consider this case: other case, when it is lower is similar) then the number of elements that are strictly greater than 1_k. From the antitonic and bijective property (involution) of the bilattice negation $\neg$ we have that there exists some element $a <_t 1_k$ in some chain in C for which we can not define $\neg a$: suppose that for all elements greater that 1_k we defined correspondent negative elements (for any $b >_t 1_k$ and $\neg b \neq a$, $\neg b$ is in a chain with $\neg\neg b = b$). If it is D-bilattice than $\neg 1_k = 1_k$, so that from $a <_t 1_k$ must hold (from the antitonic property of $\neg$) that $\neg a >_t \neg 1_k = 1_k$, but this element $a' = \neg a$ is greater than 1_k so that already has some different element b from a with $\neg a = \neg b$, i.e., $a = \neg\neg a = \neg\neg b = b$ what is a contradiction. Thus for such Boolean algebras we are not able to define the bilattice negation $\neg$, and such algebras can not be bilattices.
$\square$

2.1 The smallest nontrivial D-bilattice

The smallest non trivial bilattice is a 4-valued Belnap's bilattice, particularly important for the knowledge systems with incomplete and inconsistent information. It can be seen as the minimal many-valued logic system where we are able to deal with incomplete and inconsistent information: in [37] is presented an autoepistemic logic programming, with the Moore's modal operator, where an intuitive and natural approach is used to resolve inconsistency, with the simple monotonic (w.r.t. the knowledge ordering) "immediate consequence operator" for the least-fixpoint computation of their Herbrand models.
The following propositions show that Belnap's bilattice has the perfect duality properties and that the duality extension of the original Belnap's bilattice is truth-functionally complete.

Proposition 3 *Belnap's bilattice $\mathcal{B}_4$ is a perfect D-bilattice.*

Proof: The intuitionistic implications (relative pseudo-complements) are the following [40]:

$\rightharpoonup$	t	$\bot$	f	$\top$		$\rightharpoondown$	t	$\bot$	f	$\top$
t	t	$\bot$	f	$\top$			$\top$	f	f	$\top$
$\bot$	t	t	$\top$	$\top$			$\top$	$\top$	$\top$	$\top$
f	t	t	t	t			t	t	$\top$	$\top$
$\top$	t	$\bot$	$\bot$	t			t	$\bot$	f	$\top$

We define the modal dual operation ∂ as follows:
$\partial t = \top,\ \partial\top = t,\ \partial f = \bot$, and $\partial\bot = f$, with $\neg_t x =_{def} x \rightharpoonup f = \neg\partial\neg\partial x$.
It is easy to verify that it is an involution, $\partial\partial = id$, and that hold
$\partial(x \wedge y) = \partial x \otimes \partial y,\ \ \partial(x \rightharpoonup y) = \partial x \rightharpoondown \partial y$, and $\partial \sim x = \neg\partial x$, and
$\partial(x \otimes y) = \partial x \wedge \partial y,\ \ \partial(x \rightharpoondown y) = \partial x \rightharpoonup \partial y$, and $\partial\neg x = \sim \partial x$.
It is perfect D-lattice because holds that $\neg = \neg_t \sim$.
Alternatively, we can show that Belnap's bilattice is the Boolean algebra w.r.t. the negation $\neg_t$ (holds that $\neg_t\neg_t$ is an identity, and $\neg_t x \vee x = t$, $\neg_t x \wedge x = f$) with the even number of atoms $\{\bot, \top\}$, thus by Proposition 2 it is perfect D-lattice.
$\square$
Just because of this isomorphism ∂, the t-lattice (or equivalently, k-lattice) completely defines the D-bilattice.
We can se that the conflation (knowledge negation) operator $\sim$ is the epistemic belief operator " it is believed that" for a bilattice, which extends the 2-valued belief of the autoepistemic logic as follows:

- if A is true than "it is believed that A", i.e., $\sim A$, is true.
- if A is false than "it is believed that A", i.e., $\sim A$, is false.
- if A is unknown than "it is believed that A", i.e., $\sim A$, is inconsistent: it is really inconsistent to believe in something that is unknown.

- if A is inconsistent (that is, *both* true and false) than "it is believed that A", i.e., $\sim A$, is unknown: really, we can not tell nothing about believing in something that is inconsistent.

Notice that in this 4-valued framework by assigning to some inconsistent fact the value $\top$ we avoid the *explosive inconsistency*; it is a good feature of Belnap's based autoepistemic logic, where $x \wedge \neg x \neq f$ (for example, $\top \wedge \neg\top = \top \neq f$), differently from the classic 2-valued logic. That is the reason that we are able to use it in the cases where mutually-inconsistent information can happen, as, for example in data integration of different data sources [27, 37]. So, as in the 2-valued case, the *epistemic* negation $\neg$ is the (intuitionistic) negation of the belief, that is $\neg = \neg_t \sim$. Dually, for the knowledge ordered lattice, holds that the (epistemic) knowledge negation is the negation of the knowledge-belief, that is $\sim = \neg_k \neg$.
While Belnap's bilattice is Boolean algebra w.r.t. the intuitionistic negation $\neg_t$ (where $x \wedge \neg_t x = f$, $x \vee \neg_t x = t$ and $x \rightharpoonup y = \neg_t x \vee y$), it is easy to verify that, w.r.t. the bilattice epistemic negation $\neg = \neg_t \sim$ the $x \rightharpoonup y$ is different from $\neg x \vee y$, and the more strong requirement holds:

Proposition 4 *The implication operator $\rightharpoonup$ and the duality operator ∂ on the bilattice $\mathcal{B}_4$ cannot be written in terms of the existing bilattice functions $\wedge, \vee, \otimes, \oplus$ and $\neg$ defined on $\mathcal{B}_4$.*

Proof: The intuitionistic negation $\neg_t$ is defined by $\neg_t x = x \rightharpoonup f$. Let us show that it cannot be written in terms of $\wedge, \vee, \otimes, \oplus$ and $\neg$. We have that all functions in $\wedge, \vee, \otimes, \oplus$ and $\neg$ distribute with respect to the operation $\otimes$, so that if κ is any modal operator constructed from them we will have $\kappa(\perp) = \kappa(f \otimes t) = \kappa(f) \otimes k(t)$ (see Proposition 4.1 in [25]). However, $\neg_t \perp = \top$, while $\neg_t f \otimes \neg_t t = t \otimes f = \perp \neq \top$.
Also, $\partial \perp = f$, while $\partial f \otimes \partial t = \perp \otimes \top = \perp \neq f$.
□
We are able to define other modal operators over D-bilattices, as, for example, Moore's [31] modal operator M, where Mx is intended to capture the notion of "I know that x". In the 4-valued framework this is related to the following unary mapping on the bilattice $\mathcal{B}_4$ (in [25]): $M(x) = t$, if $x \in \{t, \top\}$; f otherwise.

Proposition 5 *In Belnap's bilattice Moore's modal operator is defined by* $M(x) = \partial((\top \rightharpoonup x) \wedge \neg\partial(\top \rightharpoonup x))$.

Now we will prove the truth-functionally completeness.

Proposition 6 *Every modal operator on the bilattice $\mathcal{B}_4$ can be written as a combination of the operators $\wedge, \rightharpoonup, \neg, \partial$ and the constant function $\perp$.*

Proof: For the truth-functionality completeness we have that: the Moore's modal operator is defined by $M(x) = \partial((\partial\neg\partial\perp \rightharpoonup x) \wedge \neg\partial(\partial\neg\partial\perp \rightharpoonup x))$, while the consensus is defined by $x \oplus y = \partial(\neg(\neg x \wedge \neg y))$, thus, from Proposition 4.2 [25] this proof is concluded.
□

The upshot of this proposition is that no additional operators are needed, that is, the set of operators in this D-bilattice is *truth-complete*. That is, for *any* logic program (also with Moore's or other modal operators) it is enough to use the conjunction, epistemic negation, intuitionistic implication and duality operator.

2.2 Duality in world-based bilattices

Ginsberg [23] defined a *world-based* bilattices, considering a collection of worlds W, where by world we mean some possible way of how things might be:

Definition 3 *[23] A pair $[U,V] \in \mathcal{P}(W) \times \mathcal{P}(W)$ of subsets of W (here $\mathcal{P}(W)$ denotes the powerset of the set W) express the truth of a sentence p, with $\leq_t, \leq_k$ truth and knowledge preorders relatively, as follows:*
1. U is a set of worlds where p is true, V is a set of worlds where p is false, $P = U \bigcap V$ is a set where p is inconsistent (both true and false), and $W - (U \bigcup V)$ where p is unknown.
2. $[U,V] \leq_t [U_1,V_1]$ iff $U \subseteq U_1$ and $V_1 \subseteq V$
3. $[U,V] \leq_k [U_1,V_1]$ iff $U \subseteq U_1$ and $V \subseteq V_1$.
The bilattice operations associated with $\leq_t$ and $\leq_k$ are:
4. $[U,V] \wedge [U_1,V_1] \ = \ [U \bigcap U_1, V \bigcup V_1]$, $[U,V] \vee [U_1,V_1] \ = \ [U \bigcup U_1, V \bigcap V_1]$
5. $[U,V] \otimes [U_1,V_1] \ = \ [U \bigcap U_1, V \bigcap V_1]$, $[U,V] \oplus [U_1,V_1] \ = \ [U \bigcup U_1, V \bigcup V_1]$
6. $\neg[U,V] \ = \ [V,U]$.
Let us denote by $\mathcal{B}_W$ the set $\mathcal{P}(W) \times \mathcal{P}(W)$, then $(\mathcal{B}_W, \wedge, \vee, \otimes, \oplus, \neg)$ is a bilattice.

If we take that W is equal to Herbrand base H_P of a logic program P, then a member $[U,V] \in \mathcal{B}_W$, where U is the subset of true ground atoms and V the subset of false ground atoms w.r.t some Herbrand interpretation $I = v_B : H_P \to \mathcal{B}_4$, then this member is a set-based representation of this Herbrand interpretation, and $\mathcal{B}_W$ is isomorphic (that is, equivalent) to the functional space $\mathcal{B}_4^{H_P}$. Let us show that this functional space $\mathcal{B}^{H_P}$, that is, the world-based bilattice $\mathcal{B}_W$ is a D-bilattice.

Proposition 7 *The world-based bilattice is a perfect D-bilattice composed by two lattices, a t-lattice $(\mathcal{B}_W, \alpha_t)$, with $\alpha_t = \{\wedge, \rightharpoonup, \sim\}$, and a k-lattice $(\mathcal{B}_W, \alpha_k)$, with $\alpha_k = \{\otimes, \neg\!\!\!\rightharpoonup, \neg\}$, such that:*
1. Intuitionistic implication: $[U,V] \rightharpoonup [U_1,V_1] \ = \ [\overline{U} \bigcup U_1, \overline{V} \bigcap V_1]$.
2. Duality operation: $\partial[U,V] = [U,\overline{V}]$.
3. Knowledge negation (modal belief operator): $\sim [U,V] = [\overline{V},\overline{U}]$,
where $\overline{S} = W - S$ is the set complement of S.

