January Solutions

Problem 1 (The Three-Cap Problem)

One cold winter morning in Maine a mother insists that her three daughters, Ann, Betty, and Cindy wear caps to school and so she randomly picks 3 caps from a pile of 5 caps, three of which are white and two are red. The girls stand in line, one behind the other, waiting for the mother to put on their caps. Cindy stands in the back and can see both of the other girls; Betty stands in the middle and can see Ann who stands in the front; but Ann standing in the front cannot see either of the other girls. The mother then puts a cap on each of the girl's heads. For fun, the mother asks the girls if they can guess the color of their own caps. (The girls know their mother picked the caps from the 5 caps, three white and two red.) Cindy, who is in the back, says she has no idea the color of her own cap. Betty, hearing what Cindy says, also says she has now idea the color of her cap. But Ann, hearing what Cindi and Betty say, says she *does* knows the color of her cap. How is Ann able to determine the color of her cap ? What is the color of Ann's cap ?

Solution

If there is one thing useful in solving complicated problems, it is *pictures.* Let's make a simple drawing illustrating the three girls standing in line with Cindi (C) in the back, Betty (B) in the middle, and Ann (A) in the front, keeping in mind that none of the girls can see their own caps, but Cindi can see the caps of Betty and Ann, Cindi can see the cap of Ann, and Ann cannot see any of the caps. If we now call R = RED CAP and W = WHITE CAP, the possible colors of the caps on the girl's heads are :

Cindi → Bettty → Ann
R → R → W
R → W → R
R → W → W
W → R → R
W → R → W
W → W → R
W → W → W

Now, believe it or not, the fact Cindy (in the back) has *no idea* the color of her own cap tells us *a lot* since if Betty and Ann *both* wore red hats, then Cindi would *know* her cap was white. Hence, we can eliminate the possibilities that both B and C wear red caps and are left with *six* possibilities:

Cindi → Betty → Ann
R → R → W
R → W → R
R → W → W
W → R → W
W → W → R
W → W → W

But now, Betty starts to think ! (These girls are *smart*.) Upon hearing what Cindi says, Betty knows that both she and Ann do not both wear red, and so if Ann wore red, Betty would announce she was wearing *white*. But Betty announces she doesn't know her own color and so Betty's announcement tells Ann (in fact it tells all three girls!) that Ann must be wearing white! In other words, we throw out possibilities 3 and 7, leaving the following possibilities for the cap colors:

Cindi → Betty → Ann
R → R → W
R → W → W
W → R → W
W → W → W

Hence, we know that A (Ann) must be wearing a white cap, but do not know the colors of the caps of C (Cindi) or B (Betty).

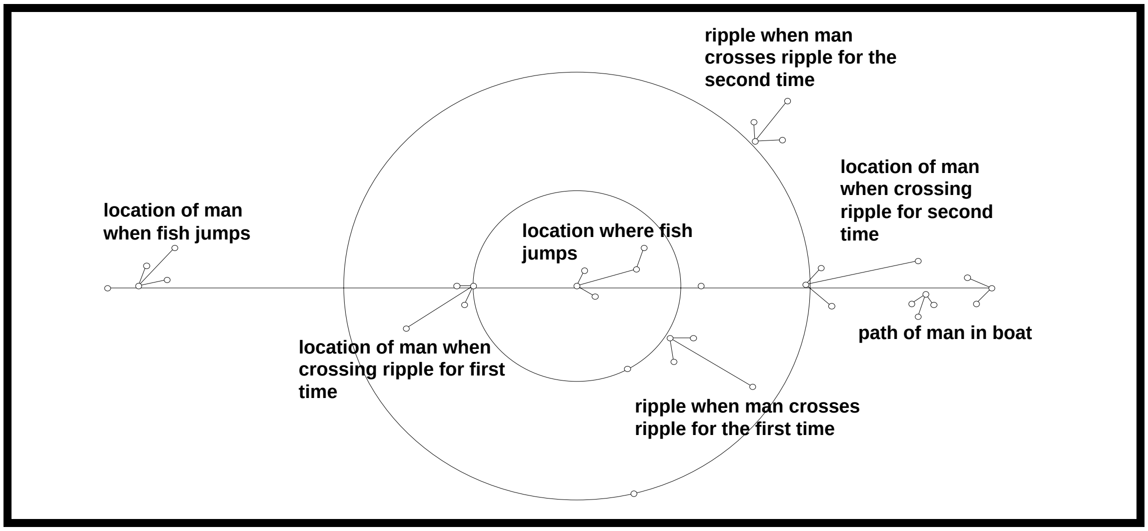
ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΦΨΣΠΔΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 2 (The Ripple Problem)

