April Solutions

Problem 1 (Hermit Problem)

A hermit has 200,000 hairs on his head and gets it cut to a length of 2 inches. After that he never cuts it again. The hair grows at a rate of 0.02 inches per day and he is losing his hair at the rate of 50 hairs a day. Determine the day when the total length of his hair will be a maximum ?

Solution

On n days after the hermit cuts his hair, he will have 200,000 $-50n$ hairs on his head, each of which will be 2 + $2n/100$ inches long. Hence, the total length of hair on his head after $n$days will be

$$\begin{aligned} L &= (200,000 - 50n)\,(2 + \tfrac{2n}{100}) \\ &= 400,000 - 100n + (200,000)\,\tfrac{2n}{100} - n^2 \\ &= -n^2 + 3900\,n + 400,000 \end{aligned}$$

inches. This expression is a quadratic with a negative coefficient of n^2 and so it describes a downward turning parabola which has a maximum point after $n = \frac{3900}{2}$ = 1950 days. Hence, 1950 days (5.3 years) after the hermit cuts his hair the total length of his hair will be a maximum. The hermit will be bald $(L = 0)$ in 4000 days or about 11 years.

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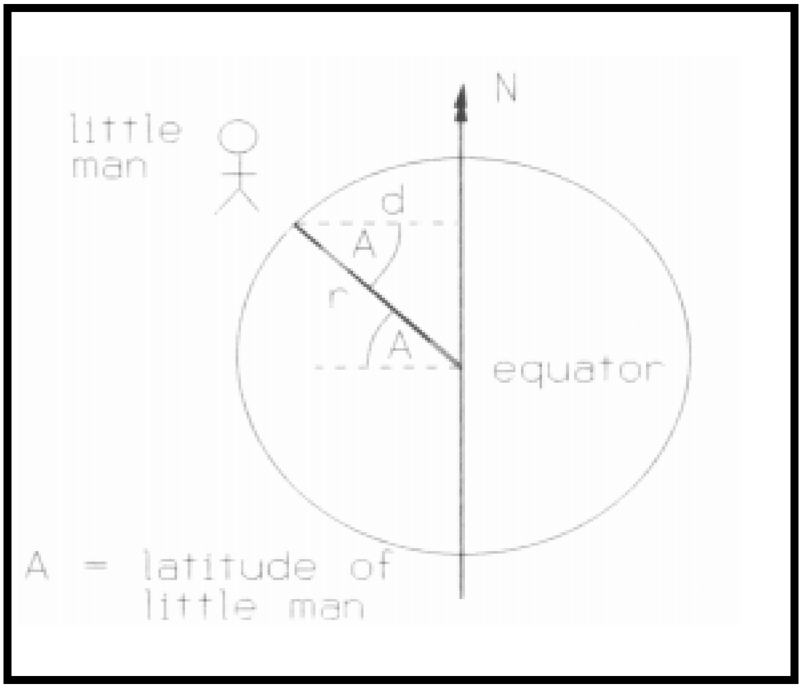
Problem 2 (The Great Balloon Race)

Harry has joined the great *Around-the-World* balloon race but has never been in a balloon. He is clever however and reads the fine print in the contest rules that say nothing about the latitude of which he must travel. So he decides to take his balloon around the world (east to west) at a *constant latitude* of 89 degrees north. Can you tell Harry how far he will travel ? How far will a balloonist travel going around the world at a constant latitude θ ? The distance around the world at the equator is 25,000 miles.

Solution Consider the drawing below that shows a person standing on the surface of the earth at a latitude of A degrees. (In Maine, we are at a northern latitude of roughly 45° depending on exactly where in Maine you are located.) Now, if the little man in the figure starts to walk straight east (or west) staying at the same latitude, then the distance he would travel before getting back to where he started would be $2\pi d$ (circumference of a circle with radius d), where d is the distance from where the man is located to the axis of rotation of the earth (the line between the north and south poles). The question of course is what is this distance d ? Well, if you know trigonometry, you know in this case by looking at the figure that the cosine of the latitude A is cos A $= d/r$, where r is the radius of the earth. Hence, the distance a person travels going around the earth at a constant latitude of A degrees is

$$\textbf{Distance traveled} = 2\pi d = 2\pi r \cos A = C \cos A$$

where $C = 2\pi r$ is the circumference of the earth (i.e. the distance around the earth at the equator). In other words, the distance around the earth at a constant latitude A degrees is the distance around the earth at the equator times the cosine of the latitude. And since the circumference of the earth is roughly $C = 25,000$ miles, we have that the distance around the earth at 89° is 25,000 $cos\,(89°) \approx 25,000\,(0.017) = 425$ miles. You can get out your calculator and compute some distances at other latitudes; say like in Maine where the latitude is 45° .



