December Solutions

Problem 1 (Merry Christmas)

How many ways can you spell MERRY CHRISTMAS (ignore the space between the words) by moving either to the left, right, or down in the following pyramid.

M
M E M
M E R E M
M E R R R E M
M E R R Y R R E M
M E R R Y C Y R R E M
M E R R Y C H C Y R R E M
M E R R Y C H R H C Y R R E M
M E R R Y C H R I R H C Y R R E M
M E R R Y C H R I S I R H C Y R R E M
M E R R Y C H R I S T S I R H C Y R R E M
M E R R Y C H R I S T M T S I R H C Y R R E M
M E R R Y C H R I S T M A M T S I R H C Y R R E M
M E R R Y C H R I S T M A S A M T S I R H C Y R R E M

Solution

If you take smaller pyramids, it is not hard to count the number of ways to spell the words AT, CAT, and SCATS in the triangles

		S
	C	**S C S**
A	**C A C**	**S C A C S**
A T A	**C A T A C**	**S C A T A C S**
3 ways = 2^2-1	**7 ways = 2^3-1**	**15 ways = 2^4-1**

getting 3, 7, and 15 ways. You might see a pattern and suspect the number of ways of spelling an n letter word is $2^n - 1$. Thus, you might guess that the number of ways of spelling MERRY CHRISTMAS is 2^{14} - 1 = 16,384 - 1 = 16,383. To prove this is the correct number, you only have to realize one thing. Calling T_n the number of ways of spelling an n letter word, and T_{n+1} the number of ways of spelling an $n+1$ letter word, we verify the relationship T_{n+1} = 2 T_n + 1. To see why this is true, look at the above leftmost triangle that spells AT $(n = 2)$, and then look at the triangle to its right $(n = 3)$ that spells CAT. Now, if you spell the words CAT *backwards*, in the "CAT" pyramid starting from the second to the last letter A, then you can see there are *two* backwards paths from A to C, with the exception of the A in the middle column, in which case there are *three* paths (one to the left, one to the right, and one up). Hence, we have the formula $T_{n+1} = 2T_n + 1$. You can observe for yourself that this pyramid leads to the same formula.

Knowing this relation and since $T_1 = 1$ (only one path for a one letter word), we can simply compute the values of T_n for different values of n. Doing this, we get

$\boldsymbol{n}$	$\boldsymbol{T_n}$	$\boldsymbol{2^n - 1}$	
1	1	$2^1 - 1 = 2 - 1 = 1$	A
2	$2(1) + 1 = 3$	$2^2 - 1 = 4 - 1 = 3$	AT
3	$2(3) + 1 = 7$	$2^3 - 1 = 8 - 1 = 7$	CAT
4	$2(7) + 1 = 15$	$2^4 - 1 = 16 - 1 = 15$	SCAT
5	$2(15) + 1 = 31$	$2^5 - 1 = 32 - 1 = 31$	ORONO
6	$2(31) + 1 = 63$	$2^6 - 1 = 64 - 1 = 63$	BANGOR
7	$2(63) + 1 = 127$	$2^7 - 1 = 128 - 1 = 127$	OLD TOWN
8	$2(127) + 1 = 255$	$2^8 - 1 = 256 - 1 = 255$	PORTLAND
9	$2(255) + 1 = 511$	$2^9 - 1 = 512 - 1 = 511$	...
10	$2(511) + 1 = 1,023$	$2^{10} - 1 = 1,024 - 1 =$ 1,023	
11	$2($1,023$) + 1 = 2,047$	$2^{11} - 1 = 2,048 - 1 =$ 2,047	
12	$2($2,047$) + 1 = 4,095$	$2^{12} - 1 = 4,096 - 1 =$ 4,095	
13	$2($4,095$) + 1 = 8,191$	$2^{13} - 1 = 8,192 - 1 = 8,191$	
14	$2($8,191$) + 1 = 16,383$	$2^{14} - 1 = 16,384 - 1 = 16,383$	
15	$2($16,383$) + 1 = 32,767$	$2^{15} - 1 = 32,768 - 1 = 32,767$	
16	$2($32,767$) + 1 = 65,535$	$2^{16} - 1 = 65,536 - 1 = 65,535$	
	.. ...	...	...