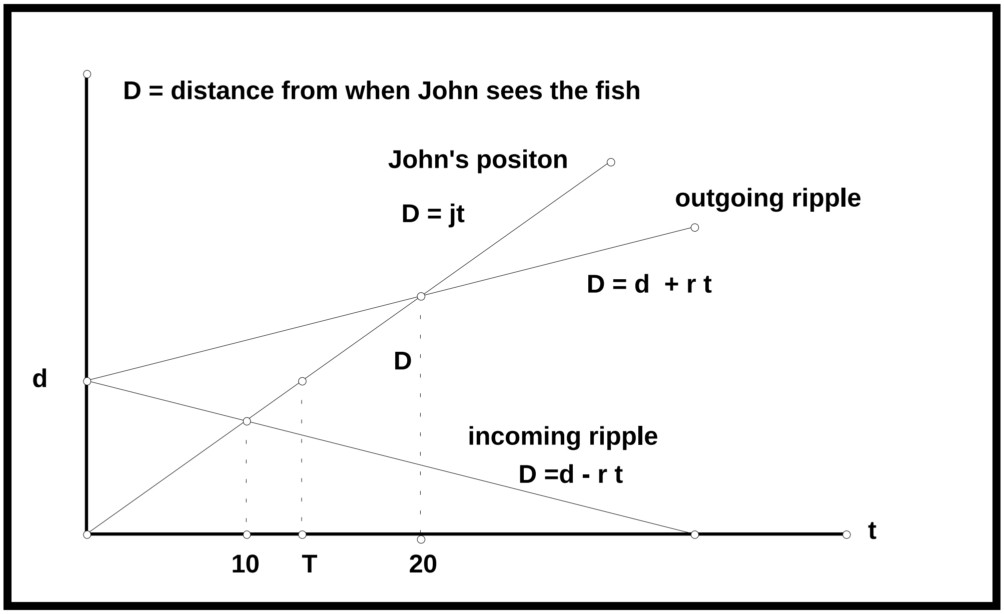
John is paddling across a lake in his canoe directly ahead of him a fish jumps out of the water creating a circular ripple which moves outward at a constant rate. Exactly 10 seconds after the fish jumps John paddles across the ripple coming towards him and after 20 seconds he crosses the ripple for the second time. How long after the fish jumps does John cross the location where the fish jumped ?

Solution

We show below John's position in the canoe when the fish jumps, his location when he meets the ripple from the fish for the first time, and John's position in the canoe when he crosses the larger circular ripple for the second time.



We basically use the well-known equation that distance is velocity times time over and over. We start by calling D the distance the fish is ahead of John when the fish jumps, T the unknown time when John reaches the location where the fish jumps, and r and j the velocities of the circular ripple in the water and John's velocity in the canoe, respectively. We first draw a graph showing the position of John's canoe and the location of the ripples, one ripple moving towards John and the other the same direction as John.