Proof: For intuitionistic implication holds that:
$[U,V] \rightharpoonup [U_1,V_1] \ = \ \bigvee\{[U_2,V_2] \mid [U_2,V_2] \wedge [U,V] \leq_t [U_1,V_1]\}$
$= \ \bigvee\{[U_2,V_2] \mid U_2 \bigcap U \subseteq U_1$ and $V_2 \bigcup V \supseteq V_1\} \ = \ [max\{U_2 \mid U_2 \bigcap U \subseteq U_1\}, min\{V_2 \mid V_2 \bigcup V \supseteq V_1\}] \ = \ [\overline{U} \bigcup U_1, \overline{V} \bigcap V_1]$. It is easy to verify that ∂ is an involution and an isomorphism between $(\mathcal{B}_W, \alpha_t)$ and $(\mathcal{B}_W, \alpha_k)$. So, $\sim [U,V] = \partial\neg\partial[U,V] = [\overline{V},\overline{U}]$, and holds $\neg = \neg_t \sim$, or $\neg_t = \neg \sim = \neg\partial\neg\partial$.
Alternatively, we can show that the world-based bilattice is the Boolean algebra w.r.t.

the negation $\neg_t[U,V] = [\overline{U}, \overline{V}]$ (holds that $\neg_t\neg_t$ is an identity, and $\neg_t[U,V] \vee [U,V] = [W, \emptyset] = 1_t$, $\neg_t[U,V] \wedge [U,V] = [\emptyset, W] = 0_t$) with the even number of atoms $|W^2|$, thus by Proposition 2 it is a perfect D-lattice.
$\square$
In Section 3 we will generalize the world-based bilattices to *any higher-order* many-valued D-bilattice.
Differently from these first two examples, the next two examples of D-latices, the belief and the confidence-level based logics, are not Boolean algebras.

2.3 Duality in belief based logic

In what follows we will use the Heyting algebra $(L, 0, 1, \wedge, \Rightarrow, \neg_w)$ where $L = [0,1]$ is the interval of *real numbers* between zero and 1, with new negation $\neg_w$, which reverses the truth ordering, $\neg_w x = 1 - x$, (here '$-$' is the mathematical substraction for real numbers) and satisfies De Morgan laws for join and meet operators.
Suppose we are attempting to assign, not classical truth values, but *probability estimates* to formulae, that is, the closed intervals $[x,y] \subseteq [0,1]$. We introduce the following truth and knowledge orderings for this interval-based bilattice:
1. Knowledge ordering: Sandewall, in [6], suggested that such truth values should be ordered by set inclusion (inverse to knowledge ordering), thus we set $[x,y] \leq_k [z,w]$ if $x \leq z$ and $w \leq y$, that is, $[z,w] \subseteq [x,y]$ (if we consider that $[z,w]$ is empty set if $z > w$), with the bottom element $0_K = [0,1]$ and the top element $1_K = [1,0]$.
2. Truth ordering: Scott suggested, in [5], that such generalized truth values should be partially ordered by *degree of truth*, that is $[x,y] \leq_t [z,w]$ if $x \leq z$ and $y \leq w$, with the bottom element $0_T = [0,0]$ and the top element $1_T = [1,1]$.
Thus we obtain the interval-based bilattice with set of truth values $L_B = \{[x,y] \mid 0 \leq x, y \leq 1\}$. Differently from Fitting, in [19], and Ginsberg, in [23], we will consider all values in L_B (also 'inconsistent' values $[x,y]$ with $x > y$).
It corresponds to a simple way of constructing bilattices, due to Ginsberg, that is, to the structure $< L_1 \times L_2, \leq_t, \leq_k >$, in the case when $L_1 = L_2$ is a complete lattice of reals from 0 to 1, with classic orderings $\leq$ of real numbers. Thus, this structure is an extended Heyting algebra $(L, 0, 1, \wedge, \Rightarrow, \neg_w)$. Think of L_1 as the lattice of values used to measure the lower boundary degree of *belief* we have in a sentence; probabilities, weighted opinions of experts, etc., and of L_2 as the lattice of values used to measure the upper boundary degree of belief. Then a member of $L_1 \times L_2$ embodies an assessment of belief degree. The ordering $\leq_t$ intuitively says that degree of truth increases if both belief boundaries increase (in [35] for example). Likewise $\leq_k$ intuitively says degree of knowledge increases if the precision goes up, in the case of 'consistent' intervals (used in [32]).

Proposition 8 *Let $(L, 0, 1, \wedge, \Rightarrow, \neg_w)$ be the interval-based extended Heyting algebra with $L = [0,1]$. The interval-based bilattice is a D-bilattice composed by two lattices, a t-lattice (L_B, α_t), with $\alpha_t = \{\wedge, \rightharpoonup, \sim\}$, and a k-lattice (L_B, α_k), with $\alpha_k = \{\otimes, \neg\!\!\rightarrow, \neg\}$, such that:*
1. Meet operation for $\leq_t$ ordering is the positive correlation probability strategy conjunction $[x,y] \wedge [x_1, y_1] \;=\; [x \wedge x_1, y \wedge y_1] = [min\{x, x_1\}, min\{y, y_1\}]$.

2. Intuitionistic implication: $[x, y] \rightharpoonup [x_1, y_1] \ = \ [x \Rightarrow x_1, y \Rightarrow y_1]$.
3. Truth negation: $\neg[x, y] = [\neg_w y, \neg_w x] = [1 - y, 1 - x]$,
4. Knowledge negation: $\sim [x, y] = [y, x]$.
5. Duality operation: $\partial[x, y] = [x, \neg_w y] = [x, 1 - y]$.

Proof: Easy to verify. Notice that it is not perfect, that is $\neg \neq \neg_t \sim$.
□
The meet and join operation in this D-bilattice correspond to the positive conjunction/disjunction probabilistic strategies. We can extend the duality principle (that ∂ is the isomorphism between truth and knowledge lattices, such that, for example, for any conjunctive strategy $\wedge_p$ holds $\partial(x \wedge_p y) = \partial x \otimes_p \partial y)$) also for other conjunction/disjunction strategies, as ignorance, negative correlation, independence and mutually-exclusive probabilistic strategies. What is important is that we maintain only one negation operator (which satisfies De Morgan laws for each of these conjunctive/disjunctive strategies). For generality we will introduce also the parameterized implications (relative pseudo-complement for given set of parameterized conjunctions)- in practice for Logic programming we can use only the intuitionistic implication of Heyting algebra based on the meet operation (which can be seen as the 'principal' conjunction strategy). But we can in future consider also the cases when to each logic rule of a logic program is associated a particular conjunctive/disjunctive strategy: in that case we can interpret the implication of this rule as a parameterized implication for this particular rule-adopted strategy. In the same way that we may consider the conjunctive/disjunctive strategies as kind of approximations of meet/join operators, also parameterized implications can be seen as approximations of the intuitionistic implication. The aim of this paper is not to adopt pro/contro statements about such possibilities: we simply consider that if the conjunctive/disjunctive probabilistic strategies are sensible operations, then also the implication probabilistic strategies (derived from the conjunctive strategies by means of the relative pseudo-complement) are sensible operations.
As first step we will define this family for the most simple case of uncertainty type, when we associate to any knowledge fact a single probability value from the set $L = [0, 1]$.

Definition 4 *We define the parameterized Heyting algebra by the following extension of the Heyting algebra [38],* $(L, 0, 1, \neg_w, \{\wedge_p \mid p \in Par\}, \{\vee_p \mid p \in Par\}, \{\Rightarrow_p \mid p \in Par\})$*, where* $Par = \{w, s, m, e\}$ *are the parameters, such that for any pair* $x, y \in L$ *(which can be considered as probabilities of two given events), the probability for the composed event is one of the following cases:*
1. weak conjunction (meet operator), $x \wedge_w y = min\{x, y\}$
2. strong conjunction, $x \wedge_s y = max\{0, x + y - 1\}$
3. multiplicative conjunction, $x \wedge_m y = x \cdot y$
4. mutually exclusive conjunction, $x \wedge_e y = 0$.
The parametric disjunctions are defined by de Morgan laws $x \vee_p y = \neg_w(\neg_w x \wedge_p \neg_w y)$, $p \in Par$*, and the parametric implications by relative pseudo complements (only the* $\Rightarrow_w$ *is an intuitionistic implication, other can be seen as its approximations)* $x \Rightarrow_p y = max\{z \in L \mid z \wedge_p x \leq y\}$*, for any* $p \in Par$.

Notice that the first three "conjunctions" are well known t-norms, while the last express the fact that if two events are mutually exclusive, the probability for their contemporary appearance is zero. The $(L, 0, 1, \neg_w, \{\wedge_p \mid p \in Par\}, \{\vee_p \mid p \in Par\}, \{\Rightarrow_p \mid p \in Par\})$ is not a lattice except for weak conjunction. It is similar to multi-adjoint lattice [14] used for logic programming with continuous semantics. For a given parameterized Heyting algebra over single probability values in $L = [0, 1]$, we are able to define the belief-based logic, parameterized by a number of possible *probabilistic strategies*, over probability-interval based bilattice L_B defined in precedence, with two orderings $\leq_t, \leq_k$ (obviously, holds that $L_B = L \times L$).