We can also see that the equation $T_n = 2^n - 1$ satisfies the difference equation $T_{n+1} = 2\,T_n + 1$ since

$$2^{n+1} - 1 = 2\,(2^{n} - 1) + 1$$

Hence, if we had a pyramid with 30 letters, the number of paths would be

$$T_{30} = 2^{30} - 1 = 1,073,741,823$$

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Problem 2 (Periodic Sequence Problem)

Show that the sequence defined by

$$\boldsymbol{a_1 = \sqrt{2} \qquad a_{n+1} = \frac{\sqrt{3}\, a_n - 1}{a_n + \sqrt{3}} \qquad (n = 1, 2, 3, \ldots.)}$$

is periodic.

Solution

This problem tests your algebraic skills with radicals. We write

$$a_1 = \sqrt{2}$$

$$a_2 = \frac{\sqrt{3}\sqrt{2}-1}{\sqrt{3}+\sqrt{2}} = \frac{(\sqrt{6}-1)(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} = \frac{4\sqrt{2}-3\sqrt{3}}{1} = 4\sqrt{2} - 3\sqrt{3}$$

$$a_3 = \frac{\sqrt{3}(4\sqrt{2}-3\sqrt{3}-1)}{(4\sqrt{2}-2\sqrt{3})+\sqrt{3}} = \frac{(2\sqrt{6}-5)(2\sqrt{2}+\sqrt{3})}{(2\sqrt{2}-\sqrt{3})(2\sqrt{2}+\sqrt{3})} = \frac{3\sqrt{3}-4\sqrt{2}}{5}$$

$$a_4 = \frac{\sqrt{3}\left(\frac{3\sqrt{3}-4\sqrt{2}}{5}\right)-1}{\left(\frac{3\sqrt{3}-4\sqrt{2}}{5}\right)+\sqrt{3}} = \frac{4-4\sqrt{6}}{8\sqrt{3}-4\sqrt{2}} = \frac{(1-\sqrt{6})(2\sqrt{3}+\sqrt{2})}{(2\sqrt{3}-\sqrt{2})(2\sqrt{3}+\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$a_5 = \frac{\sqrt{3}(-\sqrt{2}/2)-1}{-\sqrt{2}/2+\sqrt{3}} = \frac{-\sqrt{6}-2}{2\sqrt{3}-\sqrt{2}} = \frac{(-6\sqrt{2}-2)(2\sqrt{3}+\sqrt{2})}{(2\sqrt{3}-\sqrt{2})(2\sqrt{3}+\sqrt{2})} = \frac{-6\sqrt{3}-8\sqrt{2}}{10}$$

$$a_6 = \frac{\sqrt{3}\left(\frac{-3\sqrt{3}-4\sqrt{2}}{5}\right)-1}{\left(\frac{-3\sqrt{3}-4\sqrt{2}}{5}\right)+\sqrt{3}} = \frac{-14-4\sqrt{6}}{2\sqrt{3}-4\sqrt{2}} = \frac{-5\sqrt{3}-20\sqrt{2}}{-5} = 3\sqrt{3} + 4\sqrt{2}$$

$$a_7 = \frac{\sqrt{3}(3\sqrt{3}+4\sqrt{2})-1}{3\sqrt{3}+4\sqrt{2}+\sqrt{3}} = \frac{8+4\sqrt{6}}{4\sqrt{3}+4\sqrt{2}} = \frac{4\sqrt{2}(\sqrt{2}+\sqrt{3})}{4(\sqrt{2}+\sqrt{3})} = \sqrt{2} \qquad (\leftarrow a_1)$$

Whew! We were just about ready to give up. So, we know that $a_7 = a_1$. Hence, we know the sequence repeats itself every *six terms*; $a_1 = a_7 = a_{13} = a_{19} = a_{25} = \ldots$ and so on. Of course, we also have $a_2 = s_8 = a_{14} = \ldots$ and so on.

