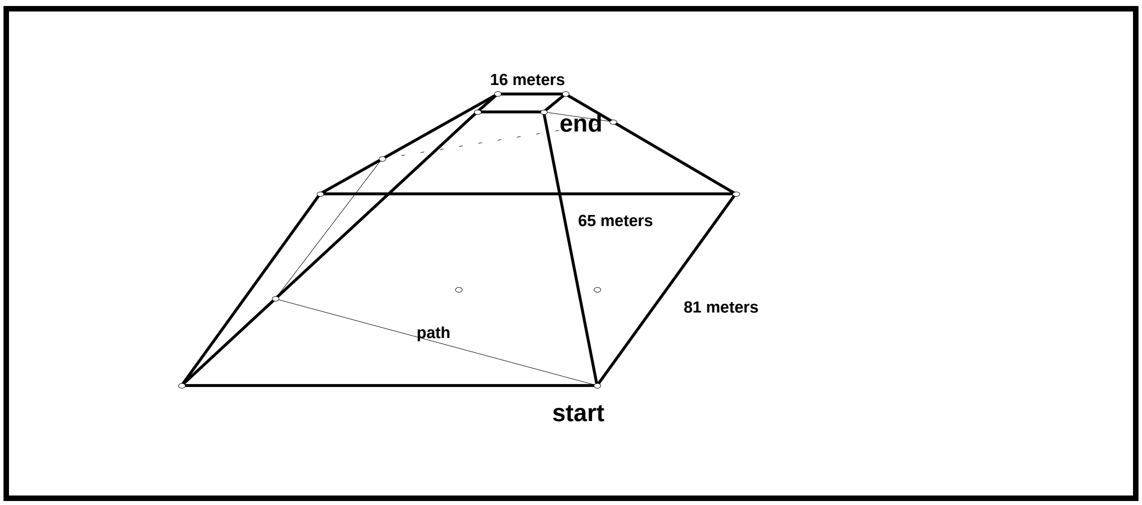
November Solutions

Problem 1 (Aztec Pyramid)

Consider an Aztec pyramid that has a square bottom and square top (like an Egyptian pyramid but with a square top). The base of the pyramid is a square with dimensions 81 by 81 meters, and the top is a square with dimensions 16 by 16 meters. The four edges that run up the sides of the pyramid are each 65 meters long. We wish to design a stairway that starts on the ground at one of the four corners of the pyramid and corkscrews up and around the four sides in such a way that it goes up at a constant rate until it reaches the top on the same edge above where it started.

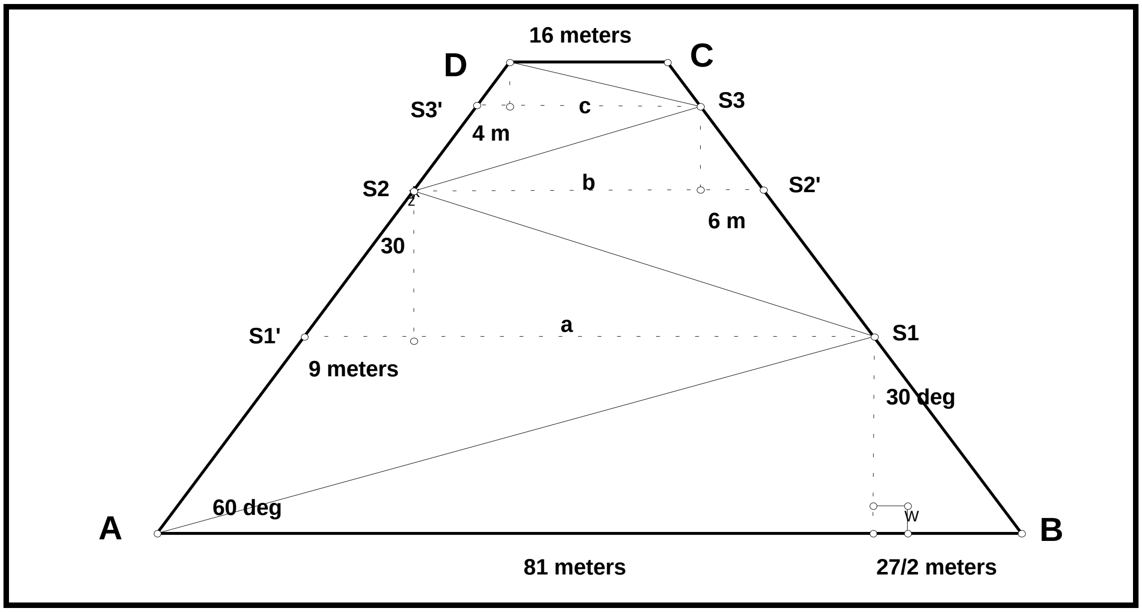


Aztec Pyramid with square bottom and top

How high up along the edges of the pyramid are the points where the stairs goes around the three corners eventually reaching the top at a height of 65 meters ?

