March Problems

Problem 1 (Arithmetic Sequence)

An arithmetic sequence has a difference between consecutive terms of 3, and the sum of squares of the first 1001 terms is equal to the sum of squares of the next 1000 terms. Find the first term in this sequence.

Solution

We use the well-known identities for adding the sum and the sum of squares of the first n natural numbers:

$$1+2+3+\ ...\ +\ (n-1)\ =\ \frac{n\,(n-1)}{2}$$
$$1\ +\ 2^2\ +\ 3^2\ +\ ...\ +\ (n-1)^2 =\ \frac{n\,(n-1)(2n-1)}{6}$$

We now call a_1, $a_2 =\ a_1 + d$, $a_3 = a_1 + d$, $a_4 = a_1 + d$, ... the elements of the unknown arithmetic sequence where $d = 3$ is the common difference in our sequence. (We leave this difference as d since it is actually less confusing than putting in a specific number.) Using the identity for adding the sum of squares of the first n natural numbers, we have

$$\begin{aligned} \mathrm{S}_n &=\ a_1^2\ +\ a_2^2 +\ ...\ +\ a_n^2 \\ &=\ a_1^2 +\ (a_1+d)^2 +\ (a_1+2d)^2 +\ ...\ +\ (a_1+(n-1)d)^2 \\ &=\ a_1^2 +\ (a_1^2 +\ 2a_1 d + d^2)\ +\ (a_1^2 +\ 2\cdot 2\,a_1 d + 4d^2) +\ ... \\ &\qquad\qquad +\ (a_1^2 +\ 2(n-1)a_1 d + (n-1)^2 d^2) \\ &=\ na_1^2 +\ 2\,[1+2+...\ +\ (n-1)]\,a_1 d\ +\ [1+2^2+...\ +\ (n-1)^2]\,d^2 \\ &=\ na_1^2 +\ n(n-1)\,a_1 d +\ \frac{n(n-1)(2n-1)}{6}\,d^2 \end{aligned}$$

If we now substitute our common difference $d = 3,$ we have

$$S_n =\ na_1^2\ +\ 3n(n-1)a_1 +\ \frac{3}{2}\,n\,(n-1)(2n-1)$$

According to our condition, the sum of squares of the first 2001 elements equals twice that of the first 1001 terms. That is $S_{2001} =\ 2\,S_{1001}$. Hence, we have a quadratic equation with huge coefficients:

$$2001\,a_1^2 +\ 12,006,000\,a_1 + 24,018,003,000$$
$$=\ 2002\,a_1^2 + 6,0076,000 a_1 + 6,009,003,000$$

which can be rewritten as

$$a_1^2 -\ 6,000,000\,a_1\ -\ 18,009,000,000\ =\ 0$$

Although the coefficients are large, we can still solve this equation getting $a_1 = -3000$ and $a_1 = 6,003,000$, of which we keep only the positive root of 6,003,000. Hence, the sequence is 6,003,000, 6,003,003, 6,003,006, ...

Note: This problem brings out the value of *computer algebra systems* (CAS) like *Mathematica* and *Maple* that would carry out all the routine, but horribly complicated, algebraic manipulations found in this problem (even with the very large numbers).

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Problem 2 (Strange Number)

What positive integer n satisfies the condition that 5^n is preceded by exactly $7,812,000$ numbers that have no common divisor with 5^n .

Solution

The solution is based on the following interesting property of numbers which get to the heart of the meaning of multiplication:

$5^0 = 1$ number less than or equal to 5 is divisible by 5
5^1 numbers less than or equal to $5^2 = 25$ are divisible by 5
5^2 numbers less than or equal to $5^3 = 125$ are divisible by 5
... ...
5^{n-1} numbers less than or equal to 5^n are divisible by 5.