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Problem 3 (Trigonometry Equation)

Solve

$$(sin\,x\,+\,sin\,2x\,+\,sin\,3x)^2\,+\,(cos\,x\,+\,cos\,2x\,+\,cos\,3x)^2\,=\,1$$

Solution:

We use the trigonometric identities

$$sin\,u\,+\,sin\,v\,=\,2\,sin\left(\frac{u+v}{2}\right)cos\left(\frac{u-v}{2}\right)$$
$$cos\,u\,+\,cos\,v\,=\,2\,cos\left(\frac{u+v}{2}\right)cos\left(\frac{u-v}{2}\right)$$

and write

$$sin\,x\,+\,sin\,3x\,=\,2\,sin\,(\tfrac{4x}{2})\,cos\,(\tfrac{2x}{2})=\,2\,sin\;2x\;cos\,x$$
$$cos\,x\,+\,cos\,3x\,=\,2\,cos\,(\tfrac{4x}{2})\,cos\,(\tfrac{2x}{2})\,=\,2\,cos\;2x\;cos\,x$$

We can now rewrite the equation in the problem as

$$(sin\,2x\,+\,2\,sin\,2x\;cos\,x)^2\,+\,(cos\,2x\,+\,2\,cos\,2x\,cos\,x)^2=1$$

which upon squaring becomes

$$sin^2 2x\,+\,4\,sin^2\,2x\,cos\,x\,+\,4\,sin^2\,2x\,cos^2\,x$$
$$+\,cos^2\,2x\,+\,4\,cos^2\,2x\,cox\,x\,+\,4\,cos^2 2x\,cos^2 x\,=\,1$$

or simply

$$4\,cos^2\,x\,(sin^2\,2x\,+\,cos^2\,2x)\,\,+\,4\,cos\,x\,(sin^2\,2x\,+\,cos^2\,2x)$$
$$+\,(sin^2\,2x\,+\,cos^2\,2x)\,=\,1$$

But using the well-known identify $sin^2\,\alpha\,+\,cos^2\,\alpha\,=\,1$ for any α, this equation simplifies to

$$cos^2\,x\,+\,\,cos\,x\,=\,0$$

or

$$cos\,x\,(cos\,x\,+\,1)\,=\,0$$

Hence, we have the roots

$$cos\,x\,=\,0\,\Rightarrow\,\,x=\,\tfrac{\pi}{2}\,\pm\,n\pi\qquad n\,=\,0,\,1,\,2,\,...$$
$$cos\,x\,=\,\,-1\,\Rightarrow\,x\,=\,\pi\pm\,2n\pi\;\;n\,=\,0,\,1,\,2,\,...$$

For example, we check the root $x=\pi/2$ which gives

$$(sin\tfrac{\pi}{2}\,+\,sin\,\pi\,+\,sin\tfrac{3\pi}{2})^2\,+\,(cos\,\tfrac{\pi}{2}+\,cos\,\pi\,+\,cos\,\tfrac{3\pi}{2})^2\,=\,\,0^2+(\text{-}1)^2=1$$

