February Problems

Problem 1 (The Rope-Around-the-World Problem)

A rope is wrapped around the earth at the equator three feet in the air. If we assume the earth is a perfect sphere with radius 8,000 miles and if we drop the rope to the ground, how much slack will there be in the rope ? You might make a guess before you solve this problem.

Solution:

If we call R the radius of the earth (in feet), then the circumference of the rope around the earth before it is dropped to the ground is

$$\mathbf{C_{rope} = 2\pi\,(R + 3)}$$

and the circumference after it is dropped to the ground is the radius of the earth, or

$$\mathbf{C_{earth} = 2\pi R}$$

Hence, the slack is the difference between the two circumferences, or

$$\textbf{Slack} = \mathbf{C_{rope} - C_{earth} = 2\pi\,(R + 3) - 2\pi R = 2\pi\,(3) = 6\pi \approx 18.85 \textbf{ feet}}$$

The interesting thing about this problem is that the amount of slack does not depend on the radius of the earth. In fact, a rope 3 feet above a basketball dropped to the basketball would have the same slack of $6\pi \approx 18.85$ feet. It is also interesting that most people guess slacks of upwards of thousands of miles. In fact, if you are jogging around a 400 meter track with a friend and if you are running on the outside 3 from your friend, then every lap you will run $2\pi\,(3) = 6\pi \approx 18.85$ feet further than your friend (since the only time you run further is on the two ends of the track which can be collapsed into a circle).

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Problem 2 (Interesting Equation)

Solve

$$\sqrt[3]{x} + \sqrt[3]{1 - x} = 2$$

Solution

You might be tempted to plot the curve $y = \sqrt[3]{x} + \sqrt[3]{1 - x}$ on your graphing calculator or personal computer and see when the curve has a height of y = 2, but this strategy fails since the curve never has a height of 2. So, we try a different strategy and raise both sides of the equation to the 3rd power and using the identity

$$(a+b)^3 = a^3 + 3a^2b + 3\,ab^2 + b^3$$

to get (after organizing terms)

$$3\sqrt[3]{x}\;\sqrt[3]{1 - x}\left(\sqrt[3]{x} + \sqrt[3]{1 - x}\right) = 8 - 1$$

were we now make the fundamental observation that the quantity inside the parenthesis is equal to 2 (the original equation), and hence we have

$$6\sqrt[3]{x}\;\sqrt[3]{1 - x} = 7$$

or simply

$$\sqrt[3]{x\,(1 - x)} = \tfrac{7}{6}$$

Raising both sides to the 3rd power again gives the simple quadratic equation

$$x^2 - x + (\tfrac{7}{6})^3 = 0$$

which we can easily solve and has two *complex* roots

$$x = \frac{1 \pm \sqrt{1-4\,(7/6)^3}}{2} = \frac{1 \pm\, i\sqrt{4\,(7/6)-1}}{2} \approx \frac{1 \pm 1.9\,i}{2}$$

The fact the equation has complex roots is why the graph of $y = \sqrt[3]{x} + \sqrt[3]{1 - x}$ in the xy-plane never attains a height of $y = 2$.

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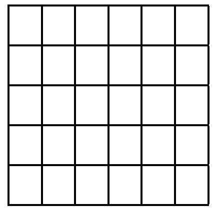
Problem 3 (Jane's Quilt)

Jane had just finished making the following quilt



and marveled at the 30 squares in the quilt. Actually, there are *more* than 30 squares in the quilt, and even more rectangles in the quilt. Can you determine the total number of squares of all sizes and the number of rectangles of all sizes in the quilt ?

Solution

This problem can be considered a bookeeping problem where we simply tabulate the squares and rectangles of different sized. We begin by counting the squares. There are various types of squares ranging from single squares we call 1×1 squares to 5×5 squares. We have

Type of Square	**Number of Squares**
1×1 squares	30
2×2 squares	20
3×3 squares	12
4×4 squares	6
5×5 squares	2
	Total 70

For rectangles, we list all rectangles of different sizes below where a 1×3 rectangle is considered the same as a 3×1 rectangle. If we simply count the rectangles of the various size, we find

Type	**# Rectangles**	**Type**	**# Rectangles**
1×1	30	3×4	17

1×2	49
1×3	38
1×4	27
1×5	16
1×6	5
2×2	20
2×3	31
2×4	22
2×5	13
2×6	4
3×3	12
3×5	10
3×6	3
4×4	6
4×5	7
4×6	2
5×5	2
5×6	1
Total	**315**

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Problem 4 (Case of the Two Quadratics)

The roots of the equation $x^2 - bx + c = 0$ are one more than the roots of the equation $x^2 + bx + c = 0$. Find the values of b and c. Do not use a calculator.