Now, since John reaches the first ripple in 10 seconds, we set the location of the incoming ripple equal to John's position, or

$$D - 10\,r = 10\,j$$

But we also know that the radius of the ripple when John reaches the ripple for the first time to be $10r$ and that the radius of the ripple when John reaches the ripple the second time to be $20r$, and so between 10 seconds and 20 seconds John covers the distance between the two ripples, he travels a distance of $10\,r + 20\,r$ = $30r$ feet. But we also know that John travels with velocity j, and so he also travels $10j$ feet. Hence, we set $30r = 10j$ or $j = 3r$, which means that John travels three times faster than the ripple. We now substitute this value into the previous equation, getting

$$D - \frac{10}{3}\,j = 10\,j \Rightarrow D = \frac{40}{3}\,j$$

But we know that John crosses the place where the fish jumps when $T = D/j$. Hence, we have

$$T = \frac{D}{j} = \frac{40}{3} = 13\,\frac{1}{3} \text{ seconds.}$$

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΦΨΣΦΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 3   (Equation with Integer Solutions)

Among the natural numbers 1, 2, 3, ...   find all solutions (if any) of the equation

$$x^2 = 1.5\ y^3$$

where we restrict x^2 and y^2 to be less than 10,000.

Solution

Since both numbers x^2 and y^2 are less than 10,000, we know that x is one of the integers 1, 2, 3, ... 99, and since $1.5y^3 = x^2 < 10{,}000$, we know that y is one of the integers 1, 2, 3, ... 18.   We could look at all these combinations of x and y, but it is best to carry out some basis analysis to narrow the number of candidates.

We argue that:

1. y^3 is an even integer (else 1.5 y^3 is not an integer)
2. hence, x^2 is a multiple of 3 (since $x^2 = 1.5\,y^3$)
3. hence, x^2 is a multiple of 9 (since it is the square of an integer)
4. hence 1.5 y^3 is a multiple of 9
5. hence y^3 is a a multiple of 6
6. hence y is a multiple of 6
7. hence y = 6, 12, or 18

So, all we have to do is make up the table

y	y^3	$1.5\,y^3$
6	216	324
12	1728	2592
18	5832	8749

and ask if there are any numbers 1.5 y^3 in the rightmost column that are perfect squares.   The answer is that 324 is the square of 18 (i.e. $324 = 18^2$) and so we have the solution x = 18, y = 6.   Note that these values satisfy the equation $18^2 = 1.5 \times 6^3$.

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΓΛΦΨΣΠΔΓΩΦΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 4 (Draining a Tank)

Here is an interesting problem. Consider a tank with a square base and a height of 15 feet with a small hole at the bottom used for draining the tank. Suppose you know it takes 1 hour to fill the tank when the hole at the bottom is plugged. Suppose you also know that when the plug is removed and water is poured into the tank (we always pour water into the tank at the same rate), it takes 5 minutes for the water to rise to 10 inches. How long will it take to empty a full tank by removing the plug ?

Solution

Since the tank has a square base and a height of 15 ft, we know the volume of the tank is $15b^2$ ft^3, where b is the width of the base. And since the tank (the bottom plugged) can be filled in 1 hr (60 minutes), we have

$$15b^2 \text{ cubic ft} = (RATE\,IN \text{ cubic ft/min})\,(60 \text{ minute}) \qquad (1)$$

where $RATE\,IN$ is the rate (cubic ft/min) at which we pour water into the tank. We also know that when the plug is removed and water is poured into the tank, the water level rises 10 inches (5/6 ft) in 5 minutes, and so we have

$$\tfrac{5}{6}\,b^2 \text{ cubic feet} = 5\,(RATE\,IN) - 5\,(RATE\,OUT) \qquad (2)$$

where $RATE\,OUT$ is the rate at which water leaves the hole at the bottom of the tank. We can now eliminate b between equations (1) and (2), getting a relationship between the incoming water rate ($RATE\,IN$) and the outgoing rate ($RATE\ OUT$). We find

$$RATE\,IN = 3\,RATE\,OUT$$

and since it takes 1 hr to fill the tank (with the hole plugged), it takes 3 hrs to empty the tank by pulling the plug.