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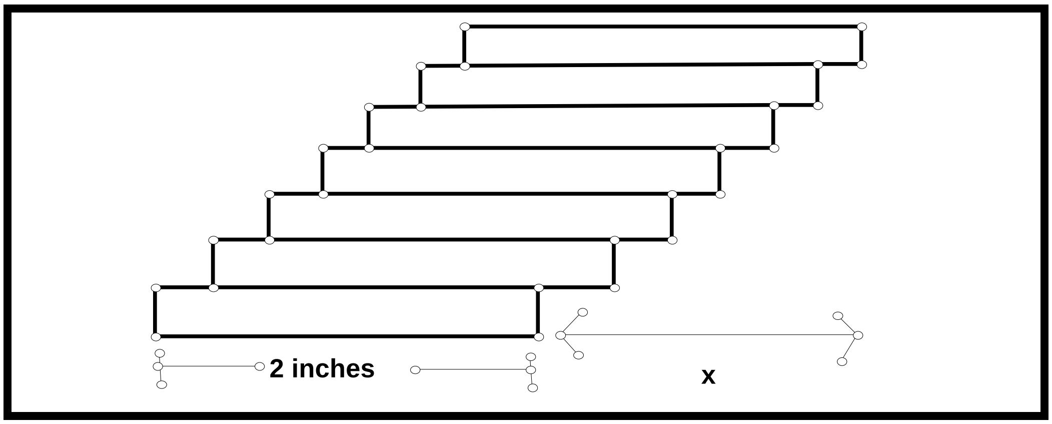
Problem 4 (Piling Dominoes)

Harry is stacking up n dominoes, each of whose size is 1 inch by 2 inches by 1/4 inches. Find the largest distance that Harry can make the top domino overhang the bottom one before the stack tips over. First find the maximum overhang for $n = 1$, 2, 3, ... dominoes.



Solution

For convenience, we pile the dominoes on a table and introduce coordinates so that $x = 0$ corresponds to being right at the edge of the table and positive x being over the edge. (See the figure below.) The goal then is to pile the dominoes at the edge of the table on top of each other so that the center of gravity of all the dominoes is located exactly at zero. (If the center of gravity is located at a positive x, then the center of gravity is over the edge of the table and the dominos will topple.)

We begin with one domino and see how far we can extend it over the edge of the table without falling off. Clearly, we can extend it halfway with and overhang of 1/2 inch. Now suppose we have 2 dominoes, and we stack the second domino on top of the first one so it extends past the bottom domino. Clearly, the two dominos will topple since no matter how little the second domino is extended past the bottom one, the center of gravity of the two dominos is positive (over the edge of the table). It appears things are hopeless. But wait! There is another strategy. Let us place the second domino *under* the first domino and not over it. In fact, if we place the second domino on the table with an overhang of 1/4 inch (i.e. it's rightmost edge located at $x = 1/4.)$ and keep the overhang of the topmost domino on the bottom one at 1/2 inch, then if you draw a nice picture you will see that the center of gravity of the two dominoes is right at the edge of the table at $x = 0$, and hence they won't topple. What's more is that the total overhang of the two dominos is 1/4 + 1/2 = 3/4 inches. Hence, the strategy is to place each new domino *under* the previous dominos in such a way that the center of gravity of all the dominos is located at $x = 0$. And what's more it is easy to find the center of gravity of n dominoes since the center of gravity of each domino is the average of its left and right hand coordinates, and the center of gravity of all the dominos is simply the average of the center of gravity of all the dominos, which is

$$\text{center of gravity of } n \text{ dominoes} \;=\; \frac{\left(\frac{L_1+R_1}{2}\right)+\left(\frac{L_2+R_2}{2}\right)+\cdots+\left(\frac{L_n+R_n}{2}\right)}{n}$$

where L_k and R_k are the left and right hand coordinates, respectively, of the n dominoes. Hence, in order that the center of gravity is located right at the edge of the table, we must have

$$\frac{\left(\frac{L_1+R_1}{2}\right)+\left(\frac{L_2+R_2}{2}\right)+\cdots+\left(\frac{L_n+R_n}{2}\right)}{n} \;=\; 0$$

or simplifying

$$(L_1 + L_2 + \cdots + L_n) + (R_1 + R_2 + \cdots + R_n) \;=\; 0$$

but the dominoes are all 1 inch long and os we have $L_k = R_k - 1,$ and so the above expression simplifies to

$$\sum_{k=1}^{n} (R_k - 1) + \sum_{k=1}^{n} R_k \;=\; 0$$

and finally

$$\sum_{k=1}^{n} R_k \;=\; \tfrac{n}{2}$$

In other words, for n dominoes, in order that the center of gravity lie right at the edge of the table we must have the sum of the coordinates of the right edges of dominos be equal to $n/2$. We can now use this relationship to solve our problem. Consider the above equation one by one for $n = 1,\ 2,\ 3,\ \ldots$ dominoes. When we have $n = 1$ domino, the equation is $R_1 = 1/2$, which says we should extend a single domino 1/2 inch over the edge of the table in order for its center of gravity be right at the table's edge (something we already knew). For $n = 2$ dominos, we have the equation $R_1 + R_2 = 2/2 = 1$. But we really don't care about the *coordinates* of the right edges, but how much each domino overhangs the domino below it. For that reason we introduce the notation h_i = overhang of domino ith on the domino below it (i.e. domino $i+1$). Hence, we can write the above equation $R_1 + R_2 = 1$ as