To make your work simple, you could use a computer algebra system like *Mathematica* or *Maple* to carry out your calculations. A computer would solve the problem in a second. If you are not familiar with computer algebra systems, go to *www.maple.com* or *www.mathematica.com* on the Internet. Also look at some of our math links at our web site *www.geocities.com/collegepark/center/1539.*

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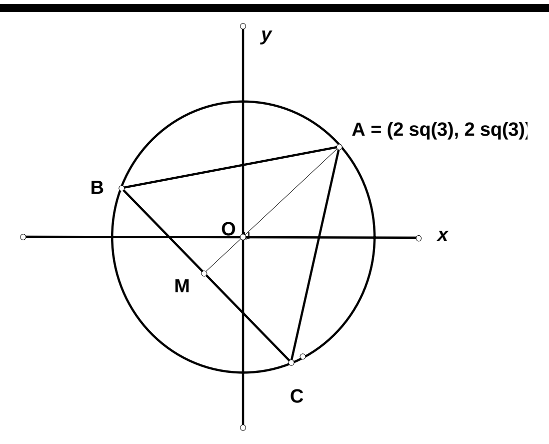
Problem 3 (Equilaterial Triangle)

One vertex of an equilaterial triangle is located at the point $(2\sqrt{3},\, 2\sqrt{3}\,)$. The centroid of the triangle is located at the origin (0, 0). What are the coordinates of the other two vertices ?

Solution

In the case of an equilaterial triangle, the centroid divides the midlines in a (2:1) proportion, where the length of a midline from a vertex to the centroid is twice the length from the centroid to the opposite side.



Equilaterial triangle with origin centroid

From this we conclude that the midpoint of the line opposite A is the point $M = (\text{-}\sqrt{3},\, \text{-}\sqrt{3}\,)$. and the slope of the line BC is -1. Hence, the equation of the line BC (actually, the line passing through BC) is

$$y = -x + 2\sqrt{3}$$

Now, since the triangle $\triangle(ABC)$ is equilaterial, we have

$$OA = OB = OC = \sqrt{(2\sqrt{3})^2 + (2\sqrt{3})^2} = \sqrt{24}$$

and hence the circle with center at the origin O passing through $A, B,$ and C is

$$x^2 + y^2 = 24$$

We now find B and C by finding the intersection of the line BC and this circle, which are

$$\begin{aligned} B &= (-3-\sqrt{3}\,, 3-\sqrt{3}\,) \\ C &= (3-\sqrt{3}\,, -3-\sqrt{3}\,). \end{aligned}$$

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Problem 4 (Albertsville to Bordertown)

The road from Albertsville to Bordertown starts with a 3 mile long uphill slope. After that the road is flat for 5 miles until there is a downward slope for the last 6 miles, with the slope of the last six miles the same down as the slope of the first three miles are up. Randy starts to walk from Albertsville to Bordertown, but halfway there he changes his mind and returns to Bordertown 3 hours and 36 minutes after he started. A little later he starts again and this time walks the entire distance to Bordertown in 3 hours and 51 minutes. He then turns around and walks all the way back to Albertsville in 3 hrs and 51 minutes. Determine Randy's uphill walking speed, downhill walking speed, and speed on the flat portion of the road.

Solution

Let u, d, and f denote Randy's speed (we'll measure this in miles/minute) while walking uphill, downhill, and on the flat portion of the road, respectively. Using the basic formula $d = vt$ for distance d traveled in terms of the speed v and elapsed time t, we know that Randy's first unsuccessful trip, in which he walks halfway from Bordersville to Albertville and turns around, takes a total of

$$\frac{3}{u} + \frac{8}{f} + \frac{3}{d} = 216$$

minutes. We also know it takes a total of 3 hrs 51 minutes (231 minutes) for Randy to walk from Albertsville to Bordertown, which says

$$\frac{3}{u} + \frac{5}{f} + \frac{6}{d} = 231 \qquad \text{(minutes)}$$

And finally since it takes 3 hrs 51 minutes (231 minutes) for Randy to walk from Bordertown to Albertsville, we have