Solution

It is easy to see that the roots of $x^2 - bx + c = 0$ are

$$x_1,\, x_2 = \frac{b}{2} \pm \frac{\sqrt{b^2-4c}}{2}$$

and the roots of $x^2 + bx + c = 0$ are

$$x_1,\, x_2 = \frac{b}{2} \pm \frac{\sqrt{b^2-4c}}{2}$$

Hence, we have

$$-\frac{b}{2} + \frac{\sqrt{b^2-4c}}{2} + 1 = \frac{b}{2} + \frac{\sqrt{b^2-4c}}{2}$$

$$-\frac{b}{2} - \frac{\sqrt{b^2-4c}}{2} + 1 = \frac{b}{2} - \frac{\sqrt{b^2-4c}}{2}$$

We can easily see that both equations imply $b = 1$ with no restrictions on c. Thus $b = 1$ and c is arbitrary.

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Problem 5 (Rolling Circles)

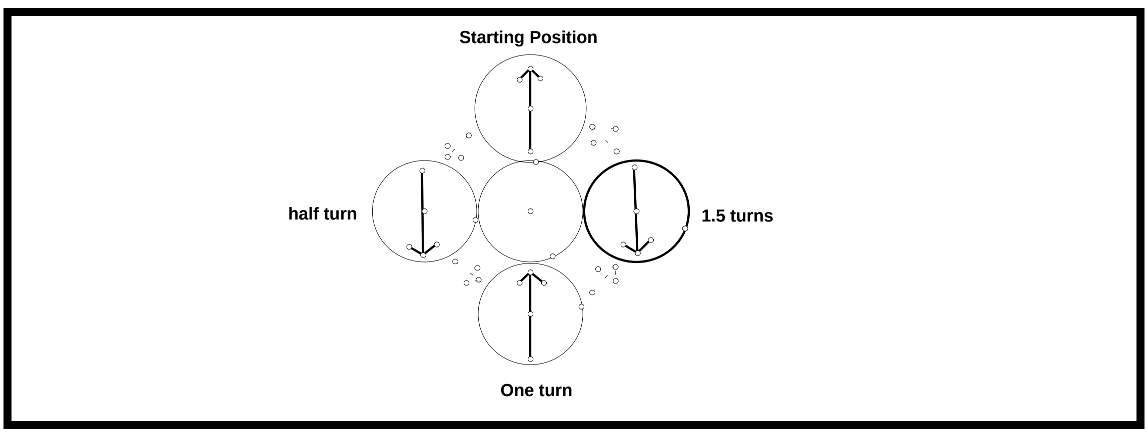
An interesting but not so easy problem is to determine the number of turns a rolling circle make around the outside of a fixed circle. For example, suppose you put two pennies side by side face up. Hold one and rotate the other circle around it counterclockwise. It is not easy to visualize the number of turns the rolling penny makes around the outside of the fixed penny. We have two questions, one you can do by physical experimentation, the other with your mind.

(a) How many turns does the rolling penny make around the outside of the fixed penny ?

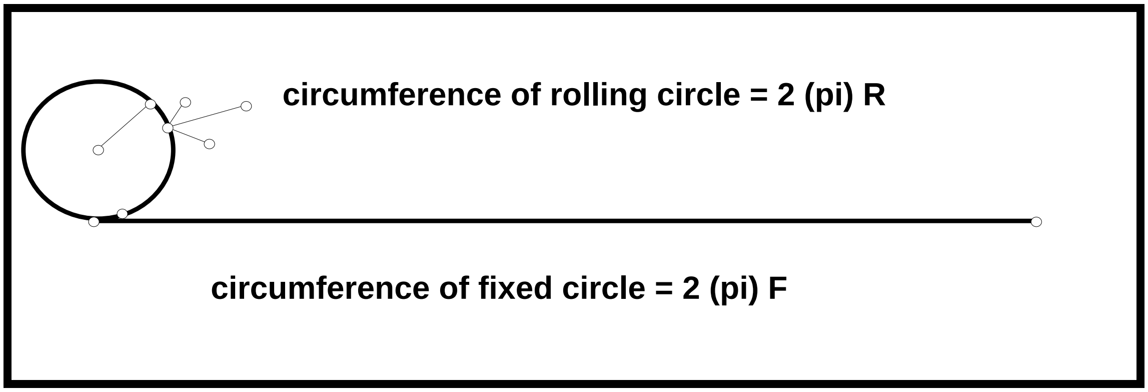
(b) What if the radius of the rolling circle is R and the radius of the fixed circle is F. How many turns T will the rolling circle make around the outside of the fixed circle ?