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΦΨΣΠΔΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 5 (Insect-Plague Problem)

An archeologist has deciphered an old scroll that says an ancient people suffered from 7 plagues of locusts that occurred at 8 year intervals. The scroll also said that after the 7th plague, the sum of the years when the plagues occurred was 420. The scroll didn't give the years of the plagues and so the archeologist never knew the exact dates. Can you tell the archeologist when the 13 plagues occurred ?

Solution

The years form an arithmetic sequence and so we know the year of the middle plague (4th plague) is the *average* of the 7 plague years, or 420/7 = 60. And since there are 7 plagues at 8 year intervals, we know the plague years are

1st plague = 60 - 3 (8) = 36 AD
2nd plague = 60 - 2 (8) = 44 AD
3rd plague = 60 - 1 (8) = 52 AD
4th (middle) plague = 60 AD
5th plague = 60 + 1 (8) = 68 AD
6th plague = 60 + 2 (8) = 76 AD
7th plague = 60 + 3 (8) = 84 AD

ΦΛΨΘΠΩΘΣΠΔϒΞΣΦΠΨΣΦΩΠΣΓΛΦΨΣΠΔΓΩΦΩΛΓΠΣΦΛΩΠΘΞϒΞΠ

Problem 6 (Missing Numbers Problem)

In the following multiplication, every letter and dot stands for one of the numbers 0, 1, 2, ... ,8, and 9 with identical letters for identical numbers. Fill in the letters and dots with numbers that make the multiplication correct.

$$
\begin{array}{rrrrrrrr}
 & & & & \mathbf{P} & \mathbf{Q} & \mathbf{Q} & \mathbf{P} \\
 & & & & \mathbf{R} & \mathbf{S} & \mathbf{T} & \mathbf{P} \\
\hline
 & & & & \mathbf{T} & \mathbf{Q} & \mathbf{Q} & \mathbf{T} \\
 & & \cdot & \cdot & \cdot & \mathbf{S} & \cdot & \\
 & \cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{R} & \mathbf{T}
\end{array}
\quad \times
$$

Solution

If you look carefully at the first multiplication of $PQQP$ by P, you see that $P^2 = T$ with *no* carry-over number, and so we can conclude that P = 0, 1, 2, or 3. But since $P^2 = T \neq P$ we can conclude $P = 2$ or 3. Also, there is no carry-over number when P is multiplied by Q and so $PQ = Q$ and hence $Q = 0$. But we can also conclude that $P \neq 2$ since if $P = 2$, we would have $PQQP \times P^2 = 2002 \times 4 = 8008$ (remember $P = P^2$), which means that $T = 8$, and contradicts the fact that T is a perfect square. Hence, we conclude that $P = 3$ and so $T = P^2 = 9$. Hence, we know

$$
\begin{array}{rrrrrrrr}
 & & & & \mathbf{3} & \mathbf{0} & \mathbf{0} & \mathbf{3} \\
 & & & & \mathbf{R} & \mathbf{S} & \mathbf{9} & \mathbf{3} \\
\hline
 & & & & \mathbf{9} & \mathbf{0} & \mathbf{0} & \mathbf{9} \\
 & & \mathbf{2} & \mathbf{7} & \mathbf{0} & \mathbf{2} & \mathbf{7} & \\
 & \cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{7} & \mathbf{9}
\end{array}
\quad \times
$$

from which we can determine the last letters S = 2 and $R = 7$. We can now fill in the rest of the dots and write

$$
\begin{array}{rr}
 & 3003 \\
 & 7293 \\
\times & \overline{} \\
 & 9009 \\
 & 27027 \\
 & 6006 \\
 & 21021 \\
\hline
 & 21900879
\end{array}
$$

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΦΨΣΦΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 7 (Magic Square Problem)

Place three numbers $a, b,$ and c in a 3×3 array as follows:

	$\boldsymbol{a}$	$\boldsymbol{b}$
$\boldsymbol{c}$		

Find the numbers in the empty boxes so the sum of the 3 rows, 3 columns, and two diagonals is always the same.