$$\frac{6}{u} + \frac{5}{f} + \frac{3}{d} = 231 \qquad \text{(minutes)}$$

We now solve the three above equations for the three speeds u, f, and d by letting $x = 1/u$, $y = 1/d$, and $z = 1/d$, which gives the three linear equations

$$3x + 8y + 3z = 216$$
$$3x + 5y + 6z = 231$$

$$6x + 5y + 3z = 231$$

Solving these equations, we find $x = 18.286$, $y = 13.286$, and $z = 18.286$. In other words

$u = 1/18.286 \approx 0.055$ miles per minute (3.28 miles per hour)
$d = 1/13.286 \approx 0.075$ miles per minute (4.52 miles per hour)
$f = 1/18.286 \approx 0.055$ miles per minute (3.28 miles per hour)

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Problem 5 (Innkeeper Problem)

After a long day hiking, three hikers stop at an inn where they order a big plate of dumplings to share between them. However, by the time the innkeeper brings the dumplings and places them on the table the hikers are sound asleep. A little later the first hiker awakes and eats one-third of the dumplings. A little later, the second hiker awakes and eats one-third of the remaining ones. Then finally, the third hiker awakes and eats one third of what is left. After that there were 8 dumplings on the plate. How many dumplings did the innkeeper prepare ?

Solution

This is one of those problems where you can start at the end and work forward. The third hiker leaves 8 dumplings on the plate, which means he found 12 dumplings on the plate and ate 4 of the 12. The second hiker left 12 dumplings, which means he found 18 dumplings on the plate and ate 6. So finally, the first hiker left 18 dumplings on the plate, which means he found 27 and ate 9. Hence, the innkeeper prepared 27 dumplings.

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Problem 6 (Equation with Logarithms)

Find the real roots of the equation

$$\boldsymbol{\sqrt{log^2 x + log\, x^2 + 1} + log\, x + 1 = 0}$$

Solution

We can write

$$log^2 x + log\, x^2 + 1 = log^2 x + 2\, log\, x + 1 = (log\, x + 1)^2$$

and so the equation becomes

$$\sqrt{(log\, x + 1)^2} + (log\, x + 1) = 0$$

$$|\, log\, x + 1\, | + (log\, x + 1) = 0$$

which has solutions

$$log\, x + 1 \leq 0 \Rightarrow log\, x \leq -1 \Rightarrow x \leq \tfrac{1}{10}$$

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΦΨΣΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 7 (Mystery Mathematician)

A famous mathematician said he was once x years old in the year x^2. Who was this mathematician ? First, however, find the possible year(s) this person was born.

Solution

There are several candidate mathematicians. We first make the simple observation that if a person had his/her xth birthday in the year x^2, then that person was born in the year $x^2 - x$. So we make a table listing the values of $x^2 - x$ for $x = 1, 2, 3, ...$ to find our canditates. We have

x	x^2	$x^2 - x$	x	x^2	$x^2 - x$	x	x^2	$x^2 - x$
1	1	0	16	256	240	31	961	930
2	4	2	17	289	272	32	1024	992
3	9	6	18	324	304	33	1089	1056
4	16	12	19	361	342	34	1156	1122
5	25	20	20	400	380	35	1225	1190
6	36	30	21	441	420	36	1296	1260
7	49	42	22	484	462	37	1369	1332
8	64	52	23	529	506	38	1444	1406
9	81	72	24	576	552	39	1521	1482
10	100	90	25	625	600	40	1600	1560
11	121	110	26	676	650	41	1681	1640
12	144	132	27	729	702	42	1764	1722
13	169	156	28	784	756	43	1849	1806
14	196	182	29	841	812	44	1936	1892
15	225	210	30	900	870	45	2025	1980
						46	2116	2070

Looking at the table, we see that a person born in the year 0, was 1 year old in the year 1, allowing that person to say he/she was $x = 1$ year old in the year $x^2 = 1$. Looking further down the table to $x = 25$, we have that a person born in the year $x^2 - x$ = 600 AD had his/her 25th birthday in the year x^2 = 625 AD, just allowing that person to say he/she was $x = 25$ years old in the year $x^2 = 625$. As you can see, persons born in any one of the years $x^2 - x = 0$, 2, 6, 12, 20, ... 1980 can all make the claim they were once x years old in the year x^2. In fact, you can make that claim if you were born in the year 1980, since you will be 45 years old in the year 2025, thus making you 45 years old in the year 45^2 = 2025.