Solution

(a) By experimentation you discover that the rolling penny makes a complete revolution when it rolls only half way around on itself, and two complete revolutions when it rolls all the way around. (For no specific reason, we roll the outside penny around the fixed penny in the counterclockwise direction.) Note in the following figure if the outside penny is rolled in the counterclockwise direction starting at the top, then it undergoes two complete revolutions as it rolls around the inside penny.



(b) To determine the number of turns a rolling circle with radius R makes as it rolls around a fixed circle with radius F, we think of unrolling the fixed circle (as if it were a hooly hoop) on a flat surface, which would have length of $2\pi F$.



Hence, if we roll the outside circle of radius R (circumference $2\pi R$) on this line, the number of turns it will make will be

 # ROTATIONS OF A CIRCLE ON A LINE OF LENGTH $2\pi F = \frac{2\pi F}{2\pi R} = \frac{F}{R}$

But, the rolling circle in our problem does *not* roll along a line, but around a circle and hence the number of rotations the rolling circle makes as it rolls around a fixed circle of radius F is one *more* than F/R, or

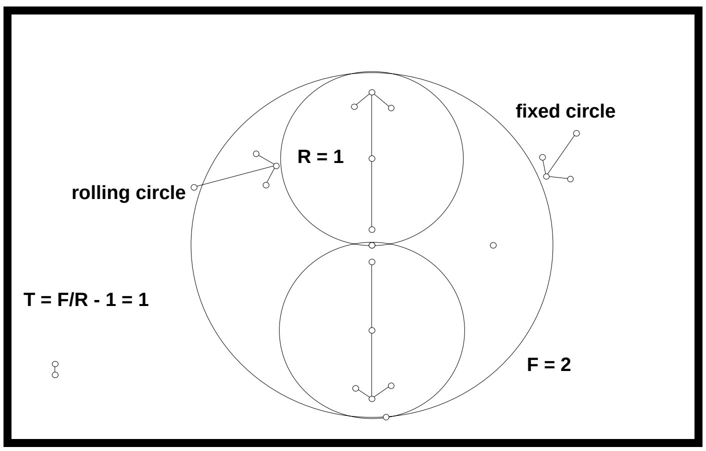
$$\boldsymbol{T = \frac{F}{R} + 1}$$

You can see this by imagining *sliding* the outside circle around the fixed circle (keeping the same point of the sliding circle in contact with the fixed circle), and noting it rotates once as a result of going around the circle. The following table gives the number of turns for different values of R and F.

R / F	**1**	**2**	**3**	**4**
1	2	3	4	5
2	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
3	$1\frac{1}{3}$	$1\frac{2}{3}$	2	$2\frac{1}{3}$
4	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2

For example, if the radius of the fixed circle was $F = 3$ inches and the radius of the rolling circle was $R = 4$ inches, then the rolling circle would undergo $T = 1\frac{3}{4}$ turns as it rolled around the fixed circle.

It is interesting to note that if the rolling circle were *inside* the fixed circle as shown in the picture below, then (by a similar argument) the number of rotations of the rolling circle will be $R/F \; - 1$. In this case, if we *slide* the inside circle counterclockwise inside the outer fixed circle, keeing the *same* point of the inner circle in contact with the outer circle, then the rotation of the inner circle is clockwise, but when we *roll* the inner circle counterclockwise inside the outer circle, the rotation is clockwise, and hence we *subtract* 1 from F/R instead of adding it.



Circle Rolling Inside a Fixed Circle

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Problem 6 (Arithmetic Sequence)

If the numbers a, b, c are terms of an arithmetic sequence, then show that the numbers

$$A = a^2 + ab + b^2 \qquad B = a^2 + ac + c^2 \qquad C = b^2 + bc + c^2$$

are also terms of an arithmetic sequence.

Solution

If a, b, and c are terms of an arithmetic sequence, then we can write

$$b = a + h \qquad\qquad c = a + 2h$$

Hence,

$$A = a^2 + a(a+h) + (a+h)^2 = 3a^2 + 3ah + h^2$$
$$B = a^2 + a(a + 2h) + (a+2h)^2 = 3a^2 + 6ah + 4h^2$$
$$C = (a+h)^2 + (a+h)(a+2h) + (a+2h)^2 = 3a^2 + 2ah + 7h^2.$$

Hence,

$$B - A = 3h^2 + 3ah$$
$$C - B = 3h^2 + 3ah$$

Hence, the differences are the same.