Hence, we have that

there are $5^2 - 5 = 4 \cdot 5$ numbers $\leq 5^2$ *not* divisible by 5
there are $5^3 - 5^2 = 4 \cdot 5^2$ numbers $\leq 5^3$ *not* divisible by 5
... ...
there are $5^n - 5^{n-1} = 4 \cdot 5^{n-1}$ numbers $\leq 5^n$ *not* divisible by 5

Thus our problem seeks the value of n which satisfies

$$5^n - 5^{n-1} = 4 \cdot 5^{n-1} = 7,812,500$$

or

$$5^{n-1} = 1,953,125 = 5^9$$

Hence, we have $n - 1 = 9,$ or $n = 10.$

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Problem 3 (Unusual Trigonometric Identity)

Find the value of $tan\, x$ when

$$tan\, x = cos\, x$$

Solution

Using the definition $tan = \frac{sin\, x}{cos\, x}$ ($x \neq \frac{\pi}{2} + k\pi$, k an arbitrary integer), we can write $sin\, x = cos^2 x$, and letting $cos^2\, x = 1 - sin^2 x$ we get a quadratic equation in $sin\, x$:

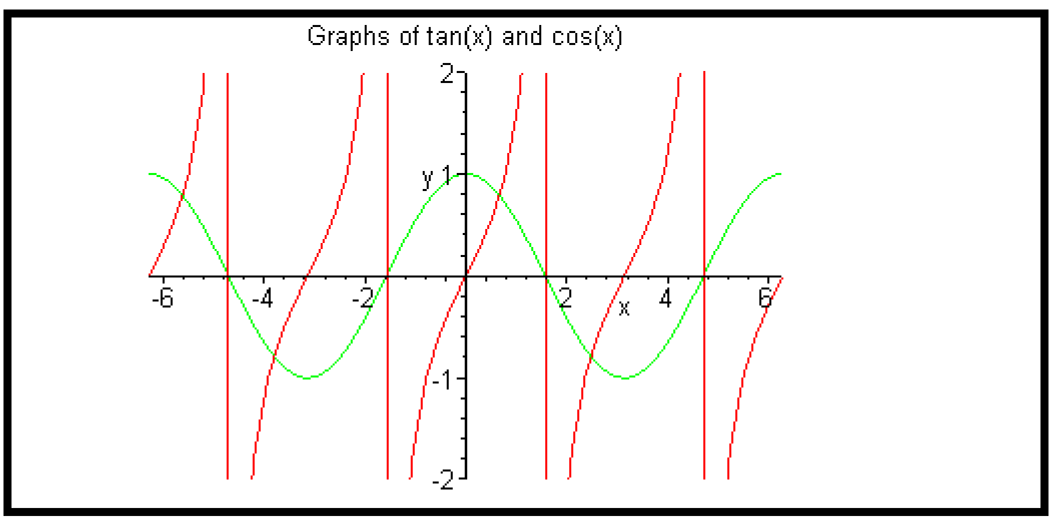
$$sin^2 x + sin\, x - 1 = 0$$

which has the two real solutions $sin\, x = \frac{-1 \pm \sqrt{5}}{2}$. But since $|\, sin\, x\,| \leq 1$, we keep only

$sin\, x = \frac{\sqrt{5} - 1}{2} \approx 0.618$, which in the interval $[0,\ 2\pi]$ we have that $x = sin^{-1}(0.618) \approx 0.66$ radians or $\pi - 0.66 \approx 2.47$ radians ($i.e.$ 38 degrees or 142 degrees). But $cos^2 x = sin\, x$ and so $cos\, x = \pm\sqrt{\frac{\sqrt{5}-1}{2}}$, and

$$tan\, x = \frac{sin\, x}{cos\, x} = \pm\sqrt{\frac{2}{\sqrt{5}-1}} = \pm\sqrt{\frac{\sqrt{5}+1}{2}} \quad (\approx \pm 0.786)$$

We can get a rough idea where in the interval $0 < x < 2\pi$ the curves $y = tan\, x$ and $y = cos\, x$ intersect. (See the following figure.) Note that the two curves intersect when $x \approx 0.66$ radians (38 degrees) and when $x \approx 2.47$ radians (142 degrees).



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Problem 4 (Rainy Summer Vacation)

In the course of a summer vacation, it rained on 7 days, either in the morning or the afternoon. If it rained in the morning, then it did not rain in the afternoon. There were a total of 5 rainless mornings and 6 rainless afternoons. How many days did the vacation last ?