So, who is the mystery mathematician ?   Well, the story goes that the French mathematician *Liouville* said it, but according to our history books, Liouville was born in 1809, and not 1806.   Hence, we'll have to find another famous mathematician.   Who knows, maybe it will be you.

If you don't have your own "equation," you can make one up.   For example, the person who is writing the words you are now reading has never been x years old in the year x^2, but was once $2x + 5$ years old in the year $3x + 1932$.   Am I giving my age away ?

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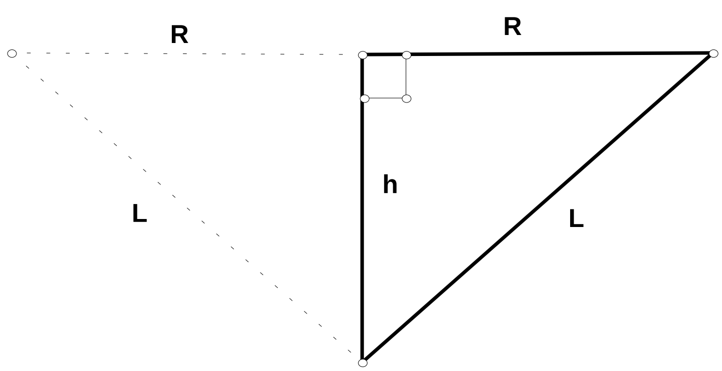
Problem 8 (Rotating a Triangle)

Consider the collection of all right triangles with the same hypotenuse, and rotate the triangles around (either) one of its legs, creating a three-dimensional solid. Find which right triangle (the relative sizes of the legs) that creates the cone with the largest volume.

Solution

We draw a triangle of length L shown below and rotate the triangle around the vertical leg, getting a cone with radius R and height h, which has volume $V = \frac{1}{3}\pi R^2 h$.



Rotating a right triangle around one of its legs gives a cone

Since $R^2 + h^2 = L^2$, we find the volume in terms of the height h as a cubic polynomial

$$\boldsymbol{V(h) = \frac{1}{3}\pi R^2 h = \frac{1}{3}\pi (L^2 - h^2) h \qquad 0 \le h < \infty}$$

which by its very nature we know has one local minimum and one local maximum. We can find the local maximum by computing the first and second derivatives, getting

$$V'(h) = \frac{1}{3}\pi (L^2 - 3h^2)$$

$$V''(h) = -\frac{1}{2}\pi h$$

Setting $V'(h) = 0$, we get $h = L/\sqrt{3}$ and since $V''(L/\sqrt{3}\,) < 0$, we have that $V(h)$ is a maximum when $h = L/\sqrt{3}$. We then find the radius R of the cone (or the second leg of the triangle) from $R^2 + h^2 = L^2$, getting

$$R = \sqrt{L^2 - h^2} = 2L/\sqrt{3}$$

Hence, the triangle with hypotenuse L with maximum volume when rotated about one of its legs is the triangle with legs

$$\text{leg 1} = L/\sqrt{3} \qquad \text{leg 2} = \sqrt{2/3}\,L/$$

where we rotate the triangle around leg 1.

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Problem 9 (Beanie Baby Problem)

Three workers work 5 hours putting together beanie babies, where they are paid a fixed amount of money for each beanie baby assembled. Suppose all together the workers earn \$470 where the first worker earns \$200, the second worker assembles one beanie baby every 4 minutes, and the third worker receives \$30 less than the second worker. All together, how many beanie babies do the three workers assemble ?