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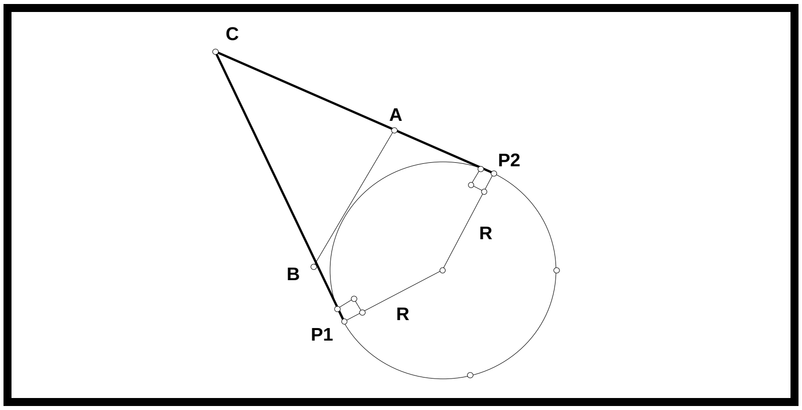
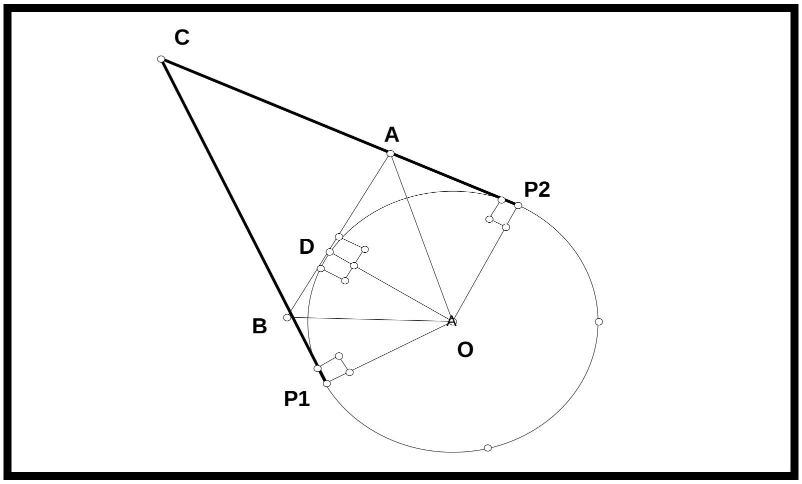
Problem 7 (Equal Perimeter Problem)

Consider the circle drawn as below with a point C outside the circle. Now draw from C the two tangent to the circle, touching the circle at the points P_1 and P_2 . Now draw a line AB connecting the tangent lines CP_1 and CP_2 and itself tangent to the circle. Show that the perimeter (distance around) of the resulting triangle ABC is the *same* no matter how the line AB connecting the tangent lines is drawn.



Solution

We first realize that the two tangent lines CP_1 and CP_2 have the same length, and that the perimeter of the triangle ABC is $AB + BC + CA$. But if we draw the point D where the line AB touches the circle and realize that the triangles $\triangle(OP_2A)$ and $\triangle(ODA)$ are congruent, as are the triangles $\triangle(OP_1B)$ and $\triangle(ODB)$, we have equality of the distance $P_1B = DB$ and $P_2A = BA$. Hence, $AB = P_1A + P_2A$, and so the perimeter of the triangle $\triangle(ABC)$ is $AB + BC + CA = CP_1 + CP_2$. Hence, the perimeter of the triangle $\triangle(ABC)$ is the same no matter now the line AB is drawn.