$$R_1 + R_2 = (R_1 - R_2) + 2R_2 = 1$$

and since $h_1 = R_1 - R_2$, as

$$h_1 + 2R_2 = 1$$

But we already know $h_1 = 1/2$ and so plugging this value into the above equation and solving for R_2, we get $R_2 = 1/4$, which in this case of $n = 2$ dominoes is the overhang of the bottom domino over the tabletop. Hence, $h_2 = 1/4$. Continuing with $n = 3$ dominoes, we have

$$R_1 + R_2 + R_3 = \tfrac{3}{2}$$

which we rewrite as

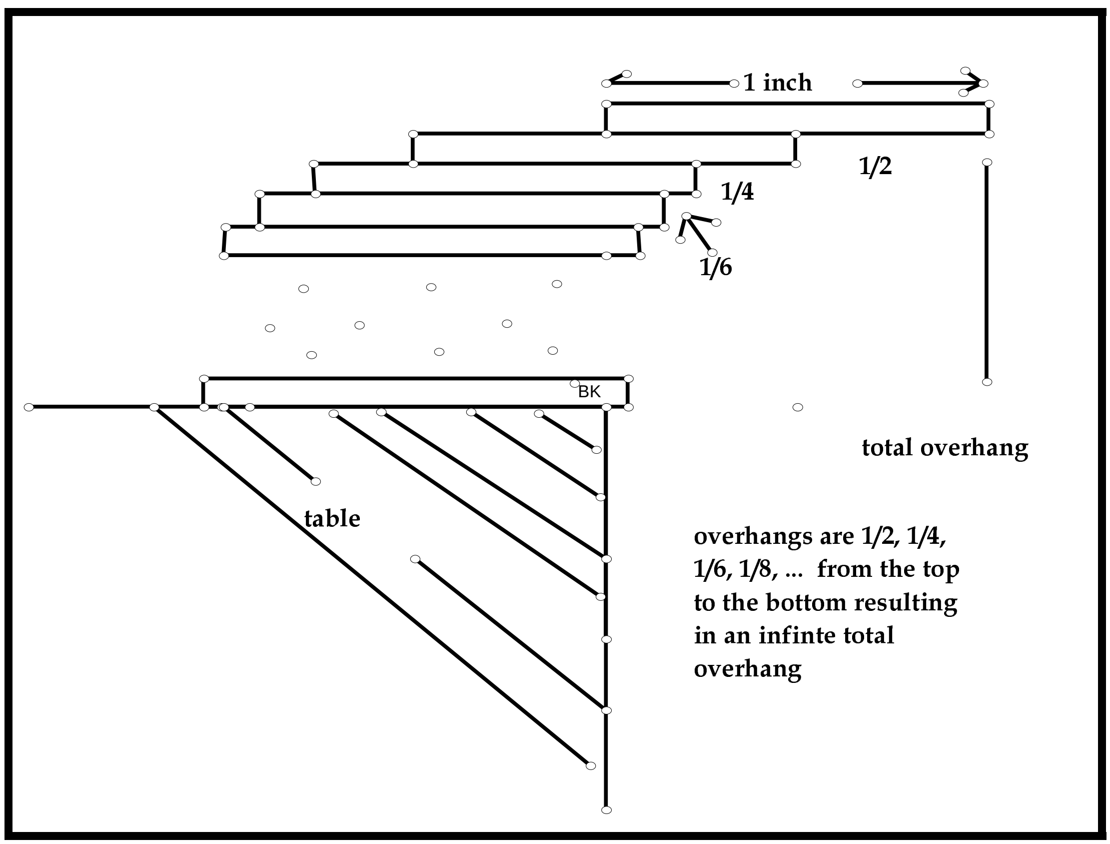
$$(R_1 - R_2) + 2\,(R_2 - R_3) + 3\,R_3 = \tfrac{3}{2}$$