Solution

The following diagram shows the how the strairs move up the faces of the four faces of the pyramid showing the points S_1, S_2, S_3, and D where the stairs goes around the corners. The point D is the top point.



Path up the pyramid

A simple calculation shows us that the base angle of the pyramid is 60° . If we also observe that the triangles $\triangle(ABS_1)$, $\triangle(S_1S'S_2)$, $\triangle(S_2S_2'S_3)$, $\triangle(S_3'S_3D)$ in the diagram are similar (they have equal angles), we have the following properties:

$$\frac{81}{a} = \frac{a}{b} = \frac{b}{c} = \frac{c}{16}$$

or

$$a^2 = 81ab \qquad b^2 = ac \qquad c^2 = 16b$$

or

$$\frac{a^2}{c^2} = \frac{81}{16}$$

Hence,

$$\frac{a}{c} = \frac{9}{4} \qquad b^2 = \frac{9}{4}c^2 \qquad b = \frac{3}{4}c$$

or

$$c = 24 \qquad b = 36 \qquad a = 54$$

From these values, we can find the four differences

$$AB - S_1S' = 81 - 54 = 27$$
$$S_1S_1' - S_2S_2' = 54 - 36 = 18$$
$$S_2S_2' - S_3S_3' = 36 - 24 = 12$$
$$S_3S_3' - CD = 24 - 16 = 8$$

from which we can use the basic properties of a 30-60 right triangle to conclude

$$
\begin{aligned}
BS_1 &= 27 \\
S_1S_2 &= 18 \\
S_2'S_3 &= 12 \\
S_3D &= 8
\end{aligned}
$$

$-------------$

Total 65 meters

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΦΨΣΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 2 (Birthday Problem)

Starting on their fourth birthdays, a brother and sister each receive books on their birthdays, always getting as many books as their age. How old will the two siblings be when they have received a total of 100 books between them over their lifetimes ?

Solution

We make a table showing the total number of books each sibling accumulates as a function of age.

birthday	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...
books	0	0	0	4	9	15	22	30	39	49	60	72	85	99	100+

Since the total number of books accumulated by both siblings is the sum of two numbers in the second row of the table, we see that the only two numbers which sum to 100 are 15 and 85, which implies the siblings are 6 and 13 years old.

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΓΩΦΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 3 (Friday-the-13th Problem)

This month (November, 1998) the 13th falls on Friday, making it the third Friday-the-13th this year (February and March of this year also had Friday-the-13ths). When will Friday-the-13th fall during the next two years in 1999 and 2000 ? Keep in mind that the year 2000 is a leap year.

Solution

The number of days in a month is 30 or 31, with the exception of February, which has 28 days in a non leap year, and 29 in a leap year. Leap years occur every 4th year (years divisible by 4) with the exception of the first year in every century (century years), which are *not* leap years, unless they are divisible by 400. In other words, 1700, 1800, and 1900 were not leap years, but 2000 will be a leap year, 2100, 2200, 2300 will not be leap years, but 2400 will be a leap year. Our Gregorian calendar repeats itself every 400 years, and so using this strategy for leap years, we have 97 leap years every 400 years, which means we are saying the number of days in a year averages out to $365 \frac{97}{400} = 365.2425$, which is close to the real number of days in a year as determined by astronomers.

But we digress, to find the next Friday-the-13th after the present November Friday-the-13th, we know that a 30 day month is 4 weeks and 2 days, and a 31 day month is 4 weeks and 3 days. Hence, in a 30-day month, the day of the week the 13th occurs on the *next* month is two days *later* (like from a Friday to a Sunday), and in a 31 day month, the day of the week the 13th occurs on the next month is *three* days later (like from a Friday to a Monday). So, we make the following table to illustrate the total number of day's lag following the Friday-the-13th in November, 1998 and the day in every month on which the 13th falls. We look for the first month following the Friday-the-13th when the total day's lag is a multiple of 7, like a 7, 14, 21, 28, ... After we hit a multiple of 7, we know that month will be a Friday-the-13th month, and so we start counting the days lag from that point until we get the next multiple of 7. We see that during the years 1999 and 2000, there will be two Friday-the-13ths, one in August, 1999 and the other in October of 2000.