Solution

If we call p_1, p_2, and p_3 the pay of the three workers, respectively, then we have been given the three conditions:

$$\begin{aligned} &\boldsymbol{p_1 + p_2 + p_3 = \$470} \\ &\boldsymbol{p_1 = \$200} \\ &\boldsymbol{p_3 + \$30 = p_2} \end{aligned}$$

which means the three workers earn $p_1 = \$200$, $p_2 = \$150$, and $p_3 = \$120$, respectively. We are also given the fact that the second worker can assemble one beanie baby every 4 minutes which means this worker assembles 125 beanie babies in 5 hrs. But this worker makes $p_2 = \$150$ and so we have

(125 beanie babies) (pay per beanie baby) = \$150

which means this worker makes for each beanie baby assembled:

$$\text{pay per beanie baby} = \tfrac{150}{125} = \$1.20$$

Since all workers make the same amount of \$1.20 for each beanie baby assembled, we can find the number of beanie babies assembled by the three workers by simply dividing their money earned by \$1.20, or

Worker 1: beanie babies assembled = $\frac{\$200}{\$1.20} = 166.67$ beanie babies
Worker 2: beanie babies assembled = $\frac{\$150}{\$1.20} = 125$ beanie babies
Worker 3: beanie babies assembled = $\frac{\$120}{\$1.20} = 100$ beanie babies

Hence, the total number of beanie babied assembled is the sum of the above numbers, or 391.67.

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Problem 10 (Pyramids from Spheres)

Here's a fun problem. Take 14 spheres, each of radius 1, and stack them in the form of a pyramid (like they do in grocery stores when they sometimes stack oranges), putting 9 spheres at the base, 4 on top of those, and one at the very top.

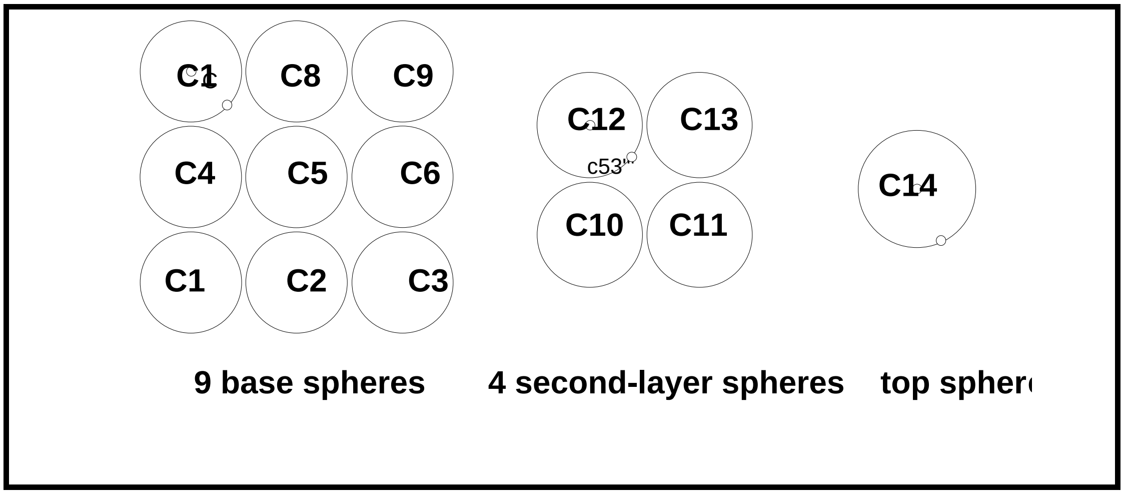

Figure 1: Forming a pyramid from spheres

What proportion of the volume of the "covering" pyramid is filled with the spheres ?

Solution

We know that the base of the pyramid is a square with corner points A, B, C, and D as shown in Figure 2 below.