Proposition 9 *The interval-based parameterized bilattice is a* p-D-bilattice *(parameterized D-bilattice extended by set of other conjunctions different from meet (positive correlation) operator) composed by two p-lattices, a p-t-lattice (L_B, α_t), with $\alpha_t = \{\{\wedge_p, \rightharpoonup_p \mid p \in Par_B\}, \sim\}$, and a p-k-lattice (L_B, α_k), with $\alpha_k = \{\{\otimes_p, \rightharpoondown_p \mid p \in Par_B\}, \neg\}$, where $Par_B = \{pc, ig, nc, in, ex\}$ are parameters, such that for any pair $[x, y], [x_1, y_1] \in L_B$ the following hold*
1. Positive correlation strategy conjunction (meet operation), $[x, y] \wedge_{pc} [x_1, y_1] = [x \wedge_w x_1, y \wedge_w y_1]$
2. Ignorance strategy conjunction, $[x, y] \wedge_{ig} [x_1, y_1] = [x \wedge_s x_1, y \wedge_w y_1]$
3. Negative correlation strategy conjunction, $[x, y] \wedge_{nc} [x_1, y_1] = [x \wedge_s x_1, y \wedge_s y_1]$
4. Independence strategy conjunction, $[x, y] \wedge_{in} [x_1, y_1] = [x \wedge_m x_1, y \wedge_m y_1]$
5. Mutually-exclusive strategy conjunction, $[x, y] \wedge_{ex} [x_1, y_1] = [x \wedge_e x_1, y \wedge_e y_1] = 0_T$
6. The parametric disjunctions are defined by de Morgan laws $[x, y] \vee_p [x_1, y_1] = \neg(\neg[x, y] \wedge_p \neg[x_1, y_1])$*, for any $p \in Par_B$, and the parametric implications by relative pseudo complements*
$[x, y] \rightharpoonup_p [x_1, y_1] = max\{[z, w] \in L_B \mid [x, y] \wedge_p [z, w] \leq_t [x_1, y_1]\}$.
7. The knowledge conjunction/disjunction strategies and implications are defined by
$[x, y] \otimes_p [x_1, y_1] = \partial(\partial[x, y] \wedge_p \partial[x_1, y_1])$, $[x, y] \oplus_p [x_1, y_1] = \partial(\partial[x, y] \vee_p \partial[x_1, y_1])$,
$[x, y] \rightharpoondown_p [x_1, y_1] = \partial(\partial[x, y] \rightharpoonup_p \partial[x_1, y_1])$, *for each $p \in Par_B$.*

Proof: It is easy to verify, based on proofs of precedent propositions. We obtain that for any pair $[x, y], [x_1, y_1] \in L_B$ hold, for example:
$[x, y] \wedge_{pc} [x_1, y_1] = [x \wedge_w x_1, y \wedge_w y_1] = [min\{x, x_1\}, min\{y, y_1\}]$, that is, the positive correlation strategy (see subsection 1.3).
Analogously we can verify that all other conjunction strategies correspond to the definitions in subsection 1.3. Moreover we have that for parameterized implications hold:
1. Positive correlation implication, $[x, y] \rightharpoonup_{pc} [x_1, y_1] = [x \Rightarrow_w x_1, y \Rightarrow_w y_1]$.
2. Ignorance strategy implication, $[x, y] \rightharpoonup_{ig} [x_1, y_1] = [x \Rightarrow_s x_1, y \Rightarrow_w y_1]$.
3. Negative correlation implication, $[x, y] \rightharpoonup_{nc} [x_1, y_1] = [x \Rightarrow_s x_1, y \Rightarrow_s y_1]$.
4. Independence strategy implication, $[x, y] \rightharpoonup_{in} [x_1, y_1] = [x \Rightarrow_m x_1, y \Rightarrow_m y_1]$.
5. Mutually-exclusive implication, $[x, y] \rightharpoonup_{ex} [x_1, y_1] = [x \Rightarrow_e x_1, y \Rightarrow_e y_1] = 1_T$.
Let us verify that knowledge conjunctions/disjunctions satisfy De Morgan laws w.r.t. the $\sim$ (knowledge negation). First, notice that holds $\neg = \partial \sim \partial$, thus for any $p \in Par_B$
$\sim (\sim [x, y] \otimes_p \sim [x_1, y_1]) =\sim ([y, x] \otimes_p [y_1, x_1]) =\sim \partial(\partial[y, x] \wedge_p \partial[y_1, x_1])$
$= \partial\neg(\partial[y, x] \wedge_p \partial[y_1, x_1]) = \partial(\neg\partial[y, x] \vee_p \neg\partial[y_1, x_1]) = \partial(\partial \sim [y, x] \vee_p \partial \sim$

$[y_1, x_1]) = \partial(\partial[x, y] \vee_p \partial[x_1, y_1]) = [x, y] \oplus_p [x_1, y_1]$. $\square$
From this proposition it is easy to find all parameterized knowledge-dimension operations. For instance, for the ignorance strategy we have that
$[x, y] \otimes_{ig} [x_1, y_1] = \partial(\partial[x, y] \wedge_{ig} \partial[x_1, y_1]) = \partial([x, \neg_w y] \wedge_{ig} [x_1, \neg_w y_1])$
$= \partial[x \wedge_s x_1, \neg_w y \wedge_w \neg_w y_1] = [x \wedge_s x_1, \neg_w(\neg_w y \wedge_w \neg_w y_1)] = [x \wedge_s x_1, y \vee_w y_1]$,
$[x, y] \oplus_{ig} [x_1, y_1] =\sim (\sim [x, y] \otimes_{ig} \sim [x_1, y_1]) =\sim ([y, x] \otimes_{ig} [y_1, x_1])$
$=\sim [y \wedge_s y_1, \neg_w(\neg_w x \wedge_w \neg_w x_1)] = [x \vee_w x_1), y \wedge_s y_1]$, and
$[x, y] \rightharpoondown_{ig} [x_1, y_1] = \partial(\partial[x, y] \rightharpoonup_{ig} \partial[x_1, y_1]) = \partial([x, \neg_w y] \rightharpoonup_{ig} [x_1, \neg_w y_1])$
$= \partial[x \Rightarrow_s x_1, \neg_w y \Rightarrow_w \neg_w y_1] = [x \Rightarrow_s x_1, \neg_w(\neg_w y \Rightarrow_w \neg_w y_1)]$.
For the interval-probability logic programming, the parameterized implications can be used for definitions of *satisfaction relation* for rules of such probabilistic logic programs: each of them preserves the principle of truth conservation from body into the head of rules, so that can be used in each fixed point algorithm for computation of models for these logic programs.
It is interesting that the strong intuitionistic negation $\neg_{nc}[x, y] = [x, y] \Rightarrow_{nc} [0, 0] = [1 - x, 1 - y]$, such that $\neg_{nc} = \neg \sim$, plays a central role as *default negation* operator in Paraconsistent logic programming [10], able to deal with both uncertain and contradiction information.

2.4 Duality in confidence-level based logic

Let us consider now the case when we are dealing with incomplete knowledge, where different evidence may contradict one other, so that is convenient to explicitly representing both belief and doubt (differently from interval based valuations, where the doubt is just equal to the epistemic negation (complement) of the belief), as it was argued in [13, 24].
In this case our lattice is the set of pairs of intervals, that is, a cartesian product $L_C = L_B \times L_B$, so that each of its members $([x, y], [z, v]) \in L_C$ is a *confidence level*, that is a couple of a belief $[x, y]$ and a doubt $[z, v]$, which we associate to the facts of our uncertain and incomplete knowledge base. Such confidence level is *consistent* [35] if $x + z \leq 1$.
In this lattice the truth ordering $\leq_t^C$ which increases the belief and decreases the doubt of facts is used to define the monotonic fixpoint operator for model theoretic characterization of confidence level logic programs (see [23, 35] for more details), that is
$([x, y], [z, v]) \leq_t^C ([x_1, y_1], [z_1, v_1])$ iff $[x, y] \leq_t [x_1, y_1]$ and $[z_1, v_1] \leq_t [z, v]$.
The meet operation (conjunction) and the join operation (disjunction) for this truth ordering $\leq_t^C$ and the epistemic negation $\neg^C$, which reverses this truth ordering, of this lattice L_C are defined by Ginsberg [23], with $\neg^C([x, y], [z, v]) = ([z, v], [x, y])$.
The bottom and the top truth members of this lattice L_C are $0_T^C = ([0, 0], [1, 1])$, and $1_T^C = ([1, 1], [0, 0]) = \neg^C 0_T^C$. The knowledge ordering $\leq_k^C$ in L_C is defined by
$([x, y], [z, v]) \leq_k^C ([x_1, y_1], [z_1, v_1])$ iff $[x, y] \leq_t [x_1, y_1]$ and $[z, v] \leq_t [z_1, v_1]$.
The meet operation (conjunction) and the join operation (disjunction) for this knowledge ordering $\leq_k^C$ and the knowledge negation $\sim^C$, which reverses this knowledge ordering, are defined by $\sim^C ([x, y], [z, v]) = (\neg[z, v], \neg[x, y]) = ([1 - v, 1 - z], [1 - y, 1 - x]$. The bottom and the top knowledge members of this lattice L_C are $0_K^C = ([0, 0], [0, 0])$, and $1_K^C = ([1, 1], [1, 1]) =\sim^C 0_K^C$.

Proposition 10 *The confidence-level parameterized bilattice is a p-D-bilattice composed by two p-lattices, a p-t-lattice* (L_C, α_t)*, with* $\alpha_t = \{\{\wedge_p^C, \rightharpoonup_p^C \mid p \in Par_B\}, \sim^C\}$*, and a p-k-lattice* (L_C, α_k)*, with* $\alpha_k = \{\{\otimes_p^C, \rightharpoondown_p^C \mid p \in Par_B\}, \neg^C\}$*, where* $Par_B = \{pc, ig, nc, in, ex\}$ *are parameters, such that for any pair* $([x, y], [z, v]), ([x_1, y_1], [z_1, v_1]) \in L_C$ *,* $p \in Par_B$ *:*
1. The parametric conjunctions (positive correlation is the meet operator) are defined by $([x, y], [z, v]) \wedge_p^C ([x_1, y_1], [z_1, v_1]) = ([x, y] \wedge_p [x_1, y_1],\ [z, v] \vee_p [z_1, v_1])$*,*
2. The parametric disjunctions are defined by de Morgan laws
$([x, y], [z, v]) \vee_p^C ([x_1, y_1], [z_1, v_1]) = \neg^C(\neg^C([x, y], [z, v]) \wedge_p^C \neg^C([x_1, y_1], [z_1, v_1]))$
$= ([x, y] \vee_p [x_1, y_1],\ [z, v] \wedge_p [z_1, v_1])$*,*
3. The parametric implications by relative pseudo complements
$([x, y], [z, v]) \rightharpoonup_p^C ([x_1, y_1], [z_1, v_1])$
$= max\{([a, b], [c, d]) \in L_C \mid ([x, y], [z, v])\ \wedge_p^C\ ([a, b], [c, d]) \leq_t^C ([x_1, y_1], [z_1, v_1])\}$
$= ([x, y] \rightharpoonup_p [x_1, y_1],\ \neg(\neg[z, v] \rightharpoonup_p \neg[z_1, v_1]))$*.*
4. Duality operation: $\partial^C([x, y], [z, v]) = ([x, y], \neg[z, v])$*.*
7. The knowledge conjunction/disjunction strategies and implications are defined by
$([x, y], [z, v]) \otimes_p^C ([x_1, y_1], [z_1, v_1]) = \partial^C(\partial^C([x, y], [z, v]) \wedge_p^C \partial^C([x_1, y_1], [z_1, v_1]))$*,*
$([x, y], [z, v]) \oplus_p^C ([x_1, y_1], [z_1, v_1]) = \partial^C(\partial^C([x, y], [z, v]) \vee_p^C \partial^C([x_1, y_1], [z_1, v_1]))$*,*
$([x, y], [z, v]) \rightharpoondown_p^C ([x_1, y_1], [z_1, v_1]) = \partial^C(\partial^C([x, y], [z, v]) \rightharpoonup_p^C \partial^C([x_1, y_1], [z_1, v_1]))$*,*
for each $p \in Par_B$*.*