or

$$h_1 + 2h_2 + 3R_3 = \tfrac{3}{2}$$

But we know $h_1 = 1/2$, $h_2 = 1/4$, we can solve for R_3 (which will be h_3), getting $R_3 = 1/6$, and with $n = 3$ dominoes is the overhang of the bottom domino over the table. Hence, with 3 dominos the overhangs (starting with the top domino over the second to the top domino) are $h_1 = 1/2$, $h_2 = 1/4$, $h_3 = 1/6$. Continuing this process with more and more dominoes, we discover the overhang of n dominoes will be 1/2, 1/4, 1/6, 1/8, ... $1/2n$. And if we are given an unlimited number of dominoes, the distance the leading edge of the top domino overhangs the edge of the table is

$$\text{total overhang } = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \tfrac{1}{8} + \cdots$$



But this series is the harmonic series which diverges, and so it is theoretically possible to stack the dominoes so the overhang of the top domino is infinite, all the while keeping the center of gravity of the dominoes at the edge of the table!! Believe it or not! If this seems to be a contradiction, keep in mind that there are a *lot* of dominoes (like an infinite number) near the bottom of the stack that barely stick out over the edge of the table!

It is interesting to observe that if we are given only 4 dominoes, the overhang of the top domino will be

$$\text{overhang of 4 dominoes} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} \approx 1.04 > 1$$

which means the top domino will completely overhang the bottom one (the left coordinate of the top domino will be 0.04). You can see for yourself if you can stack real dominos that do this. The above formula says that the domino resting on the table should have an overhang of 1/8th the length of the domino, the next domino should have an overhang of 1/6th the length of the domino, the third an overhang of 1/4th the length, and the top one an overhang of 1/2 the its length.

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Problem 5 (Interesting Arithmetic Sequence)

Consider an arithmetic sequence with first term a and common difference d. Find the mth and nth terms of the sequence where the sum of the first m terms is n and the sum of the first n terms is m.

$a)$ Find the arithmetic sequence.

$b)$ After that, find the sum of the first $m + n$ terms.

Solution

$a)$ We are given the sum S_m of the first m terms is n, and that the sum S_n of the first n terms is m, and hence

$$S_m = ma + \frac{m(m-1)}{2} d = n$$
$$S_n = na + \frac{n(n-1)}{2} d = m$$

If $m \neq n,$ we solve for d and a, in these two equations getting the arithmetic sequence whose first term is $a = \frac{m^2+n^2+mn-m-n}{mn}$ and common difference is d = -2 $\frac{m+n}{mn}$. For example, if we let $m = 1,$ $n = 2,$ we find $a = 2,$ $d =$ -3, or

$$a = 2$$
$$a + d = 2 - 3 = -1$$
$$a + 2d = 2 + 2(-3) = -4$$
$$a + 3d = 2 + 3(-3) = -7$$

$b)$

$$S_{m+n} = (m+n)\,a + (m+n)(m+n-1)\,\frac{d}{2}$$
$$= [ma + m(m-1)\,\tfrac{d}{2}] + [na + n(n-1)\tfrac{d}{2}] + mna$$
$$= S_m + S_n + mna$$
$$= n + m - 2(m+n)$$
$$= -(m+n)$$

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Problem 6 (Only Very Smart People Apply)

Ok, here we go. We start with a cube whose edges are 1 foot long which is inscribed inside a sphere. ($i.e.$ the 8 corners of the cube touch the inside of the sphere. Now a strange thing happens. Each face of the cube begins to grow outward, in such a way that pyramids are created, whose bases are the faces of the cube and the tops of the pyramids lie directly over the middle of their respective faces. These pyramids grow outward from the cube until the tops of the pyramids touch the sphere. What is the volume of this new 6-spinned object inscribed inside the sphere ?