Solution

Let n be the number of rainy mornings and hence $7-n$ is the number of rainy afternoons. We draw the following table to organize our thoughts.

*	**Mornings**	**Afternoons**
Rainy	n	$7-n$
No Rain	5	6
Total	$n+5$	$13-n$

Now, since the number of mornings is the same as the number of afternoons, we have $n+5=13-n,$ and so the number of rainy mornings is $n=4$, and thus the length of the vacation is $n+5=4+5=9$ days. (We could have also found this by computing $13-n=13-4=9.$)

Now, you might argue that the number of mornings is not equal to the number of afternoons since the vacation may start in a morning and end in a morning. (Or it could start in an afternoon and end in an afternoon.) But in either case the difference between the number of mornings and afternoons is always 1 in absolute value. Hence, we can write $|\,n+5-(13-n)\,|=1$ or $|\,2n-8\,|=1,$ which has no integer solution for n. In other words, the number of mornings and afternoons must be the same in this problem.

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Problem 5 (Mystery Triangle)

We are given an unusual triangle with the following properties

$i)$ the perimeter of the triangle is an integer as well as the area of the triangle

$ii)$ if we were to list the lengths of the 3 sides of the triangle from shortest to longest length, followed by the area of the triangle, we would get four terms of an arithmetic sequence

Determine the three sides and the area of the triangle.

Solution

We apply the general Heron formula for the area of a triangle, which is

$$A = \sqrt{s\,(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$, and $a, b, c,$ are the lengths of the sides. We can simplify matters, however, by calling a the middle size side of the triangle and d the common difference of the arithmetic sequence relating the three sides which we now can label as $a-d$, a, $a+d$ (some problems are hard using one notation become simple if the notation is chosen in the right way). Substituting these values into Heron's formula, we get the area of the triangle in terms of a and d to be

$$A = \sqrt{\frac{(a-d)+a+(a+d)}{2} \cdot \frac{-(a-d)+a+(a+d)}{2} \cdot \frac{(a-d)-a+(a+d)}{2} \cdot \frac{(a-d)+a-(a+d)}{2}}$$

$$= \sqrt{\frac{3a^2\,(a+2d)(a-2d)}{16}}$$

But the problem states that the area is the 4th term in the arithmetic sequence and so we have

$$a+2d = \sqrt{\frac{3a^2\,(a+2d)(a-2d)}{16}}$$

or

$$(a+2d)^2 = \frac{3a^2\,(a+2d)(a-2d)}{16}$$

And canceling the common factor $a+2d$, we find

$$d = \frac{3a^3-16a}{2\,(3a^2+16)}$$

But d must be an integer and so the numerator $3a^3 - 16a$ must be an even integer, which holds only if a itself is even and hence can be written in the form $a = 2x$. Substituting this value into the formula for d, and solving for d gives (after a little algebra)

$$d = \frac{3x^3 - 4x}{3x^2+4} = x - \frac{8x}{3x^2+4}$$

Now, in order that d be an integer, we must have $8x > 3x^2 + 4$, which implies $x(8 - 3x) \geq 4$ which implies $3x < 8$ or $x < 3$. Hence, we must have $x = 1$ or 2. But $x = 1$ makes d negative, so we have $x = 2$, which gives $a = 4$, $d = 1$. In other words, the sides of the triangle $a - d, a, a + d$ are 3, 4, and 5, and the area is 6.

Note that we have found one of the simplest triangles in geometry; the simple 3 : 4 : 5 right triangle.

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Problem 6 (Greatest Integer Function)

The greatest integer $[\,x\,]$ of a non negative real number x is defined to be the greatest integer less than or equal to the number. For example, we have [5.5] = 5, [4.2] = 4, [3] = 3, and so on. Show that the quantity

$$[u + v] - [\,u\,] - [\,v\,]$$

is always 0 or 1.