Proof: It is easy to verify. In fact, given $\alpha = ([x, y], [z, v])$, $\beta = ([x_1, y_1], [z_1, v_1])$, then, for conjunctions (and disjunctions) the proof is equal to the Th.4.1 in [35], that is $\alpha \wedge_p^C \beta =$
1. Positive correlation (meet operation),
$([min\{x, x_1\}, min\{y, y_1\}],\ [max\{z, z_1\}, max\{v, v_1\}])$,
2. Ignorance, $([max\{0, x + x_1 - 1\}, min\{y, y_1\}], [max\{z, z_1\},\ min\{1, v + v_1\}])$,
3. Negative correlation strategy conjunction,
$([max\{0, x + x_1 - 1\}, max\{0, y + y_1 - 1\}],\ [min\{1, z + z_1\}, min\{1, v + v_1\}])$,
4. Independence, $([x \cdot x_1, y \cdot y_1],\ [z + z_1 - z \cdot z_1, v + v_1 - v \cdot v_1])$,
5. 0_T^C, for mutually-exclusive strategy conjunction;
While for implication we have that $\alpha \rightharpoonup_p^C \beta\ =$
1. Positive correlation, $([x \Rightarrow_w x_1, y \Rightarrow_w y_1], [\neg(\neg z \Rightarrow_w \neg z_1), \neg(\neg v \Rightarrow_w \neg v_1))$
2. Ignorance strategy, $([x \Rightarrow_s x_1, y \Rightarrow_w y_1], [\neg(\neg z \Rightarrow_w \neg z_1), \neg(\neg v \Rightarrow_s \neg v_1))$
3. Negative correlation, $([x \Rightarrow_s x_1, y \Rightarrow_s y_1], [\neg(\neg z \Rightarrow_s \neg z_1), \neg(\neg v \Rightarrow_s \neg v_1))$
4. Independence strategy, $([x \Rightarrow_m x_1, y \Rightarrow_m y_1], [\neg(\neg z \Rightarrow_m \neg z_1), \neg(\neg v \Rightarrow_m \neg v_1))$
5. 1_T^C, for mutually-exclusive strategy implication.
For the knowledge conjunctions/disjunctions and implications we obtain ($p \in Par_B$) $\alpha \otimes_p^C \beta = ([x, y] \wedge_p [x_1, y_1], [v, z] \wedge_p [v_1, z_1])$, $\alpha \oplus_p^C \beta = ([x, y] \vee_p [x_1, y_1], [v, z] \vee_p [v_1, z_1])$, and $\alpha \rightharpoondown_p^C \beta = ([x, y] \rightharpoonup_p [x_1, y_1], [v, z] \rightharpoonup_p [v_1, z_1])$. $\square$
As in the case of the interval-probability logic programming, also for confidence level probability logic programming, the parameterized truth implications can be used for definitions of *satisfaction relation* for rules of such programs: each of them preserves the principle of truth conservation from body into the head of rules, so that can be used in each fixed point algorithm for computation of models for these logic programs.

3 Duality in Higher-order bilattices

Higher-order bilattices are bilattices whose elements are functions instead of constants. They appear in logic programs with higher-order Herbrand interpretations [39].
In the last part of this section we wil present an example for the Probabilistic Temporal Programs (databases).
First we will define the canonical higher-order D-bilattice, obtained from the basic D-bilattices, which are distributive lattices.
From the Birkhoff's representation theorem [8] for distributive lattices, every finite (thus complete) distributive lattice is isomorphic to the lattice of lower sets of the poset of join-irreducible elements. In what follows we denote by $\downarrow x$ the downward closed set $\{y \mid y \leq x\}$, will denote by $A \Rightarrow B$, or B^A, the set of all functions from A to B, and by im the mapping isomorphism $im : \mathbf{2}^X \simeq \mathcal{P}(X)$, such that for any function $f \in \mathbf{2}^X$, $imf = \{x \in X | f(x) = 1\} \subseteq X$.

Proposition 11 *[8]* 0-LIFTED BIRKHOFF ISOMORPHISM: *Let X be a complete distributive lattice, then we define the following mapping $\downarrow^+ \colon X \to \mathcal{P}(X)$: for any $x \in X$,*
$\downarrow^+ x \ = \downarrow x \bigcap \widehat{X}$, *where* $\widehat{X} = \{y \mid y \in X$ *and* y *is join-irriducible* $\} \bigcup \{0\}$.
We define the set $X^+ = \{\downarrow^+ x \,| x \in X\ \} \subseteq \mathcal{P}(X)$, so that $\downarrow^+ \bigvee = id_{X^+} : X^+ \to X^+$ and $\bigvee \downarrow^+ = id_X : X \to X$. Thus, the operator $\downarrow^+$ is inverse of the supremum operation $\bigvee : X^+ \to X$. The set $(X^+, \subseteq)$ is a complete lattice, such that there is the following 0-lifted Birkhoff isomorphism $\downarrow^+ \colon (X, \leq, \wedge, \vee) \simeq (X^+, \subseteq, \bigcap, \bigcup)$.

Proof: Let us show the homomorphic property of $\downarrow^+$:
$\downarrow^+ (x \wedge y) = \downarrow (x \wedge y) \bigcap \widehat{X} = (\downarrow x \bigcap \downarrow y) \bigcap \widehat{X} = (\downarrow x \bigcap \widehat{X}) \bigcap (\downarrow y) \bigcap \widehat{X}) =$
$= \downarrow^+ x \bigcap \downarrow^+ y$, and
$\downarrow^+ (x \vee y) = \downarrow (x \vee y) \bigcap \widehat{X} = (\downarrow x \bigcup \downarrow y) \bigcap \widehat{X} = (\downarrow x \bigcap \widehat{X}) \bigcup (\downarrow y) \bigcap \widehat{X}) =$
$= \downarrow^+ x \bigcup \downarrow^+ y$. The isomorphic propertu holds from Bikhoff's theorem.
□
The name lifted is used to denote the difference from the original Birkhoff's isomorphism: that is, we have that for any $x \in X$, $0 \in \downarrow^+ x$, so that $\downarrow^+ x$ is never empty set (it is lifted by bottom element 0).
Notice that when X is distributive lattice then $(X^+, \subseteq, \bigcap, \bigcup)$ is a *subalgebra* of the powerset Boolean algebra $(\mathcal{P}(X), \subseteq, \bigcap, \bigcup)$, differently from the case when X is not distributive. Thus, we have
$\downarrow^+ \colon (X, \leq, \wedge, \vee) \simeq (X^+, \subseteq, \bigcap, \bigcup) \ \subseteq \ (\mathcal{P}(X), \subseteq, \bigcap, \bigcup)$.

Proposition 12 *(Higher-order D-bilattices):*
Let $\mathcal{B}$ be a D-bilattice with $\widehat{\mathcal{B}}_t$ the set of join-irreducible elements w.r.t. $\leq_t$ with 0_t, and $\widehat{\mathcal{B}}_k$ the set of join-irreducible elements w.r.t. $\leq_k$ with 0_k. Then the higher-order bilattice $\mathcal{B}_\delta = \{\delta_x \in \mathbf{2}^{\mathcal{B}} \mid x \in B$ and $im\delta_x = \downarrow_t x \bigcap \widehat{\mathcal{B}}_t = \downarrow_t^+ x\} \subseteq \mathbf{2}^{\mathcal{B}}$, such that
1. $\delta_x \leq_t \delta_y$ iff $x \leq_t y$, and $\delta_x \leq_k \delta_y$ iff $x \leq_k y$,
2. for any unary operator $\odot \in (\neg, \sim, \partial\}$, $\odot\delta_x =_{def} \delta_{\odot x}$
3. for any binary operator $\odot \in (\wedge, \vee, \otimes, \oplus\}$, $\delta_x \odot \delta_y =_{def} \delta_{x \odot y}$,
is a D-bilattice isomorphic to the original D-bilattice $\mathcal{B}$.

Analogously, the bilattice $\overline{\mathcal{B}_\delta} = \{\overline{\delta}_x \in \mathbf{2}^{\mathcal{B}} \mid x \in B$ *and* $im\overline{\delta}_x = \downarrow_k x \bigcap \widehat{\mathcal{B}}_k = \downarrow_k^+ x\}$, *which satisfies the points 1,2 and 3 above, is a D-bilattice isomorphic to the original D-bilattice* $\mathcal{B}$.

Proof: Easy to verify. Let us consider the negation in $\mathcal{B}_\delta$. If $\delta_x \leq_t \delta_y$ then $x \leq_t y$, thus $\neg x \geq_t \neg y$, thus $\delta_{\neg x} \geq_t \delta_{\neg y}$, i.e., $\neg\delta_x \geq_t \neg\delta_y$, and $\neg$ in this higher order bilattice $\mathcal{B}_\delta$ is antitonic. The bottom element 0_t in $\mathcal{B}_\delta$ is the function $\delta_{0_t} : \mathcal{B} \rightarrow \mathbf{2}$, such that $\delta_{0_t}(x) = 1$ iff $x \in \{0_t\}$, i.e., $im\delta_{0_t} = \{0_t\}$, while the top element 1_k is the function $\delta_{1_t} : \mathcal{B} \rightarrow \mathbf{2}$, such that $\delta_{1_t}(x) = 1$ iff $x \in \widehat{\mathcal{B}}_t$, i.e., $im\delta_{0_t} = \widehat{\mathcal{B}}_t$. And $\neg\delta_{0_t} = \delta_{\neg 0_t} = \delta_{1_t}$ and viceversa, with $\neg\neg\delta_x = \neg\delta_{\neg x} = \delta_{\neg\neg x} = \delta_x$, that is this negation is an involution and, consequently the bilattice negation.
$\square$
It is easy to verify that the meet and join operators in these higher-order D-bilatice have the intersection and Union set-based representations respectively, That is, $im(\delta_x \wedge \delta_y) = im\delta_x \bigcap im\delta_y$, $im(\delta_x \vee \delta_y) = im\delta_x \bigcup im\delta_y$ and $im(\overline{\delta}_x \wedge \overline{\delta}_y) = im\overline{\delta}_x \bigcap im\overline{\delta}_y$, $im(\overline{\delta}_x \vee \overline{\delta}_y) = im\overline{\delta}_x \bigcup im\overline{\delta}_y$, which will be used for the representation theorem of the D-bilattices. Consequently we define the following two sets:
$\mathcal{B}_\delta^+ = \{\downarrow_t^+ x = im\delta_x] \mid x \in \mathcal{B}\}$ and $\overline{\mathcal{B}}_\delta^+ = \{\downarrow_k^+ x = im\overline{\delta}_x] \mid x \in \mathcal{B}\}$.