1998	**Days**	**Lag**
Jan	31	3
Feb	28	0
March	31	3
April	30	2
May	31	3

1999	**Days**	**Lag**	
Jan	31	3	5 Wed
Feb	28	0	8 Sat
March	31	3	8 Sat
April	30	2	11 Tue
May	31	3	13 Thur

2000	**Days**	**Lag**		
Jan	31	3	(13)	Th
Feb	29	1	(16)	Sun
Mar	31	3	(17)	Mon
April	30	2	(20)	Th
May	31	3	(22)	Sat

June	30	2		June	30	2	16 Sun	June	30	2	(25)	Tue
July	31	3		July	31	3	18 Tue	July	31	3	(27)	Th
Aug	31	3		**Aug**	**31**	**3**	**21 Fri**	Aug	31	3	(30)	Sun
Sept	30	2		Sept	30	2	3 Mon	Sept	30	2	(33)	Wed
Oct	31	3		Oct	31	3	5 Wed	**Oct**	**31**	**3**	**(35)**	**Fri**
Nov	**30**	**2**	**0 Fri**	Nov	30	2	8 Sat	Nov	31	3	(3)	Mon
Dec	31	3	2 Sun	Dec	31	3	10 Mon	Dec	31	3	(5)	Wed

If you continue this analysis, you will discover that in a non leap year, the different possibilities for Friday-the-13ths in one year is one of the following:

{Jan, Oct}
{Feb, March, Nov}
{April, July}
{May}
{June}
{August}
{Sept, Dec}

In the case of a leap year, the months when a Friday-the-13th occurs will be one of the seven possibilities:

{Jan, April, July}
{Feb, August}
{March, Nov}
{Sept, Dec}
{May}
{June}
{Oct}

The year 1998 was interesting since it had three Friday-the-13ths, which is the most number of Friday-the-13ths that can occur in a year. You will also discover that every year has at least one Friday-the-13th. If you were really ambitious, you could write a computer program and determine the occurances of Friday-the-13th over the next 400 years.

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΦΨΣΠΔΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 4 (Ali Baba Problem)

Ali Baba (from the Arabian Nights) has returned from his cave, which is full of gold and platinum, where he wants to fill his sack. His sack can hold at most 200 pounds of gold or 40 pounds of platinum. The maximum total weight he can carry in his sack is 100 pounds. He wants to use the gold and platinum to buy camels, where he can purchase 20 camels for every pound of gold, and 60 camels for every pound of platinum. How many pounds of gold and platinum should Ali Baba put in the sack to obtain the most number of camels ?

Solution

Let $g \geq 0$ be the number of pounds of gold Ali Baba puts in his sack, and $p \geq 0$ be the number of pounds of platinum he puts in his sack. Since his sack can hold at most 200 pounds of gold, we have that $g \leq 200$, and since his sack can hold at most 40 pounds of platinum, we have $p \leq 40$. And finally, since the total amount of gold and platinum his sack can hold is 100 pounds, we have that $g + p \leq 100$. Now, since Ali Baba can buy 20 camels for a pound of gold, and 60 camels for a pound of platinum, we the number of camels C Ali Baba can buy is $C = 20g + 40p$.