Solution

We let u and v be arbitrary real numbers. Hence we can write

$u = [\,u\,] + r_1$ where r_1 is the fractional part of u
$v = [\,v\,] + r_2$ where r_2 is the fractional part of v

But the integer part of $u + v$, i.e. $[\,u + v\,]$, is the sum of the integer parts of u and v, i.e. $[\,u\,] + [\,v\,]$, plus the integer part of $r_1 + r_1$, i.e. $[\,r_1 + r_2\,]$. That is

$$[\,u + v\,] = [\,u\,] + [\,v\,] + [\,r_1 + r_2\,]$$

Therefore, we have

$$[\,u + v\,] - [\,u\,] - [\,v\,] = [\,r_1 + r_2\,]$$

But we know $[\,r_1 + r_2\,]$ is either 0 or 1, which solves the problem.

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Problem 7 (Two Operations Problem)

We can perform two operations on a number; doubling and adding 1. What is the least number of operations that can transform the number 0 into 100 ? What is the least number of operations that transform 0 into an arbitrary positive integer n ?

Solution

Given the final number of 100, we find the smallest number of steps by going backwards from 100 to zero by using the inverse operations of subtracting 1 or by dividing by 2. The strategy is to divide by 2 when a number is even and subtract 1 when a number is odd. Clearly, this strategy will have the smallest number of steps. For example, starting at 100, we have the numbers 100, 50, 25, 24, 12, 6, 3, 2, 1, 0, which we perform in 9 steps. We now go from 1 to 100 in the opposite order getting 0, 1, 2, 3, 6, 12, 24, 25, 50, 100. This same process can be carried out for any positive integer. For example, to transform 0 into 235, we first construct the sequence 235, 234, 117, 116, 58, 29, 28, 14, 7, 6, 3, 2, 1, 0. Hence, we can go from 0 to 235 in the opposite order. Basically, the strategy is to get on the "highway" of numbers that lead to n by repeated doubling. An easy number would be 256 since then after adding our initial one, we could then just keep doubling. We would have 0, 1, 2, 4, 8, 16, 32, 64, 128, 256.

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Problem 8 (Tricked You!)

Show that if the top 26 cards of a deck 52 playing cards has more red cards than there are black cards in the bottom 26 cards, then there are in the deck at least 5 consecutive cards of the same color! Believe it or not!!

Solution

A statement of the form: *if A then B* is a true statement logically if the premise A is *false*. For example, the nonsense statement: if the moon is made of blue cheese, then $1+1=3$ is a true statement regardless of whether $1+1=3$ is true. In our problem, the premise that the top 26 cards of a deck of 52 playing cards has more red cards than there are black cards in the bottom 26 cards is a false statement (they are always the same) and so the overall statement is true.

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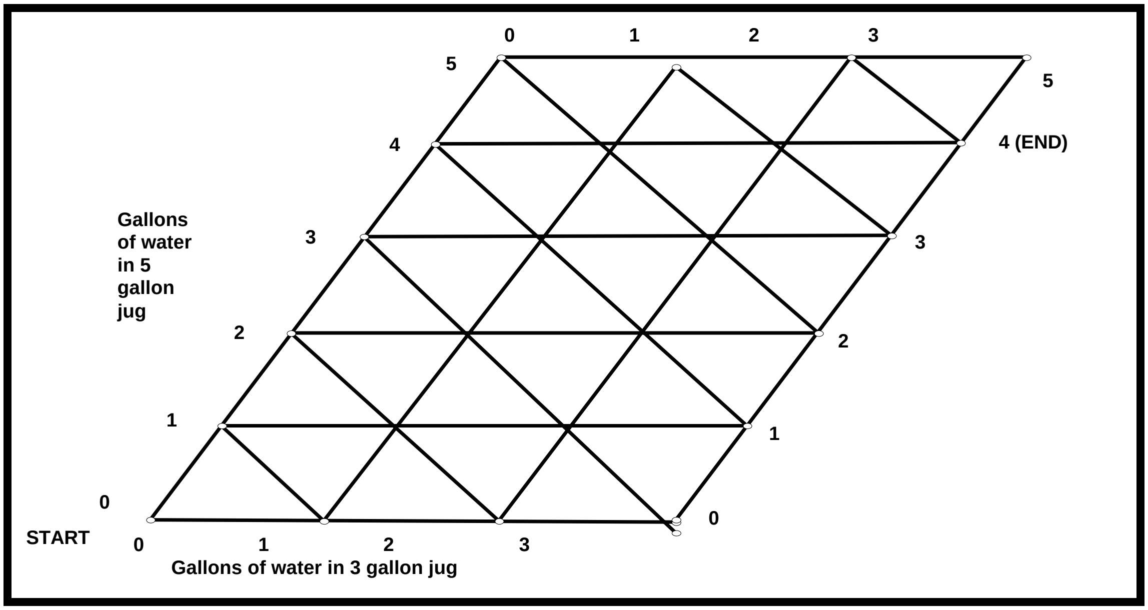
Problem 9 (The Jug Problem)