3.1 Minimal higher-order D-bilattice

The minimal non banal D-bilattice of the higher-order type is the following functional version of the Belnap's bilattice: Higher-order Belnap's bilattice. We define the canonical functional version of the Belnap's bilattice based on the Proposition 12 $\overline{\mathcal{B}_\delta} = \{\overline{\delta}_x : \mathcal{B}_4 \rightarrow \mathbf{2},\ x \in \mathcal{B}_4\}$, where
$im\overline{\delta}_\perp = \{\perp\}$, $im\overline{\delta}_f = \{\perp, f\}$, $im\overline{\delta}_t = \{\perp, t\}$, $im\overline{\delta}_\top = \{\perp, f, t\}$.
It is a bilattice with the following knowledge and truth orderings $\leq_k$, $\leq_t$
$\overline{\delta}_x \leq_k \overline{\delta}_y$ iff $im\overline{\delta}_x \subseteq im\overline{\delta}_y$ iff $x \leq_k y$, and $\overline{\delta}_x \leq_t \overline{\delta}_y$ iff $x \leq_t y$.
Let us give an intuitive explanation [37] of what the Higher-order bilattice $\overline{\mathcal{B}_\delta}$ and its knowledge ordering *mean* in the fixpoint semantics for logic programs with incomplete and possibly inconsistent information. In order to obtain a new bilattice abstraction rationality, useful to manage logic programs with *inconsistencies*, we need to consider more deeply the *fundamental phenomena* in such one framework. In the process of derivation of new facts, for a given logic program, based on the 'immediate consequence operator', we have the following truth transformations for ground atoms in a Herbrand base of such a program (at the beginning, the initial many-valued interpretation assign the unknown value $\overline{\delta}_\perp$ to all ground atoms of the Herbrand base of a logic program):
1. When a ground atom passes from *unknown* to *true* logic value (it means that the value of an atom was unknown and in the next iteration it becomes true), without generating inconsistence; then we assign to this atom the value $\overline{\delta}_t$. Let us denote this action by $\uparrow_t$: $(\perp \leq_k t)$, which is just the knowledge ordering in the set $im\overline{\delta}_t = \downarrow_k^+ t$. This action is 'knowledge increasing'.
2. When a ground atom, tries to pass from unknown to *both* true and false value, generating an inconsistency, then is applied the *inconsistency repairing*, that is, the actual value of the literal of this atom, in a body of a violated clause with built-in

predicate, is replaced by *possible* value, that is $\overline{\delta}_\top$. Let us denote this action by the pair $\uparrow_\top$: $(\bot \leq_k f\ ,\ \bot \leq_k t)$, which is just the knowledge ordering in the set $im\overline{\delta}_\top = \downarrow_k^+ \top$. This action is 'knowledge increasing', because we pass from the value $\bot$ to the value $\top$. Notice that this transformation *does not change* the truth ordering.
3. When a ground atom passes from *unknown* to *false* logic value, without generating an inconsistence. Let us denote this action by $\uparrow_f$: $(\bot \leq_k f)$, which is just the knowledge ordering in the set $im\overline{\delta}_f = \downarrow_k^+ f$. It is knowledge increasing.
4. For ground atom which did not change its initial state we denote this action by the identity $\uparrow_\bot$: $(\bot \leq_k \bot)$, which is just the knowledge ordering in the set $im\overline{\delta}_\bot = \downarrow_k^+ \bot$.

It is evident how this dynamic phenomena, during computation of the least fixpoint model of a logic program, is more precisely represented by replacing the ordinary Belnap's bilattice $\mathcal{B}_4$ by its higher order type $\overline{\mathcal{B}_\delta}$, where each logic value corresponds to some of the actions above.
Notice that the set $\overline{\mathcal{B}}_\delta^+ = \{\downarrow_k^+ x = im\overline{\delta}_x] \mid x \in \mathcal{B}_4\}$ is the set of proper ideals (downward closed w.r.t. the knowledge ordering) of the Belnap's bilattice. It is not a simple coincidence, but the consequence of the general definition of the join-semilattice for Higher-order Herbrand interpretation types [39]. The elements of $\overline{\mathcal{B}}_\delta^+$ are *closed sets*, and more over, the knowledge-lattice $(\overline{\mathcal{B}}_\delta^+, \subseteq)$ with set intersection and set union (meet and join operators respectively), is the canonical (set-based) Belnap's knowledge-lattice.

3.2 Duality in higher-order Herbrand interpretations

The higher-order types of Herbrand interpretations for logic programs, where we are not able to associate a fixed logic value to a given ground atom of a Herbrand base, arise often in practice when we have to deal with uncertain information. In such cases we associate some *degree of belief* to ground atoms, which can be simple probability, probability interval, or other more complex data structures, as for example in Bayesian logic programs where for a different kind of atoms we associate also different (from probability) kind of *measures*.
The higher-order Herbrand interpretations of logic programs (for example Databases), produce models where the true values for ground atoms are not truth constants but functions. In this section we will give the general definitions for such higher-order Herbrand interpretation types for logic programs and their models.
Now we can formally introduce the definition for higher-order Herbrand interpretation types:

Definition 5 *(Higher-order Herbrand interpretation types [39]):*
Let H be a Herbrand base, then, the higher-order Herbrand interpretations are defined by $I_{abs} : H \to T$, where T denotes the functional space $W_1 \Rightarrow (...(W_n \Rightarrow \mathbf{2})...)$, where $\mathbf{2} = \{0, 1\}$ is the set of classic truth values, denoted also as $(...((\mathbf{2}^{W_n})^{W_{n-1}})...)^{W_1}$, and $W_i,\ i \in [1, n],\ n \geq 1$, the sets of parameters. In the case $n = 1$, $T = (W_1 \Rightarrow \mathbf{2})$, we will denote this interpretation by $I_{abs} : H \to \mathbf{2}^{W_1}$.

In the rest of the paper we will use only the simplest higher type $T = (\mathcal{W} \Rightarrow \mathbf{2})$ for higher Herbrand interpretations, where $\mathcal{W}$ is equal to the D-bilattice $\mathcal{B}$ of algebraic logic values, that is, when $T = \mathbf{2}^{\mathcal{B}}$ is the higher-order D-bilattice from the Proposition 12. Now we will introduce the transformation, called *flattening*, where the set of values of the D-bilattice $\mathcal{B}$, is fused into the Herbrand base by enlarging original predicates (in H of the old theory) with new logic attributes taken from the D-bilattice $\mathcal{B}$.

Definition 6 *(Flattening) A higher-order Herbrand interpretation $I_{abs} : H \rightarrow \mathcal{B}_\delta \subseteq \mathbf{2}^{\mathcal{B}}$, can be flattened into the ordinary Herbrand interpretation $I_F : H_F \rightarrow \mathbf{2}$, where $H_F = \{r_F(\boldsymbol{d}, \upsilon) \mid r(\boldsymbol{d}) \in H \text{ and } \upsilon \in \mathcal{B}\}$, is the Herbrand base of predicates r_F, obtained as extension of original predicates r by parameters, such that for any $r_F(\boldsymbol{d}, \upsilon) \in H_F,\ \upsilon \in \mathcal{B}$, holds that $I_F(r_F(\boldsymbol{d}, \upsilon)) = I_{abs}(r(\boldsymbol{d}))(\upsilon)$.*

We obtain the following "exponent" diagram for interpretations

$$\begin{tikzcd}
\mathbf{2}^{\mathcal{B}} \times \mathcal{B} \arrow[rr, "eval"] & & \mathbf{2} \\
H \times \mathcal{B} \arrow[u, shift right=2, "I_{abs}"] \arrow[u, shift left=2, "id_B"'] & & H_F \arrow[ll, "is^{-1}"] \arrow[u, "I_F"']
\end{tikzcd}$$

which commutes, with $I_F = [I_{abs}]^{-1} \circ is^{-1}$, where $[I_{abs}]^{-1} = eval \circ (I_{abs} \otimes id_B)$, where $is : H \times \mathcal{B} \simeq H_F$ is a bijection such that $(r(\mathbf{d}), \upsilon) = is^{-1}(r_F(\mathbf{d}, \upsilon))$, and $r_F(\mathbf{d}, \upsilon) = is(r(\mathbf{d}), \upsilon))$, $[\_]$ is the curring (λ abstraction) for functions, $[\_]^{-1}$ is its inverse, and $eval$ is the evaluation of a function from $\mathbf{2}^{\mathcal{B}}$ for values in $\mathcal{B}_4$. Thus,
$I_F(r_F(\mathbf{d}, \upsilon)) = eval \circ (I_{abs} \times id_B) \circ is^{-1}(r(\mathbf{d}), \upsilon) = eval \circ (I_{abs} \otimes id_B)((r(\mathbf{d}), (r(\mathbf{d}), \upsilon))$
$= eval \circ (I(r(\mathbf{d})), \upsilon) = I_{abs}(r(\mathbf{d}))(\upsilon)$.
If we consider a many-valued logic program P with a Herbrand base H and a many-valued interpretation $I_{mv} : H \rightarrow \mathcal{B}$, then such a many-valued interpretation can be replaced by the higher-order Herbrand interpretation $I : H \rightarrow \mathcal{B}_\delta$, by replacing the original bilattice values $\upsilon \in \mathcal{B}$ with correspondent functional elements $\delta_\upsilon \in \mathcal{B}_\delta$.
Thus, by flattening we obtain an ordinary Herbrand interpretation $I_F : H_F \rightarrow \mathbf{2}$. Such a transformation is an "ontological encapsulation" [41] of many-valued logic programs into 2-valued "meta" logic programs. For excample, the Fitting's 4-valued "immediate consequence operator" $\Phi_P : \mathcal{B}^H \rightarrow \mathcal{B}^H$, based on the Belnap's D-bilattice $\mathcal{B}$, monotonic w.r.t. the *knowledge* ordering in $\mathcal{B}_4$, can be transformed in an analog operator $\Psi_P : \mathbf{2}^{H_F} \rightarrow \mathbf{2}^{H_F}$, which is monotonic w.r.t. the *truth* ordering (see for more technical details in [41, 39]).
Such result is intimately connected with the duality of simple D-bilattices $\mathcal{B}$ and its functional higher-order version $\mathcal{B}_\delta$, and its extension to the following functional space:

Lemma 3 *If $\mathcal{B}$ is a D-bilattice then the functional space $\mathcal{B}_\delta^H$ is a D-bilattice.*

Proof: From the isomorphism $\delta : \mathcal{B} \simeq \mathcal{B}_\delta$ in the Proposition 12 and the Lemma 4.
$\square$

EXAMPLE: **Temporal-probabilistic logic**:

For a more exhaustive presentation of TP-logic/database we refer to the work in [2, 44], so we will restrict our attention only to necessary concepts for the work presented in this paper, but we will use the same terminology as those used in [2, 44]. We denote by S_τ the set of all valid time points of a calendar of a type $\mathcal{T}$.
Temporal Constraint: A temporal constraint C over calendar $\mathcal{T}$ is defined inductively:
1. Any atomic temporal constraint, ($\mathbf{t}$ op c_i), where c_i is the time-value and $op \in \{\leq, <, =, \neq, >, \geq\}$, is a temporal constraint.
2. if C_1 and C_2 are temporal constraints, then $C_1 \wedge C_2,\ C_1 \vee C_2$, and $\neg C_1$ are temporal constraints.
The extension (the set of time points) of a temporal constraint C is denoted by $sol(C_i)$.
TP-case statement and TP-tuple: A TP-case statement
$\gamma = \{\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i\rangle \mid 1 \leq i \leq n,\ n \geq 1\}$, is an expression where C_i and D_i are temporal constraints, L_i and U_i are lower and upper probability boundaries, and $\delta_i : sol(D_i) \rightarrow [0,1]$ is a probability distribution, so that for any $\mathbf{t} \in sol(D_i)$, $\delta_i(D_i, \mathbf{t})$ is a probability value at $\mathbf{t}$; and the following conditions are satisfied for all $1 \leq i \leq n$:
1. $0 \leq L_i \leq U_i \leq 1, \quad sol(C_i) \subseteq sol(D_i)$.
2. $\mid sol(C_i) \mid \geq 1$, that is, at least one time point satisfy C_i and D_i.
3. $\forall (j \leq n).(\ i \neq j$ implies $sol(C_i) \bigcap sol(C_j) = \{\})$.
Let $A = \{A_1, .., A_k\}$ be an ordinary relational schema, correspondent to the predicate $r(x_1, ..., x_k)$, and $\mathbf{d} = \{d_1, .., d_k\}$ its ordinary tuple from a database domain, then $tp = (\mathbf{d}, \gamma)$ is a TP-tuple over a schema A.
Intuitively, γ gives the probability p for each $\mathbf{t} \in S_\tau$, that the event with data $\mathbf{d}$ occurs at time $\mathbf{t}$; as default, we consider that for each $\mathbf{t} \in S_\tau - \bigcup_{\gamma_i \in \gamma} sol(C_i)$ the probability is zero; that is $\bigcup_{\gamma_i \in \gamma} sol(C_i)$ represents the temporal uncertainty for this event. Other important assumption is that each event in a TP-Database is uniquely determined by its data tuple $\mathbf{d}$. A TP-relation over a relational schema A is a multiset of TP-tuples over a schema A.
Their equivalent, explicit, definition by *ordinary* tuples and relations is given by so called *annotated relations*: an annotated tuple provides probabilistic information ($[L, U]$) for one data tuple $\mathbf{d}$ at one point in time $\mathbf{t}$.
Annotated relations: Let $tp = (\mathbf{d}, \gamma)$ be a TP-tuple over a relational schema A, where $\gamma = \{\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i\rangle \mid 1 \leq i \leq n,\ n \geq 1\}$.
Then the annotated relation for this TP-tuple tp is defined as the set
$\{(\mathbf{d}, \mathbf{t}, L_i \cdot x, U_i \cdot x) \mid \mathbf{t} \in \bigcup_{\gamma_i \in \gamma} sol(C_i),\ \ x = \delta_i(D_i, \mathbf{t})\ \}$.
Intuitively, in the above definition, x represents the percentage of $sol(D_i)$'s probability which is associated with time point $\mathbf{t}$ according to δ_i. If the original relation for a schema A is denoted by the predicate $r(x_1, ..., x_k)$, the correspondent predicate for annotated relation will be denoted by $r_F(x_1, ..., x_k, \mathbf{t}, L_t, U_t)$, where are added the temporal attribute and lower and upper probability attributes.
Annotated relations, as we will show in advance, can be seen as a particular case of the flattening (Definition 6) of the higher-order Herbrand interpretations for TP-tuples,

introduced in what follows.
These TP-tuples and TP-relations are implicitly defined by means of temporal constraints C_i in γ and are not ordinary relations, that is, they cannot be defined by ordinary 2-valued predicates. In fact, to each ground atom $r(\mathbf{d})$ we associate the TP-case statement $\gamma = \{\gamma_i = \langle C_i, D_i, L_i, U_i, \delta_i \rangle \mid 1 \leq i \leq n,\ n \geq 1\}$, and this mapping for each ground atom in a Herbrand base, can be equivalently represented by the higher-order Herbrand interpretation [44]:
$I_{abs} : H \rightarrow T$, where $T = (\mathcal{T} \Rightarrow (L_B \Rightarrow \mathbf{2}))$, that is, $T = (\mathbf{2}^{L_B})^{\mathcal{T}}$, where L_B is the interval-based lattice used for a belief based logic, so that for any $r(\mathbf{d})$ in a Herbrand base H, $I_{abs}(r(\mathbf{d}))$ is a function from $\mathcal{T}$ to the set of functions in $\mathbf{2}^{L_B}$, such that,
$I_{abs}(r(\mathbf{d}))(\mathbf{t})(L_i \cdot x, U_i \cdot x) = 1$, (i.e., true), iff $\mathbf{t} \in \bigcup_{\gamma_i \in \gamma} sol(C_i)$, $x = \delta_i(D_i, \mathbf{t})$.
This fact simply tells us that the probability of the event, represented by the atom $r(\mathbf{d})$ in a database, in the time-instance $\mathbf{t}$, is between values $L_i \cdot x$, and $U_i \cdot x$.
If we enlarge every predicate r in a database by a temporal attribute $\mathbf{t}$, so that instead of original atoms $r(\mathbf{d})$ we introduce new *temporal atoms* $r'(\mathbf{d}, \mathbf{t})$, we obtain a new Herbrand base $H_T = \{r'(\mathbf{d}, \mathbf{t}) | r(\mathbf{d}) \in H \text{ and } \mathbf{t} \in \mathcal{T}\}$, with the following higher-order interpretation:
$I'_{abs} : H_T \rightarrow T'$, where $T' = (L_B \Rightarrow \mathbf{2})$, that is, $T = \mathbf{2}^{L_B}$ is a higher-order belief-based D-bilattice.
More precisely, we have the higher-order Herbrand interpretation
$I'_{abs} : H_T \rightarrow \widetilde{L_B} \subseteq \mathbf{2}^{L_B}$, where $\widetilde{L_B}$ is the canonical higher-order D-bilattice of the belief D-bilattice L_B in Proposition 8.
Consequently, we have the higher-order Herbrand interpretations, any time when we are using higher-order D-bilattices.

4 Duality for the fixpoint semantic of Logic programs

If P is a many-valued logic program with a Herbrand base H_P, then the ordering relations and operations in a bilattice $\mathcal{B}$ are propagated to the function space $\mathcal{B}^{H_P}$, that is the set of all Herbrand interpretations (functions), $I = v_B : H_P \rightarrow \mathcal{B}$.

Definition 7 *Ordering relations are defined on the Function space $\mathcal{B}^{H_P}$ pointwise, as follows: for any two Herbrand interpretations $v_B, w_B \in \mathcal{B}^{H_P}$*
1. $v_B \leq_t w_B$ if $v_B(A) \leq_t w_B(A)$ for all $A \in H_P$.
2. $v_B \leq_k w_B$ if $v_B(A) \leq_k w_B(A)$ for all $A \in H_P$.
Bilattice meet $\wedge, \otimes$ and join $\vee, \oplus$ operations for these two orderings are defined in pointwise fashion (for ex., $(v_B \wedge w_B)(A) = v_B(A) \wedge w_B(A)$, etc.), and also negation is defined in pointwise fashion, that is, $\neg v_B : H_P \rightarrow \mathcal{B}$ is a mapping such that $(\neg v_B)(A) = \neg(v_B(A))$ for all $A \in H_P$.

Now we will show that if the basic bilattice $\mathcal{B}$ is a D-bilattice, then also the higher level bilattice (Functional space of Herbrand interpretations) is a D-bilattice.

Lemma 4 *The Function space bilattice is a D-bilattice composed by two lattices, a t-lattice $(\mathcal{B}^{H_P}, \alpha_t)$, with $\alpha_t = \{\wedge, \rightharpoonup, \sim\}$, and a k-lattice $(\mathcal{B}^{H_P}, \alpha_k)$, with $\alpha_k = \{\otimes, \rightharpoondown$*

$, \neg\}$*, such that:*
1. Implication: for truth $v_B \rightharpoonup w_B = max_t\{u_B \mid u_B \wedge v_B \leq_t w_B\}$ *,*
for knowledge $v_B \rightharpoondown w_B = max_k\{u_B \mid u_B \otimes v_B \leq_k w_B\}$ *, that is,*
$(v_B \rightharpoonup w_B)(A) = v_B(A) \rightharpoonup w_B(A)$*, and* $(v_B \rightharpoondown w_B)(A) = v_B(A) \rightharpoondown w_B(A)$*, for all* $A \in H_P$.
2. Duality operation: ∂v_B *is pointwise defined by* $(\partial v_B)(A) = \partial(v_B(A))$ *for all* $A \in H_P$.
3. Knowledge negation: $\sim v_B$ *is pointwise defined by* $(\sim v_B)(A) = \sim (v_B(A))$ *for all* $A \in H_P$.

Proof: For all $A \in H_P$ we have that $(v_B \rightharpoonup w_B)(A) = max_t\{u_B \mid (u_B \wedge v_B) \leq_t w_B\}(A) = max_t\{u_B(A) \mid (u_B \wedge v_B)(A) \leq_t w_B(A)\} = max_t\{u_B(A) \mid u_B(A) \wedge v_B(A) \leq_t w_B(A)\} = v_B(A) \rightharpoonup w_B(A)$.
Let us prove that duality homomorphism holds for implications, $v_B \rightharpoondown w_B = \partial(\partial v_B \rightharpoonup \partial w_B)$; for all other it is easy to verify in a pointwise fashion. So, we have that for all $A \in H_P$ (consider that ∂ is operator which *preserves* orderings):
$\partial(v_B \rightharpoondown w_B)(A) = \partial(max_k\{u_B \mid u_B \otimes v_B \leq_k w_B\})(A) = \partial(max_k\{u_B(A) \mid (u_B \otimes v_B)(A) \leq_k w_B(A)\}) = max_t\{\partial u_B(A) \mid (u_B \otimes v_B)(A) \leq_k w_B(A)\}$
$= max_t\{\partial u_B(A) \mid \partial(u_B \otimes v_B)(A) \leq_t \partial w_B(A)\} = max_t\{\partial u_B(A) \mid (\partial u_B \wedge \partial v_B)(A) \leq_t \partial w_B(A)\} = (max_t\{\partial u_B \mid (\partial u_B \wedge \partial v_B) \leq_t \partial w_B\})(A) = (\partial v_B \rightharpoonup \partial w_B)(A)$,
that is, $\partial(v_B \rightharpoondown w_B) = \partial v_B \rightharpoonup \partial w_B$, and, consequently ($\partial$ is an involution operation), $v_B \rightharpoondown w_B = \partial(\partial v_B \rightharpoonup \partial w_B)$. $\square$
Now we will show that full duality holds for a fixpoint semantics of logic programs [21] based on D-bilattice.