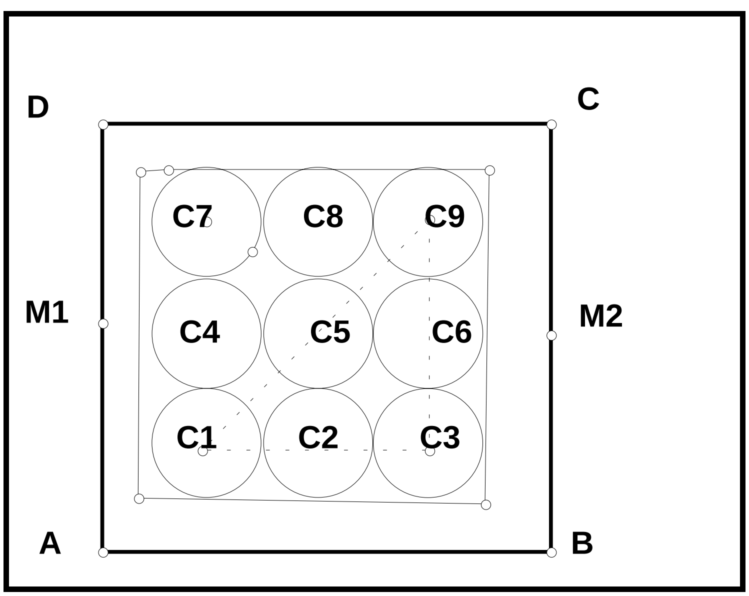

Figure 2: Base of the pyramid

If we can find the altitude and base of the pyramid, we can find its volume and hence the solution to the problem. To find the altitude, we consider the triangle $\triangle(C_1C_3C_9)$ shown in the above figure. We conclude that , $C_1C_3 = C_3C_9 = 4$ inches and thus $C_1C_9 = 4\sqrt{2}$. We now draw the vertical triangle shown in Figure 3.

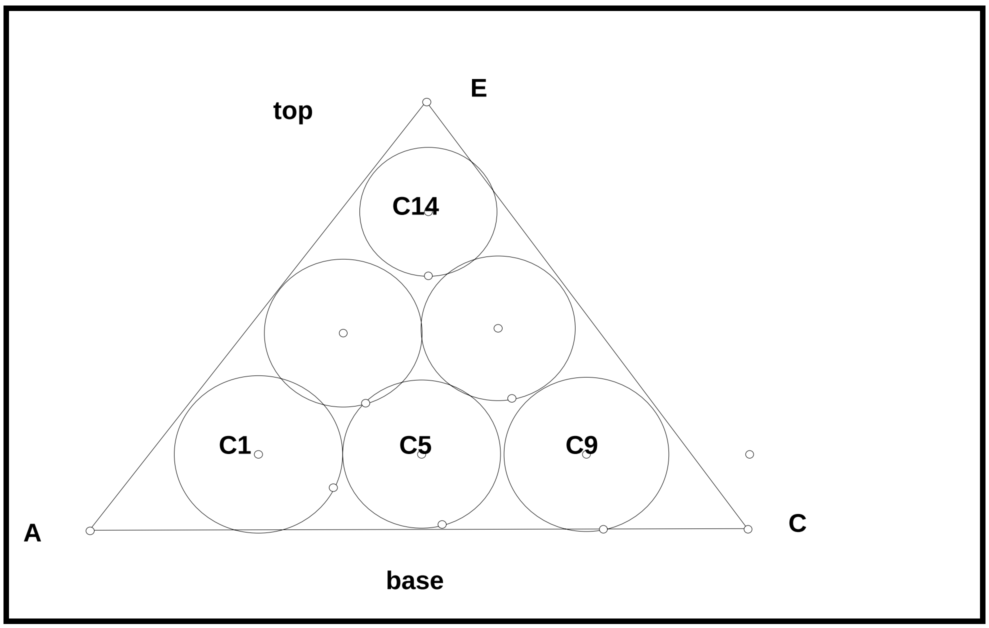

Figure 3: Vertical triangle

and make the following conclusions

$$C_1C_{14} = C_9C_{14} = 4 \text{ inches}$$
$$C_1C_5 = \tfrac{1}{2}\, C_1C_9 = 2\sqrt{2} \text{ inches}$$
$$C_5C_{14} \text{ is perpendicular to } C_1C_9$$