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Problem 8 (The Plane Problem)

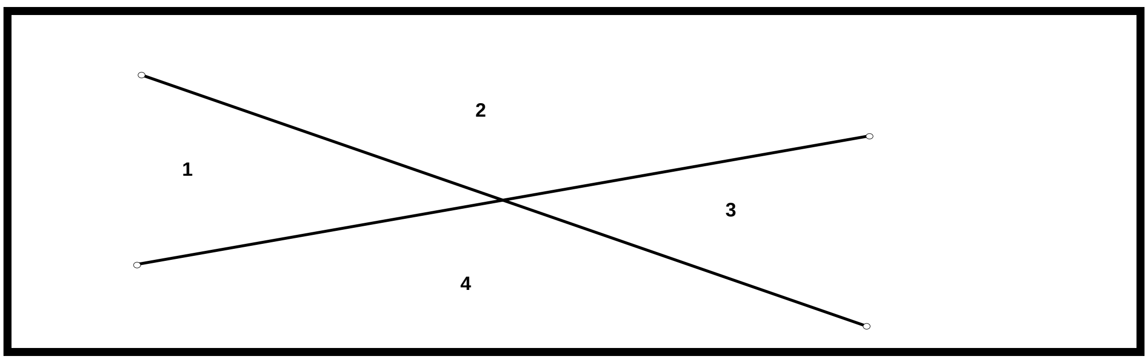
There are two parts to this problem. The first part is not terrible hard but if you can solve the second part, you are very smart indeed.

(a) What is the maximum number of parts you can subdivide the plane by n lines, for $n = 1, 2, 3, 4,$ and 5 lines?

(b) What is the general formula for the maximum number of parts you can subdivide a plane by n lines ?

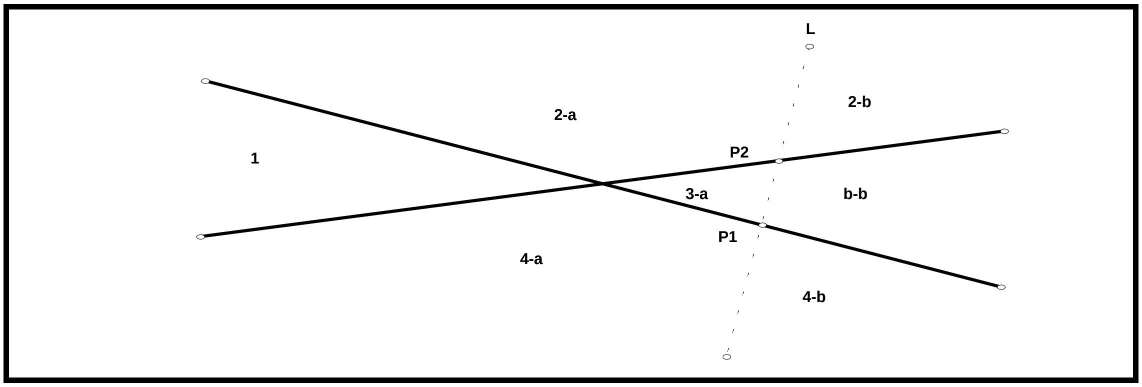
Solution

(a) We start by drawing one line that divides the plane into 2 parts. We then draw a second line not parallel to the first line which will then subdivide the plane into 4 parts as shown below.



Two non parallel lines subdivide the plane into 4 subregions

It is clear the second line should not be parallel to the first line or else it would subdivide the plane into only 2 or 3 parts, depending on whether the lines were identical or distinct. It should also be clear that all additional lines be drawn so they are not parallel to existing lines and not pass through previous points of intersection. Consider now the addition of a third line L shown below which intersects the two previous lines at the points P1 and P2, and subdivides the previous 3 regions 2, 3 and 4, into 6 regions 2-a, 2-b, 3-a, 3-b, 4-a, and 4-b, giving rise to a total of 7 regions.



Addition of the 3rd line subdivides 3 regions into 6 regions

In general, if we have already drawn $n-1$ lines giving rise to S_{n-1} subregions, then the addition of the $n\,th$ line will intersect the previous n-1 lines and pass through n previous regions, dividing these n regions into $2n$ regions. Hence, the total new number of regions S_n will be

$$\boldsymbol{S_n = S_{n\text{-}1} + n \qquad n = 2, 3, \ldots}$$

And since $S_1 = 2$, we can determine S_n for any n. The following table illustrates S_n for different values of n.