Proposition 13 *Let* $\Phi_q : \mathcal{B}^{H_P} \rightarrow \mathcal{B}^{H_P}$ *be an "immediate consequence operator" for a logic program P, monotonic w.r.t. the ordering* $\leq_q$, $q \in \{t, k\}$*, with a least fixpoint* $u = \Phi_q(u)$*. Then there exists the "immediate consequence operator"* $\Phi_{\overline{q}} = \partial\Phi_q\partial : \mathcal{B}^{H_P} \rightarrow \mathcal{B}^{H_P}$*, of the dual program* $\partial P = \{\partial\varphi \mid \varphi \in P\}$*, monotonic w.r.t. the dual ordering* $\leq_{\overline{q}}$ *, with a least fixpoint* $u' = \partial u \partial$*.*

Proof: By monotonicity of Φ we have that $v \leq_q w$ implies $\Phi_q(v) \leq_q \Phi_q(w)$. So, by duality $v' = \partial v \leq_{\overline{q}} \partial w = w'$ and $\partial\Phi_q(v) \leq_{\overline{q}} \partial\Phi_q(w)$, that is
$\Phi_{\overline{q}}(v') = \partial\Phi_q\partial(\partial v) \leq_{\overline{q}} \partial\Phi_q\partial(\partial w) = \Phi_{\overline{q}}(w')$.
$\square$

EXAMPLE: In what follows, for any Herbrand interpretation, of a Logic Program P based on Belnap's D-bilattice $\mathcal{B}_4$, $v \in \mathcal{B}_4^{H_P}$, we denote by $\overline{v}$ its standard unique extension to all ground formulae, such that for any $A \in H_P$, and the ground formulae ψ, ϕ, $\overline{v}(\sim A) = \sim v(A)$, $\overline{v}(\neg A) = \neg v(A)$, $\overline{v}(\partial A) = \partial v(A)$, and $\overline{v}(\psi \wedge \phi) = \overline{v}(\psi) \wedge \overline{v}(\phi)$, etc..
It is known that a general logic program P (with epistemic negation in the body of rules) has the least fixpoint semantics [18] of the immediate consequence operator Φ_k, monotonic with respect to the *knowledge* ordering, such that $v_{i+1} = \Phi_k(v_i) \geq_k v_i$, $i = 0, 1, 2, ..$ with v_0 the bottom knowledge member in $\mathcal{B}^{H_P}$ which assigns to all atoms in H_P the unknown value $\perp$ of Belnap's 4-valued bilattice, determined as follows: for a ground atom A, let $S = \{A \leftharpoondown B_{j1}, .., B_{jk_j} \in P^*,\ j = 1, 2, .., n\}$ be the set of all

clauses in a grounded program P^*, then
(1) $v_{i+1}(A) = \overline{v}_i((B_{11} \wedge .. \wedge B_{1k_1}) \vee ... \vee (B_{n1} \wedge .. \wedge B_{nk_n}))$.
We have that the dual grounded program is $\partial P^* = \{\partial(A \leftharpoondown B_{j1} \wedge .. \wedge B_{jk_j}) \mid A \leftharpoondown B_{j1}, .., B_{jk_j} \in P^*\} = \{\partial A \leftharpoondown \partial B_{j1} \otimes .. \otimes \partial B_{jk_j} \mid A \leftharpoondown B_{j1}, .., B_{jk_j} \in P^*\}$. So, the dual immediate consequence operator $\Phi_t = \partial \Phi_k \partial$ is monotonic with respect to the *truth* ordering, such that $v'_{i+1} = \Phi_t(v'_i)$, $i = 0, 1, 2, ..$ with v'_0 the bottom truth member in $\mathcal{B}^{H_P}$ which assigns to all atoms in H_P the false value f of Belnap's 4-valued bilattice.
Let us show that the next interpretation v'_{i+1} is determined as follows: for a ground atom A, let $S = \{\partial A \leftharpoondown \partial B_{j1} \otimes .. \otimes \partial B_{jk_j} \in \partial P^*,\ j = 1, 2, .., n\}$ be the set of all clauses in a grounded program ∂P^*, then
(2) $\overline{v}'_{i+1}(\partial A) = \overline{v}'_i((\partial B_{11} \otimes .. \otimes \partial B_{1k_1}) \oplus ... \oplus (\partial B_{n1} \otimes .. \otimes \partial B_{nk_n}))$
We can apply the operator ∂ to both sides of the equation (1). For the left side we obtain
$\partial v_{i+1}(A) = \partial \Phi_k v_i(A) = \partial \Phi_k(\partial \overline{v}'_i \partial)(A) = \Phi_t \overline{v}'_i(\partial A) = \overline{v}'_{i+1}(\partial A)$.
While for the right side we obtain
$\partial \overline{v}_i((B_{11} \wedge .. \wedge B_{1k_1}) \vee ... \vee (B_{n1} \wedge .. \wedge B_{nk_n})) =$
$= \partial((v_i(B_{11}) \wedge .. \wedge v_i(B_{1k_1})) \vee ... \vee (v_i(B_{n1}) \wedge .. \wedge v_i(B_{nk_n}))) =$
$= (\partial v_i(B_{11}) \otimes .. \otimes \partial v_i(B_{1k_1})) \oplus ... \oplus (\partial v_i(B_{n1}) \otimes .. \otimes \partial v_i(B_{nk_n})) =$
$= (\partial v_i \partial(\partial B_{11}) \otimes .. \otimes \partial v_i \partial(\partial B_{1k_1})) \oplus ... \oplus (\partial v_i \partial(\partial B_{n1}) \otimes .. \otimes \partial v_i \partial(\partial B_{nk_n})) =$
$= (\overline{v}'_i(\partial B_{11}) \otimes .. \otimes \overline{v}'_i(\partial B_{1k_1})) \oplus ... \oplus (\overline{v}'_i(\partial B_{n1}) \otimes .. \otimes \overline{v}'_i(\partial B_{nk_n})) =$ (from the fact that $\overline{v}'_i = \partial v_i \partial$)
$= \overline{v}'_i((\partial B_{11} \otimes .. \otimes \partial B_{1k_1}) \oplus ... \oplus (\partial B_{n1} \otimes .. \otimes \partial B_{nk_n}))$.
That is, we obtain (2).

5 Autoreferential Representation Theorem for D-bilattices

In mathematics, *Stone's representation theorem* for Boolean algebras [30], of Marshall H. Stone, is the duality between the category of Boolean algebras and the category of Stone spaces, i.e., totally disconnected compact Hausdorff topological spaces. In detail, the Stone space of a Boolean algebra BA is the set of all 2-valued homomorphisms on BA. Stone's original idea was to model an element a of Boolean algebra by the set of homomorphisms h into the two-element Boolean algebra $\mathbf{2} = \{0, 1\}$ of truth-values, such that $h(a) = 1$. Such homomorphisms are completely determined by the inverse image $h^{-1}(1)$, which can be easily seen to be an *ultrafilter* (maximal filter).
Every Boolean algebra is isomorphic to the algebra of clopen (i.e., simultaneously closed and open) subsets of its Stone space. The Stone's *isomorphism*
$is : (BA, +, \cdot, \backslash, 0, 1) \simeq (C_L(\mathcal{P}(S)), \bigcup, \bigcap, -, \emptyset, S)$
for any set S, where $\mathcal{P}(S)$ is the powerset of S, and $C_L(\mathcal{P}(S))$ its subset of clopen sets (here "$-$" denotes the set complement w.r.t. S), maps any element a of BA to the set of homomorphisms that map a to 1. The powerset algebra is the *canonical* extension of Boolean algebra: any Boolean algebra *can be represented* by this canonical algebras of *sets* whose structure is simpler to describe and much better understood.
The Stone representation theorem cannot be proven within the Zermelo-Fraenkel axioms. This theorem was proved by M.Stone in 1934. Stone's theorem has since been

the model for many other similar representation theorems, as, for example, Priestley,s duality for bounded distributive lattices, where $h^{-1}(1)$ is a *prime filter* of the lattice, and known duality theorems for semilattices.
Generally, all Stone-like representation theorems for algebraic logics are based on the assumption that the possible worlds, in this relational Kripke-like semantics, are *filters* of their Lindenbaum algebras: it is original approach used for the 2-valued algebraic modal logics. So that each possible world is modeled as a *set of sets* of values.
In what follows we will not follow this standard approach but will use *different semantics* for possible worlds, which is *autoreferential* w.r.t. the lattice of algebraic logic values where the set of possible worlds is equal to the set of elements of this lattice.
Autoreferential semantics for possible worlds is based on the consideration that each possible world represents a level of *credibility*, so that only the propositions with the right logic value (i.e., level of credibility) can be accepted by this world (more about this approach can be find in [42, 43]).
This new approach becomes more clear when we consider, for example, the way how to choose the Higher-order D-bilattice, in Proposition 12, $\mathcal{B}_\delta \subset \mathbf{2}^{\mathcal{B}}$ as a strict subset of the set of all mappings from the lattice $(\mathcal{B}, \leq_k)$ to the lattice $(\mathbf{2}, \leq)$. To show it, let us first define the inclusive homomorphism $i_2 : (\mathbf{2}, \leq) \hookrightarrow (\mathcal{B}, \leq_k)$ which preserves orderings, that is, $i_2(0) = 0_k$ and $i_2(1) = 1_k$ (maps bottom-to-bottom and top-to-top elements). Now, the question of how to obtain Higher-order bilattices *is inverse* w.r.t. the inclusive homomorphism. That is we are able to define the family of inverse homomorphisms from $(\mathcal{B}, \leq_k)$ to $(\mathbf{2}, \leq)$. The natural choice is defined by Proposition 12, such that, by Proposition 11, we obtain the isomorphisms
$\downarrow_t^+ : (\mathcal{B}, \leq_t, \wedge, \vee) \simeq (\mathcal{B}_\delta^+, \subseteq, \bigcap, \bigcup)$ and $\downarrow_k^+ : (\mathcal{B}, \leq_k, \otimes, \oplus) \simeq (\overline{\mathcal{B}}_\delta^+, \subseteq, \bigcap, \bigcup)$,
where $\mathcal{B}_\delta^+ = \{\downarrow_t^+ x = im\delta_x] \mid x \in \mathcal{B}\}$ and $\overline{\mathcal{B}}_\delta^+ = \{\downarrow_k^+ x = im\overline{\delta}_x] \mid x \in \mathcal{B}\}$.

EXAMPLE: For the Belnap's D-lattice we have that (see also the subsection 3.1),
$\mathcal{B}_\delta^+ = \{\{f\}, \{f, \bot\}, \{f, \top\}, \{f, \bot, \top\}\}$, while $\overline{\mathcal{B}}_\delta^+ = \{\{\bot\}, \{\bot, f\}, \{\bot, t\}, \{\bot, f, t\}\}$.
$\square$
In order to extend these canonical lattices $(\mathcal{B}_\delta^+, \subseteq)$ and $(\overline{\mathcal{B}}_\delta^+, \subseteq)$ to intuitionistic implication (and negation) operators, we have to provide the closure property [12] for elements of these lattices.