Solution

We label the 6 unknown entries in the table u, v, w, x, y, and z as shown in the following table:

$\boldsymbol{u}$	$\boldsymbol{a}$	$\boldsymbol{b}$
$\boldsymbol{v}$	$\boldsymbol{w}$	$\boldsymbol{x}$
$\boldsymbol{c}$	$\boldsymbol{y}$	$\boldsymbol{z}$

Our goal is to find equations these variables and then solve them in terms of the given numbers a, b, and c. There is more than one way to do this, but we set

Row 1 = Column 1: $u + a + b = u + v + c$

$$\therefore \quad \boldsymbol{v = a + b - c}$$

Row 2 = Anti-Diagonal: $v + w + x = c + w + b$

$$\therefore \quad \boldsymbol{x = c + b - v = c + b - (a + b - c) = 2c - a}$$

Row 2 = Column 2 : $v + w + x = a + w + y$

$$\therefore \quad \begin{aligned} \boldsymbol{y} &= \boldsymbol{(a + b - c) + (2c - a) - a} \\ &= \boldsymbol{-a + b + c} \end{aligned}$$

Diagonal = Row 3: $u + w + z = c + y + z$

$$\therefore \quad u + w = -a + b + 2c$$

Diagonal = Column 2: $u + w + z = a + w + (-a + b + c)$

$$\therefore \quad u + z = b + c$$

Diagonal = Column 1: $u + w + z = u + v + c = u + (a + b - c) + c$

$$\therefore \quad w + z = a + b$$

It is a simple matter now to solve these last three simultaneous equations for $u, w,$ and z, getting

$$u = \frac{-2a + b + 3c}{2}$$

$$w = \frac{b + c}{2}$$

$$z = \frac{2a + b - c}{2}$$

Hence, we have the table

$\frac{-2a+b+3c}{2}$	a	b
$a + b - c$	$\frac{b+c}{2}$	$2c - a$
c	$-a + b + c$	$\frac{2a+b-c}{2}$

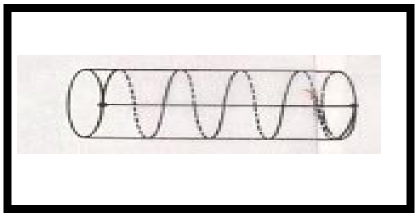
If you check, you will find that the sum of every row, every column, and the two diagonals has the same sum of

Common sum = $\frac{3b+3c}{2}$

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΦΨΣΠΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

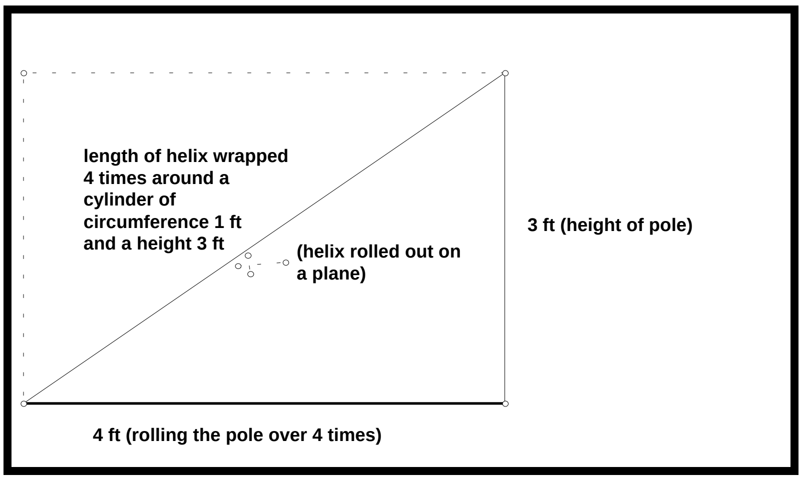
Problem 8 (Barber Pole)

A barber paints a thin strip of paint in the shape of a helix 4 times around a cylinder which has a height of 3 feet and circumference of 1 foot. If the strip starts and ends on the same cylindrical element (along a same line on the cylinder -- see diagram below), what is the length of the strip of paint ?