Harry is given a 3- and a 5-gallon jug. How can Harry measure off 4 gallons of water from a jug of 8 gallons ?

Solution

There is a very clever way to solve this problem by using the diagram drawn below. The idea is to start at the point (0,0) at the lower left of the diagram, which means that initially there is no water in either of the 3 or 5 gallon jugs. We then think of the diagram as a drawing of a funny looking billiard table and think of a ball rolling on the table until it hits one of the walls, where it bounces off in the usual reflective manner, moving in another line until it hits another wall. The interesting thing (and this is the real clever part) is that *each point* of contact (or the coordinates) represents *one step* in the process of pouring water from one jug into another. If the ball ever ends up at the point (3,4), this means the small jug contains 3 gallons of water (first coordinate), and the large jug (second coordinate) contains 4 gallons of water, and hence we have measured off 4 gallons of water like the problem requests. So, we start at (0,0) and move either to the right or upward (we have decided to move in the upwards direction), then we get :

Small Jug (3 gallons)	**Large Jug (5 gallons)**	**Physical Operation**
0	0	Start
0	5	Fill the large jug
3	2	Fill the small jug
0	2	Empty the small jug
2	0	Pour from the large to small jug
2	5	Fill the large jug
3	4	Pour from large to small jug (the large jug now has 4 gallons)



This diagram was originally reported in *Scientific American* by Martin Gardiner. You could physically simulate the pouring of water from one jug into another by drawing two squares and putting pennies in each square (one square can hold a maximum of 3 pennies, the other a maximum of 5 pennies).

Note: You can also ask if you can reach the point (3,4) by starting at (0,0) and moving to the right, hitting (3,0) and then reflecting to (0,3), then to (3,3), and so on. You may also wonder if you can solve more complicated problems with different sized jugs. This is a very clever diagram and shows how difficult problems can be solved if a person is clever enough.

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Problem 10 (So You Think You Know Algebra Huh ?)
Evaluate

$$\left(\frac{1\cdot2\cdot4\ +\ 2\cdot4\cdot8\ +\ 3\cdot6\cdot12\ +\cdots}{1\cdot3\cdot9\ +\ 2\cdot6\cdot18\ +\ 3\cdot9\cdot27+\cdots}\right)^{1/3}$$

Solution

This problem is quite simple if you keep your eyes open for common factors. Factoring out $1 \cdot 2 \cdot 4$ and $1 \cdot 3 \cdot 9$ in the numerator and denominator, respectively, we can write

$$\left(\frac{1\cdot2\cdot4\ +\ 2\cdot4\cdot8\ +\ 3\cdot6\cdot12\ +\cdots}{1\cdot3\cdot9\ +\ 2\cdot6\cdot18\ +\ 3\cdot9\cdot27+\cdots}\right)^{1/3} = \left(\frac{1\cdot2\cdot4\,(1+2^3+3^2+\ldots)}{1\cdot3\cdot9\,(1+2^3+3^3+\ldots)}\right)^{1/3} = \left(\frac{8}{27}\right)^{1/32} = \frac{2}{9}$$

Note: One might argue that the above expression does not exist since the two infinite series, although they cancel each other, both diverge. What do you think ? Hmmmmmmm.

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