Proposition 14 *Let $\Rightarrow$ be the relative pseudo-complement operator for sets, that is, for any two sets X and Y, $X \Rightarrow Y = max\{Z \mid Z \bigcap X \subseteq Y\}$, so, the powerset algebras $(\mathcal{P}(\downarrow_t^+ 1_t), \subseteq, \bigcap, \Rightarrow)$ and $(\mathcal{P}(\downarrow_k^+ 1_k), \subseteq, \bigcap, \Rightarrow)$ are Heyting algebras.*
The operators $C_t : \mathcal{B}_\delta^+ \to \mathcal{B}_\delta^+$ and $C_k : \overline{\mathcal{B}}_\delta^+ \to \overline{\mathcal{B}}_\delta^+$, such that $C_t = \_ \bigcup\{0_t\}$, $C_k = \_ \bigcup\{0_k\}$, are closure operators. Thus, also $(\mathcal{B}_\delta^+, \subseteq, \bigcap, \Rightarrow)$ and $(\overline{\mathcal{B}}_\delta^+, \subseteq, \bigcap, \Rightarrow)$ are Heyting algebras with $\{0_t\}$ and $\{0_k\}$ bottom elements respectively. Thus, the negation is defined by $\neg_t X = X \Rightarrow \{0_t\}$ for any $X \in \mathcal{B}_\delta^+$, and by $\neg_k X = X \Rightarrow \{0_k\}$ for any $X \in \overline{\mathcal{B}}_\delta^+$, respectively.

Proof: It is enough to prove that C_t (and C_k) are closure operators for sets. In fact:
1. $C_t(\{0_t\}) = \{0_t\} \bigcup \{0_t\} = \{0_t\}$
2. $C_t(X \bigcup Y) = X \bigcup Y \bigcup \{0_t\} = (X \bigcup \{0_t\}) \bigcup (Y \bigcup \{0_t\}) = C_t(X) \bigcup C_t(Y)$

3. $X \subseteq X \bigcup \{0_t\} = C_t(X)$
4. $C_tC_t(X) = C_t(X) \bigcup \{0_t\} = X \bigcup \{0_t\} = C_t(X)$.
Notice that all elements in $\mathcal{B}_\delta^+$ are clopen, that is, closed and open (their complement w.r.t. $\downarrow_t^+ 1_t$ is closed element). Thus operations $\bigcap$ and $\Rightarrow$ are closed over the set $\mathcal{B}_\delta^+$, and consequently $(\mathcal{B}_\delta^+, \subseteq, \bigcap, \Rightarrow)$ is a (Heyting) subalgebra of the Heyting algebra $(\mathcal{P}(\downarrow_t^+ 1_t), \subseteq, \bigcap, \Rightarrow)$.
$\square$
But as Halmos have shown [29], the structures as $(\mathcal{B}_\delta^+, \subseteq)$ (and $(\overline{\mathcal{B}}_\delta^+, \subseteq)$) in which each closed element is also open have monadic algebraic structure and can support also the closure *modal* operator $\diamond$ conjugate to itself. And that is just our case, for modal D-bilattice operators $\sim$ and $\neg$ in the truth and knowledge lattices respectively.

Proposition 15 *Let $\diamond_t$ and $\diamond_k$ be two operators on sets such that for a given set $X \in \mathcal{P}(\downarrow_t^+ 1_t)$,*
$\diamond_t X = \{\sim x \mid x \in X\}$, and for $Y \in \mathcal{P}(\downarrow_k^+ 1_k)$, $\diamond_k Y = \{\neg y \mid y \in Y\}$.
Then $(\mathcal{P}(\downarrow_t^+ 1_t), \subseteq, \bigcap, \Rightarrow, \diamond_t)$ and $(\mathcal{P}(\downarrow_k^+ 1_k), \subseteq, \bigcap, \Rightarrow, \diamond_k)$ are modal extensions of Heyting algebras.
Also $(\mathcal{B}_\delta^+, \subseteq, \bigcap, \Rightarrow, \diamond_t)$ and $(\overline{\mathcal{B}}_\delta^+, \subseteq, \bigcap, \Rightarrow, \diamond_k)$ are monadic Heyting algebras.

Proof: We have that $\diamond_t(\{0_t\}) = \{\sim 0_t\} = \{0_t\}$, so $\diamond_t$ is normal modal operator, and, for any two sets $X, Y \in \mathcal{P}(\downarrow_t^+ 1_t)$, $\diamond_t(X \bigcup Y) = \{\sim x \mid x \in X \bigcup Y\} = \{\sim x \mid x \in X \text{ or } x \in Y\} = \{\sim x \mid x \in X\} \bigcup \{\sim x \mid x \in Y\} = \diamond_t(X) \bigcup \diamond_t(Y)$, that is, $\diamond_t$ is additive.
It is easy to show that for any $X \in \mathcal{B}_\delta^+$, $\diamond_t X = \neg_t \diamond_t \neg_t X = \square_t X \in \mathcal{B}_\delta^+$, thus $\diamond_t \equiv \square_t$, that is it is conjugate to yourself w.r.t. $\mathcal{B}_\delta^+$. The same holds for $\diamond_k$ w.r.t. $\overline{\mathcal{B}}_\delta^+$, thus $(\mathcal{B}_\delta^+, \subseteq, \bigcap, \Rightarrow, \diamond_t)$ and $(\overline{\mathcal{B}}_\delta^+, \subseteq, \bigcap, \Rightarrow, \diamond_k)$ are monadic Heyting subalgebras of $(\mathcal{P}(\downarrow_t^+ 1_t), \subseteq, \bigcap, \Rightarrow, \diamond_t)$ and $(\mathcal{P}(\downarrow_k^+ 1_k), \subseteq, \bigcap, \Rightarrow, \diamond_k)$ respectively.
$\square$

Theorem 1 *(Representation Theorem for D-bilattices)*
Let $\partial : (\mathcal{B}, \leq_t, \wedge, \rightharpoonup, \sim) \cong (\mathcal{B}, \leq_k, \otimes, \rightharpoondown, \neg)$ be the duality isomorphism for a D-bilattice $\mathcal{B}$. Then the following commutative diagram holds

$$\begin{array}{ccccc}
(\mathcal{B}, \leq_t, \wedge, \rightharpoonup, \sim) & \xrightarrow{\downarrow_{te}^+} & (\mathcal{B}_\delta^+, \subseteq, \bigcap, \Rightarrow, \diamond_t) & \xrightarrow{in_t} & (\mathcal{P}(\downarrow_t^+ 1_t), \subseteq, \bigcap, \Rightarrow, \diamond_t) \\
\Big\downarrow \partial & & \Big\downarrow \partial_{\mathcal{P}}^* & & \Big\downarrow \partial_{\mathcal{P}} \\
(\mathcal{B}, \leq_k, \otimes, \rightharpoondown, \neg) & \xrightarrow{\downarrow_{ke}^+} & (\overline{\mathcal{B}}_\delta^+, \subseteq, \bigcap, \Rightarrow, \diamond_k) & \xrightarrow{in_k} & (\mathcal{P}(\downarrow_k^+ 1_k), \subseteq, \bigcap, \Rightarrow, \diamond_k)
\end{array}$$

where $\downarrow_{te}^+$ and $\downarrow_{ke}^+$ are extensions of the lattice isomorphisms $\downarrow_t^+$ and $\downarrow_k^+$ respectively, to all other algebraic operators, and in_t, in_k are injective homomorphisms; the isomorphism $\partial_{\mathcal{P}}$ is the extension of the isomorphism ∂ to sets, that is, for any set $X \in \mathcal{P}(\downarrow_t^+ 1_t)$, $\partial_{\mathcal{P}} X = \{\partial x \mid x \in X\} \in \mathcal{P}(\downarrow_k^+ 1_k)$, while $\partial_{\mathcal{P}}^*$ is its reduction to $\mathcal{B}_\delta^+$ and $\overline{\mathcal{B}}_\delta^+$ respectively.

Proof: it is easy to verify, based on the precedent propositions and definitions.
□
In this diagram we have to consider the *vertical arrows* as the D-bilattices, from left to right: original D-bilattice, its set-based *canonical isomorphic Representation*, and its powerset *canonical extension*. Notice that all arrows (homomorphism between monadic Heyting algebras) of the commutative diagram on the left are *isomorphisms*. The right part of commutative diagram represents the fact that monadic Heyting algebras of canonical isomorphic representations are *subalgebras* of the powerset canonical extensions.

6 Conclusion

In this paper we introduced a family of bilattices which satisfy the truth/knowledge duality principle. They are characterized as residuated bilattice-based algebras with epistemic selfconjugate modal operator (which models S5 modal logic), that is, as extended monadic Heyting algebras. Thus, the intuitionistic negation together with this modal operator, model the epistemic negation in the way that D-bilattices can be used for logic programming able to avoid the explosive inconsistency, differently from a standard 2-value logics. They are particularly useful for logic/database systems with incomplete information (unknown value of a bilattice can characterize such logic formulae) and with mutually-inconsistent information also, which happen often when we integrate different source databases: by assigning to them the top-knowledge ("inconsistent" or "possible") value of a bilattice we avoid the explosive inconsistency of the whole logic system.
We defined also a family of parameterized Heyting algebras L, L_B and L_C, downward compatible, based on complete distributive lattices with *truth* ordering and the epistemic negation which reverse such ordering. Coherently, the whole hierarchy of these parameterized Heyting algebras is constructed from the basic parameterized algebra L. Such methodological approach may be used also for other kinds of probabilistic types of uncertainty.
The parameterizations of these Heyting algebras is given for a number of possible *conjunction probability strategies* which can be used for the rules in probabilistic logic programming, and which are a kind of *more sceptic* algebraic approximations for the meet operation of these lattices (the basic conjunctive strategy).
By connection with a paraconsistent logic programming [10] we are able to extend the use of these parameterized intuitionistic logics to deal with both *uncertain* and *inconsistent* information. Moreover, we are able to extend such logic programming also with *nested implications*: for example by allowing the formulae with implication also in bodies of programming rules.
We hope that this coherent and incremental approach to different probabilistic approaches, used for modeling uncertainty for belief, will give a clear light on the nature of deep interdependency between different measure types for uncertainty, and different conjunctive/disjunctive probabilistic strategies.
From the theoretical point of view, we investigate also the higher-order dual bilattices, based on the ordinary bilattices where the logic values are constants (or 0-arity

functions). The elements of such higher-order bilattices are elements of a subset of a functional space Y^X, where a base bilattice is the *target* set Y (the standard construction used for a fixpoint semantics of normal logic programs, for example, where the elements of such functional space are many-valued Herbrand interpretations Y^H), or in opposite case when a base bilattice is the *source* set X (the case when we are dealing with a higher-order types of Herbrand interpretations, typical in applications with uncertain information).
Finally, we define the Autoreferential Representation Theorem for D-bilattices. We demonstrated that the *canonical* D-bilattice algebras, with intuitionistic implication and epistemic negation, are subalgebras, composed by only closed sets, of a powerset algebras which are canonical extensions of the bilattice. The set of possible worlds (base set) of these powerset algebras is just the set of logical values of the original bilattice, which gives us the *concrete* definition of the modal Kripke-style semantics for logic programs, differently from the standard Ginsberg's world-based bilattices, where the way of defining a set of possible worlds remains open to various interpretations, and, thus, have more theoretical than practical importance.

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