$\boldsymbol{n}$	$\boldsymbol{S_n}$
1	2
2	$2 + 2 = 4$
3	$4 + 3 = 7$
4	$7 + 4 = 11$
5	$11 + 5 = 16$
6	$16 + 6 = 22$
7	$22 + 7 = 29$
8	$29 + 8 = 37$
9	$37 + 9 = 46$

(b) To determine a general formula for S_n and not just one that relates S_n to S_{n-1}, we repeatedly use the formula $S_n = S_{n-1} + n$ over and over with smaller and smaller values of n (after all it holds for any n = 1, 2, ...). We can write

$$\begin{aligned} S_n &= S_{n-1} + n \\ &= S_{n-2} + (n-1) + n \end{aligned}$$

$$= S_{n-3} + (n-2) + (n-1) + n$$

$$\quad .... \qquad ... \qquad ...$$

$$= S_1 + 2 + 3 + ... + n$$

$$= 2 + 2 + 3 + ... + n$$

$$= 1 + (1 + 2 + 3 + ... + n)$$

$$= 1 + \frac{n\,(n+1)}{2}$$

$$= \frac{1}{2}\,(n^2 + n + 2) \qquad n = 2, 3, ...$$

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Problem 9 (Mary's Random Walk)

Mary lives in a town which has streets and avenues crossing each other at right angles. For convenience, we will say the streets run in a north-south direction, and the avenues run in an east-west direction. One day at exactly 12 o'clock Mary decides to take a walk, starting at the intersection of a street and avenue, whereupon she walks one block every 15 minutes. When she reaches the end of a block, she turns either to the left or right. Show, that Mary can return to her starting point only on the hour.

Solution

One thing is clear. If Mary makes a right turn at every block, then when she returns home she will have traveled an even number of blocks in the horizontal direction (either left of right) and the same even number of blocks in the vertical direction (either up or down). Hence, the total number of blocks she will have traveled is twice an even number, or a multiple of four. Therefore, if Mary started her walk on the hour, she will return to the starting point on the hour.

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Problem 10 (Orange Juice Problem)

We have four orange juice mixes with concentrations A_1, A_2, A_3, and A_4 gal orange juice/gal mix and know that we get a concentration of

$i)$ 0.15 gal orange juice/gal mix if you mix 3 gallons of the first mix, 2 gal of the second mix, and 1 gal of the third mix

$ii)$ 0.10 gal of orange juice/gal mix is you mix one gallon each of the four mixes

$iii)$ 0.10 gal orange juice/gal mix if you mix 1 gallon of the first mix and 1 gallon of the third mix

How much orange juice/gal solution would you get if you add 2 gal of the first mix and 1 gal of the fourth mix ?

Solution

Condition $i)$ says if we add 3, 2, and 1 gal of orange juice mixes of respective concentrations A_1, A_2, and A_3 gal orange juice/gal mix (which means we add a total of $3A_1 + 2A_2 + A_3$ gal orange juice), then the concentration of the resulting 3 + 2 + 1 = 6 gallons of mix will be 0.10 gal orange juice/gal mix. Hence, we have

$i)$ $3A_1 + 2A_2 + \mathrm{A}_3 = 0.15\,(3 + 2 + 1) = 0.90$ gal orange juice

Likewise, by adding a gallon of each of the four concentrations gives 0.10 gal orange juice/gal mix, and hence

$ii)$ $A_1 + A_2 + A_3 + A_4 = 0.10\,(1 + 1 + 1 + 1) = 0.40$ gal orange juice

And finally by adding one gallon of the first concentration A_1 and one gallon of the third concentration A_3, gives 0.10 gal orange juice/gal mix, and hence

$iii)$ $A_1 + A_3 = 0.10\,(1 + 1) = 0.20$ gal orange juice

The three previous equations $i)$, $ii)$, and $iii)$ constitute an underdetermined system of three linear equations with four unknowns A_1, A_1, A_3, and A_4. If we take one of the four unknowns as arbitrary, say $A_4 = \alpha$, then we can solve for the other three unknowns in terms of α using gaussian elimination, giving the solutions

$$A_1 = 0.15 + \alpha$$
$$A_2 = 0.20 - \alpha$$
$$A_3 = 0.05 - \alpha$$
$$A_4 = \alpha$$

where α is between 0 and 0.05 in order to keep all the concentrations nonnegative. Hence, there are many different concentrations A_1, A_2, A_3, and A_4 that satisfy the three conditions $i)$, $ii)$, and $iii)$. Hence, the amount of orange juice in the mix when we mix 2 gal of A_1 and one gal of A_4 is any of the values $2A_1 + A_4 =$ 0.30 + 3 α. And since we have a total of 3 gal of the mix, this means the concentration is any of the values

$$\text{concentration} = \frac{0.30 + 3\alpha}{3} = 0.10 + \alpha \text{ gal orange juice/gal solution.}$$

where $0 \le \alpha \le 0.05$. In other words, from our information we cannot determine uniquely the concentration of the mix $2A_1 + A_4$; it can be any concentration between 0.10 and 0.15 gal orange juice/gal mix.

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