Solution

We first find the radius r of the sphere, which is the distance from the center of the cube (which is the same as the center of the sphere) to one of the 8 vertices of the cube (which touches the sphere). To find this distance, we note the distace from the center of the cube one of its 6 faces is 1/2, and so from the three-dimensional Pythagorean theorem, we have

$$r^2 = (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2$$

which gives the radius of the sphere $r = \sqrt{3}/2.$ We now compute the height h of each of the 6 pyramids as

$$h = r - \tfrac{1}{2} = \tfrac{\sqrt{3}}{2} - \tfrac{1}{2} = \tfrac{\sqrt{3}-1}{2}$$

and so each pyramid has a volume of

$$V_{pyramid} = \tfrac{1}{3}\text{(area of the base) (height)} = \tfrac{1}{3} \cdot 1 \cdot \tfrac{\sqrt{3}-1}{2} = \tfrac{\sqrt{3}-1}{6}$$

Hence, the volume of the 6-spined object that grows out from the cube is the volume of the cube (which is of course 1), plus the volume of the 6 pyramids. In other words

$$V_{spined\,object} = 1 + 6\left(\tfrac{\sqrt{3}-1}{6}\right) = \sqrt{3}$$

cubic feet. Note that since volume of the sphere is $V_{sphere} = \frac{4}{3}\pi\,(\frac{\sqrt{3}}{2})^3$, and so the ratio of the volume of the 6-spined object to the volume of the sphere is

$$\text{Ratio} = \tfrac{V_{spine}}{V_{sphere}} = \tfrac{2}{\pi} \approx 0.64$$

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Problem 7 (Interesting Equation)

Betcha can't solve this one.

$$\sqrt[3]{62 + \sqrt{x}} - \sqrt[3]{6 + \sqrt{x}} = 2$$

Solution

We first write

$$\sqrt[3]{62 + \sqrt{x}} = 2 + \sqrt[3]{6 + \sqrt{x}}$$

and then cube both sides, getting

$$62 + \sqrt{x} = 8 + 6 + \sqrt{x} + 6\sqrt[3]{(6 + \sqrt{x})^2} + 12\sqrt[3]{6 + \sqrt{x}}$$

and simplifying, we get

$$6\,(6+\sqrt{x}\,)^{2/3} + 12\,(6 + \sqrt{x}\,)^{1/3} - 48 = 0$$

Now, letting $y = \sqrt[3]{6 + \sqrt{x}}$, the equation reduces to $y^2 + 2y - 8 = 0$, which has solutions $y_{1,2} = 2, \ -4.$ Taking these values one at a time we have

$$\sqrt[3]{6 + \sqrt{x}} = 2 \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4$$

$$\sqrt[3]{6 + \sqrt{x}} = -4 \Rightarrow \sqrt{x} = -70 \text{ which has no solutions.}$$

Hence, we have one solution $x = 4.$

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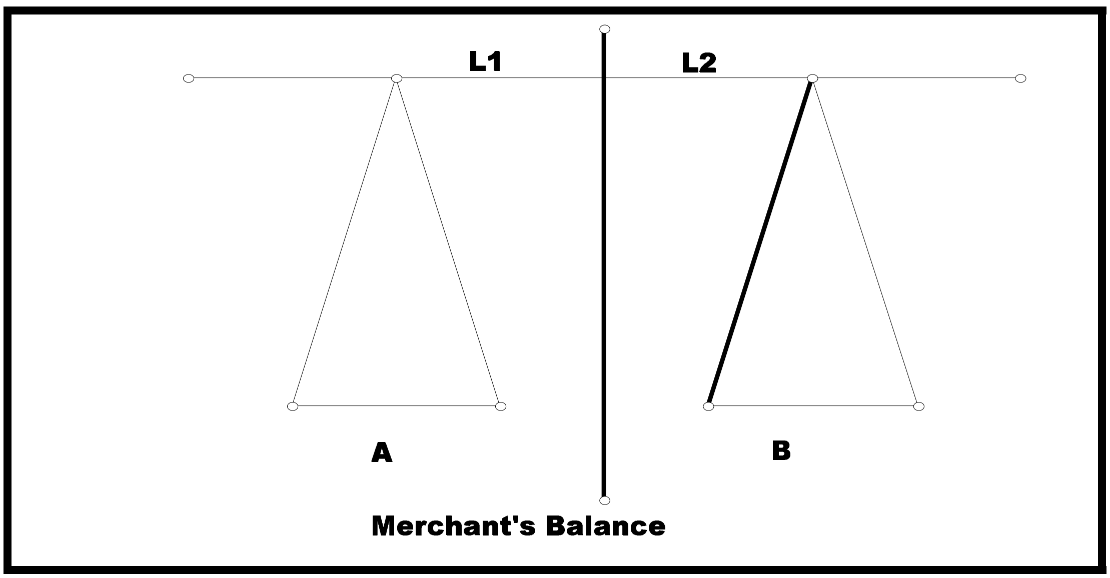
Problem 8 (The Balance Problem)