Putting all this together, we see g and p that maximize the quantity

$$C = 20g + 60p$$

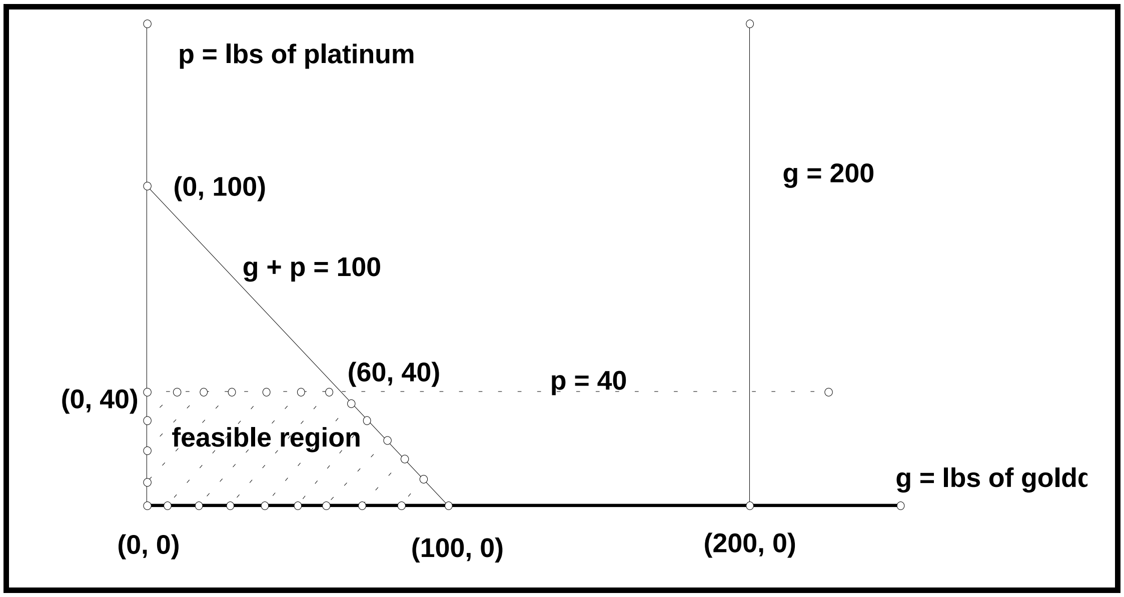
subject to the constraints

$$\begin{aligned} &g \leq 200 \\ &p \leq 40 \\ &g + p \leq 100 \end{aligned}$$

and

$$\begin{aligned} &g \geq 0 \\ &p \geq 0 \end{aligned}$$

But this is a basic linear programming problem that can be solved by the following diagram.



In the diagram, the shaded region consists of all those values of g and p that satisfy the five inequalities

$$g \leq 200 \qquad g \geq 0$$
$$p \leq 40 \qquad p \geq 0$$
$$g + p \leq 100$$

The values of g and p are the feasible number of pounds of gold and platinum that meet Ali Baba's requirements. To find the exact values of g and p that satisfy these inequalities and at the same time maximize the number of camels Ali Baba can buy, we resort to a basic result from linear programming, which states the *maximum point* (g^*, p^*) lies at one of the *corner points*. In this case, the corner points are

$$(g, p) = (0, 0)$$
$$(g, p) = (100, 0)$$
$$(g, p) = (60, 40)$$
$$(g, p) = (0, 40)$$

Evaluating the number of camels Ali Baba can buy with this many pounds of gold and platinum, we have

$$(g, p) = (0, 0) \Rightarrow C = 20g + 60p = 20(0) + 60(0) = 0 \text{ camels}$$
$$(g, p) = (100, 0) \Rightarrow C = 20g + 60p = 20(100) + 60(0) = 2000 \text{ camels}$$

$$(g,p) = (60, 40) \Rightarrow C = 20g + 60p = 20(60) + 60(40) = 3600 \text{ camels}$$
$$(g,p) = (0, 40) \Rightarrow C = 20g + 60p = 20(0) + 60(40) = 2400 \text{ camels}$$

Hence, Ali Baba should pack his sack with 60 pounds of gold and 40 pounds of platinum in his bag, which will allow him to buy 3600 camels.

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΠΔΓΩΦΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 5 (Clock Problem)

At what time between 3 *P.M.* and 4 *P.M* are the two hands of the clock pointing in exactly the same direction ?

Solution

Suppose we let T be the time in minutes past 3 o'clock. In other words, T minutes past 3 o'clock, the minute hand points to T minutes on the clock, and the hour hand points to $\frac{1}{12}T+15$ minutes. (The hour hand starts 15 minutes ahead of the minute hand, but it goes only 1/12th as fast.) Hence, the two hands point in the same direction when these values are the same, or

$$\boldsymbol{T = \frac{1}{12}T + 15}$$

Solving this equation, gives

$$T = \frac{12}{11} \times 15 = 16\,\frac{4}{11} \approx 16.363636\ldots$$

In other words, 16.363636... minutes past 3. Converting to seconds, this is approximately 16 minutes and 22 seconds past 3 o'clock.

Note: You could also find the time between 3 o'clock and 4 o'clock when the two hands point in the *opposite* direction. Then, the next problem is determine the number of times in a day when the hands of the clock point in the same direction. Most people do not get this problem right.