Hence

$$C_5C_{14} = \sqrt{4^2 - (2\sqrt{2})^2} = 2\sqrt{2} \text{ inches}$$

If we now denote by M_1 and M_2 the midpoints of AD and CD, respectively, as shown in Figure 2, we can draw the triangle shown in Figure 4 where we conclude

$$C_4C_{15} = 2\sqrt{2}$$
$$C_5B_3 = 1$$
$$C_4C_5 = 2$$

$$C_4C_{14} = \sqrt{2^2 + (2\sqrt{2})^2} = 2\sqrt{3}$$

Finally, from the similarity of the triangles drawn in bold lines, we get $EC_{14} = \sqrt{3}$.

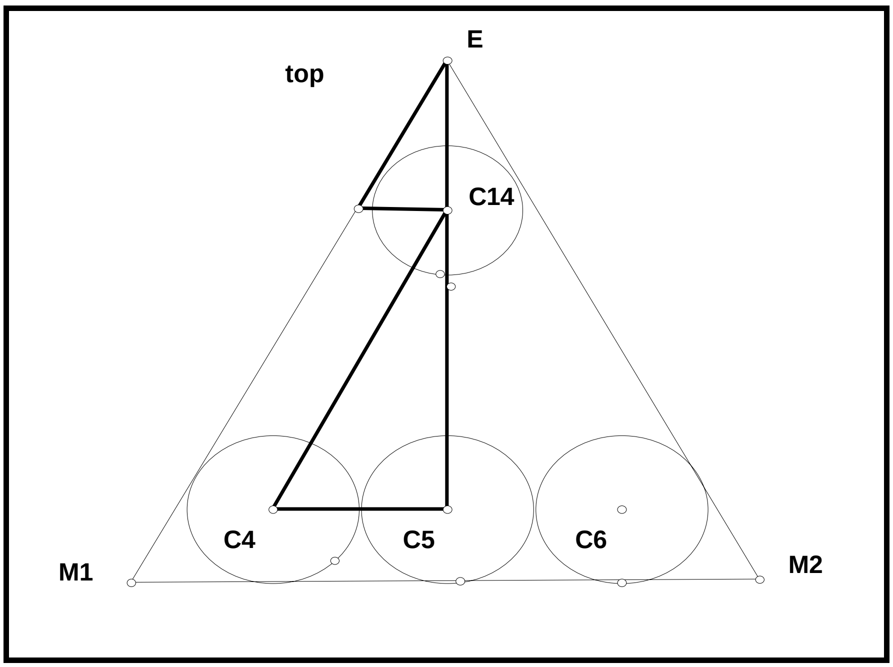

Figure 4: Finding the height of the pyramid covering the spheres

Hence, the altitude of the pyramid is

$$H = EB_5 = 1 + \sqrt{3} + 2\sqrt{2}$$

We can also find the base of the pyramid by observing the similarity of the triangles

$$\triangle(M_1B_5E) \cong \triangle(C_4C_5C_{14})$$

Calling the base of the pyramid b, we have

$$\frac{b/2}{2} = \frac{H}{2\sqrt{2}}$$

and hence

$$b = \frac{2}{\sqrt{2}} H$$

Thus, the volume of the pyramid is

$$V_{pyramid} = \frac{1}{3} b^2 H = \frac{2}{3} H^3 = \frac{2}{3} (1 + \sqrt{3} + 2\sqrt{2})^3$$

Since the volume of the 14 spheres with radius $r = 1$ is

$$V_{spheres} = 14 \left(\frac{4}{3}\pi r^3\right) = \frac{56}{3}\pi$$

the proportion of the volume of the covering pyramid taken up by the spheres is

$$proportion = \frac{V_{spheres}}{V_{pyramid}} = \frac{\frac{56}{3}\pi}{\frac{2}{3}H^3} = \frac{56\pi}{2\,(1+\sqrt{3}+2\sqrt{2})^3} \approx 0.52$$

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