A merchant has a simple balance for weighing meat which consists of two pans, pan A and pan B, located at what should be equal lengths of a balancing arm. If a customer bought 5 pounds of meat, the merchant would place a 5 lb weight in one pan and meat in the other pan until the sides balance. Unfortunately, the merchant's balance is faulty and the two arms are *not* of the same length. Suppose two customers come into the store, each requesting 5 lbs of meat. The merchant knows his balance is faulty but does not know which side of the balance arm is longer and which is shorter. But he is an honest person so he wants the total amount of produce given to these two customers to be exactly 10 lbs. To do this he places the meat for the first customer in pan A and a 5 lb weight in the other pan, and then for the second customer he does just the opposite; he puts the meat in pan B and the weight in the other pan. Was the net result of this plan fair to the merchant or did the merchant come out ahead or behind ?



Solution

We denote the lengths of the two arms by L_1 and L_2. Now, if the merchant puts the w_1 pounds of meat (he doesn't know the exact weight of the meat he is selling since the scale is faulty) for customer 1 in pan A and a 5 lb weight in pan B, until they balance, we have

Customer 1 : $L_1 w_1 = 5L_2$

And if for customer 2 the merchant places the w_2 pounds of meat in pan B and a 5 lb weight in pan A, we have

Customer 2 : $5L_1 = L_2 w_2$

Hence, the total pounds of meat the merchant is giving the two customers is

$$
\begin{aligned}
w_1 + w_2 &= 5\left(\frac{L_2}{L_1} + \frac{L_1}{L_2}\right) \\
&= 5\left(\frac{L_1^2+L_2^2}{L_1 L_2}\right) \\
&= 5\left(\frac{(L_1-L_2)^2+2L_1L_2}{L_1L_2}\right) \\
&= 5\left(\frac{(L_1-L_2)^2}{L_1L_2} + 2\right) > 10
\end{aligned}
$$

Therefore, the total pounds of meat the merchant is giving the customers is more than 10 lbs, and hence is losing out in this process.

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Problem 9 (So You Know Quadratics Huh?)

If $x_1$and x_2 are roots of the quadratic equation

$$x^2 \,+\, (a+d)\,x \,+(ad-bc) = 0 \tag{1}$$

show that the roots of

$$y^2 \,+ (a^3+d^3+\, 3abc+3bcd)\,y \,+\, (ad-bc)^3 = 0 \tag{2}$$

are equal to x_1^3 and x_2^3.

Solution

Since x_1 and x_2 are roots of the quadratic equation (1), then it follows

$$x_1+x_2 \,=\, -(a+d) \qquad\qquad x_1x_2 \,=\, ad-bc$$

We now show that $x_1^3+x_2^3$ and $x_1^3x_2^3$ satisfy the same properties for the second quadratic equation (2). We begin by computing

$$(x_1+x_2)^3 \,=\, x_1^3+3x_1^2x_2+3x_1x_2^2 \,+\, x_2^3$$

and hence

$$\begin{aligned} x_1^3+x_2^3 \quad &= (x_1+x_2)^3 \,-\, 3x_1x_2(x_1+x_2) \\ &= (a+d)^3-3\,(ad-bc)(a+d) \\ &= a^3+3a^2d+3ad^2+d^3-3a^2d\,+3abc-3ad^2+\,3bdc \\ &= a^3+d^3+\,3abc+3bdc \end{aligned}$$

which is the negative of the coefficient of y in equation (2). We also observe that

$$x_1^3\,\cdot\,x_2^3 \,=\, (ad-bc)^3$$

Hence x_1^3 and x_2^3 are the roots of equation (2).