Solution

There is a very clever solution to this problem. Think of rolling the pole over on the plane ten times where the paint strip is still wet so it leaves a trace on the plane. It is clear that the paint traces out the hypotenuse of a right triangle with legs 3 feet (height of the pole) and $4 \times 1 = 4$ feet as shown below.



Hence, from the Pythagorean theorem, the length of the paint strip is

$$\textbf{Length of paint strip} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \textbf{ feet}$$

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΦΨΣΦΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 9 (The Smith Family)

Meet the Smith family. The family consists of Mr. and Mrs. Smith and two children, a boy and girl. Mr. Smith's age is a square number and the product of the digits in his age is his wife's age. The daughter's age is the sum of the digits of the father's age, and son's age is the sum of the digits of the mother's age. How old are the members of the Smith family ?

Solution

First of all, there aren't many possibilities for Mr. Smith whose age is a perfect square. They are 25, 36, 49, 64, and maybe (but highly unlikely) 81. (A young age of 16 is not feasible either, but biologically possible.) We take each case individually:

Father's Age	Mother's Age	Daughter's Age	Son's Age	
16	6	7	6	(impossible)
25	10	7	1	(mother too young)
36	18	9	9	(mother too young)
49	**36**	**13**	**9**	**(ok)**
64	24	10	6	(mother too young)
81	8	9	8	(impossible)

The only case that makes any (biological or legal) sense is the case when the father is 49, mother 36, daughter 13, and the son 9. In all other cases, the mother is either too young to have children, younger than the daughter, or too young to legally be married.

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΩΠΣΓΛΦΨΣΠΔΓΩΦΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 10 (The Five Neighbors Problem)

On a given block of a town there are :

1: Five houses of different colors in a row.

2: In each house lives a person of a different nationality.

3: Each owner likes a specific beverage, has one pet, and likes a certain basketball team.

4: No two owners like the same beverage, have the same pet, or like the same team.

Hints:

1. The English person lives in a red house.
2. The Swedish person has a dog.
3. The Danish person drinks tea.
4. The green house is on the left of the white house.
5. The person who lives in the green house drinks coffee.
6. The person who likes the Knicks basketball team has a bird.
7. The owner of the yellow house likes the Celtics basketball team.
8. The person living in the house right in the center drinks milk.
9. The Norwegian lives in the first house.
10. The person who likes the Lakers lives next to the person who has a cat.
11. The person who has a horse lives next to the person who likes the Celtics.
12. The owner who likes the Bulls basketball team drinks Coca Cola.
13. The German is a fan of the Pistons basketball team.
14. The Norwegian lives next to the blue house.
15. The person who likes the Lakers has a neighbor who drinks water.
16. The house in the middle is red.

What are the colors of the five houses, the nationality of the persons in the houses, the favorite team of the persons in the houses, and the favorite beverage of the persons in the houses ?

Solution

For problems like this, it always helps to organize things so one can see what is going on. Below, we have given the solution by specifying the five houses, the nationality of person in the house, the team favored by the person, the favorite beverage, and the household pet.

	House 1	**House 2**	**House 3**	**House 4**	**House 5**
COLOR	Yellow(4)	Blue (2)	Red(3)	Green(3)	White(3)
NATIONALITY	Norwegian (1)	Dane	English	German	Swedish
TEAM	Celtics	Lakers	Knicks	Pistons	Bulls
BEVERAGE	water	tea	milk	coffee	Coke
PET	cat	horse	bird	fish	dog

You can check yourself to see that all the hints of the problem are satisfied. The numbers indicate the order in which we filled in the words. For example, Hint 9 told us a Norwegian lived in the first house, which was the first blank we filled. Hint 14 told us the Norwegian lived next to a blue house, and hence we filled in the color BLUE in the second house. We then used other hints to fill in the colors of the houses. From then on it was pick and choose the hints so that all the blanks were filled.

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΦΨΦΩΛΓΠΣΦΛΩΠΘΞΥΞΠ