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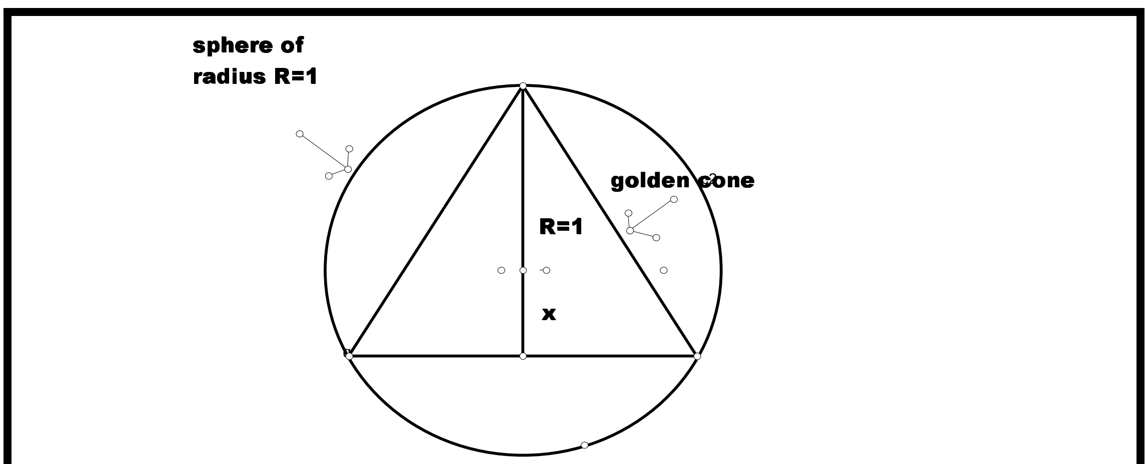
Problem 10 (The Golden Cone)

Here is a problem that will test the most nimble mind. We start with a sphere, say of radius $R = 1$, and we set a cone (like an ice cream cone) upside down and inside the sphere (the sphere is hollow) so that the pointed end of the cone (we are talking about of those old-fashioned cones with the pointed ends) touches the top of the sphere and the circular end of the cone touches the inside of the sphere somewhere on a circle in the bottom part of the sphere. (Get the picture ?) Now, here is where it gets interesting. The altitude line of the cone (the vertical line that goes down the middle of the cone) passes through the center of the sphere and we mark off that point. We then denote the remaining part of the altitude (to the bottom of the cone) by x. (See the diagram below.) Now, this cone isn't any old cone; it is what is called the "golden cone" of a sphere since we are talking about the cone whose ratio of $R = 1$ to x (the larger to the smaller parts of the altitude) is the *same* as the ratio of the total length $1 + x$ to the larger length 1. That is; $\frac{1}{x} = \frac{1+x}{1}$. So what is our question ? It is, what is the volume of this golden cone and what is the ratio of the volume of the golden cone to the volume of the sphere ? *Hint:* Remember that the volume of a (right circular) cone is $V_c = \frac{1}{3}\pi R^2 h$ where $R =$ radius of the circular end, and $h =$ altitude; and that the volume of a sphere is $V_s = \frac{4}{3}\pi R^3$, where R is the radius of the sphere.



The Golden Cone

Solution

We begin by by finding the distance x by writing the ratio $1/x = (1 + x)/1$ as the quadratic equation $x^2 + x - 1 = 0$, which has a single positive root of $x = \frac{\sqrt{5}-1}{2}$. Now by looking at the figure above, we see that if we draw a line from the center of the sphere to where the base of the cone touches the sphere the line will have length 1 and be the hypoteneuse of the right triangle with legs x and r, where r is the radius of the base of the cone. Hence, we can find the radius of the base of the cone from the equation $x^2 + r^2 = 1$, which yields

$$r^2 \;=\; 1 \;-\; \left(\frac{\sqrt{5}-1}{2}\right)^2 \;=\; \frac{\sqrt{5}-1}{2}$$

and so the volume of the cone is $V_c = \frac{1}{3}\pi r^2 h$, where $h = 1 + x$ is the height of the cone. Hence, we have

$$V_c \;=\; \tfrac{1}{3}\,\pi r^2 h \;=\; \tfrac{1}{3}\pi\,(\tfrac{\sqrt{5}-1}{2})(1+\tfrac{\sqrt{5}-1}{2}) \;=\; \tfrac{\pi}{3}$$

The volume of the sphere is $V_s = \frac{4}{3}\pi\, r^3 \;=\; \frac{4}{3}\,\pi$, and so the ratio of the volumes is

$$\text{Ratio} \;=\; \tfrac{V_c}{V_s} = \;\tfrac{(\pi/3)}{(4\pi/3)} \;=\; \tfrac{1}{4}$$

In other words the inscribed golden cone has one forth the volume of the sphere.

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