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΔΓΩΦΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 6 (Algebra Problem)

Solve

$$\frac{6}{\sqrt{x-8}-9}+\frac{1}{\sqrt{x-8}-4}+\frac{7}{\sqrt{x-8}+4}+\frac{12}{\sqrt{x-8}+9}=0$$

Solution

If we let $y = \sqrt{x-8}$ and rearrange the equation, we get

$$\left(\frac{6}{y-9}+\frac{12}{y+9}\right)+\left(\frac{1}{y-4}+\frac{7}{y+4}\right)=0$$

or

$$\frac{6\,(3y-9)}{y^2-81}+\frac{8y-24}{y^2-16}=0$$

or

$$\frac{2\,(y-3)\,[9(y^2-12)+4\,(y^2-81)\,]}{(y^2-81)\,(y^2-16)}=0$$

The roots of this equation are the roots of the numerator that are not the roots of the denominator (which are $\pm 9,\ \pm 4$). Simplifying the numerator, we find the roots of the numerator to be $y = 3,\ \pm 6.$ Hence, we have

$y = 3 \Rightarrow x = y^2 + 8 = 17$
$y = 6 \Rightarrow x = y^2 + 8 = 44$
$y = -6$ (which is not a root since $y = \sqrt{x-8}$ must be nonnegative)

Hence, the roots are $x = 17$ and 44.

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΨΣΠΔΓΩΦΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 7 (Bonnie and Clyde)

Two bandits, Bonnie and Clyde, divide a chest of gold coins between them as follows: One for Bonnie, two for Clyde, three for Bonnie, four for Clyde, and so on until they run out of coins. If they get to "20" for Clyde" but there are less than 20 coins, the game stops and Clyde gets the remaining coins. Likewise for Bonnie, if she is the last player, she gets the remaining coins. Who is better off, Bonnie or Clyde ?

Solution

It is clear that the person who collects the most coins depends on how many coins are in the chest. We can approach this problem by simply tabulating the number of coins both Bonnie and Clyde take at each step. We have

		Step		
Bonnie	$\leftarrow$	1		
		2	$\rightarrow$	**Clyde**
Bonnie	$\leftarrow$	3		
		4	$\rightarrow$	**Clyde**
Bonnie	$\leftarrow$	5		
		6	$\rightarrow$	**Clyde**
		.		
Bonnie	$\leftarrow$	$2n$-1		
		$2n$	$\rightarrow$	**Clyde**
		$\dots$		

On the nth turn Bonnie says "I'll take $2n$-1 coins", giving her a *total* take of

$$\mathbf{1 + 3 + 5 + \cdots + (2\boldsymbol{n}\text{-}1) = \boldsymbol{n}^2} \qquad \text{(\textbf{Bonnie's take})}$$

coins. Now let's assume that on her n *th* turn Bonnie takes her $2n$- 1 coins, but leaves less than $2n$ coins for Clyde, which means Clyde takes the remainder of coins r, where $0 \le r < 2n$. This means Clyde's total take of coins is

$$\mathbf{2 + 4 + 6 + \cdots + (2\boldsymbol{n} - 2) + \boldsymbol{r} = \boldsymbol{n}^2 - \boldsymbol{n} + \boldsymbol{r}} \qquad \text{(\textbf{Clyde's take})}$$

Comparing these two quantities, we conclude that

- Bonnie ends up with more coins than Clyde if $n > r$

- Bonnie and Clyde end up with the same amount if $n = r$
- Clyde ends up with more coins if $n < r$

where n is the total number of times each player has taken coins from the chest.

Now, let's consider the second situation when on his nth turn, Clyde takes $2n$ coins from the chest, but leaves only a remainder of r coins for Bonnie, where $0 \leq r < 2n+1$. In this case, Clyde's total take is

$$\mathbf{2 + 4 + 6 + \ldots + 2n = n^2 + n} \qquad \textbf{(Clyde's take)}$$

whereas Bonnie's take is

$$\mathbf{1 + 3 + 5 + \ldots + (2n\text{-}1) + r = n^2 + r} \qquad \textbf{(Bonnie's take)}$$

Comparing these two quantities, we conclude that

- Bonnie ends up with more coins than Clyde if $r > n$
- Bonnie and Clyde end up with the same amount if $n = r$
- Clyde ends up with more coins if $r < n$

In conclusion, we see that the person who picks the last remainder of coins ends up with the most coins if and only if the remainder of coins is greater than the number of times the other person has taken coins from the chest. When the remainder of coins is the same as the number of turns of each player, the two players end up with the same number of coins.

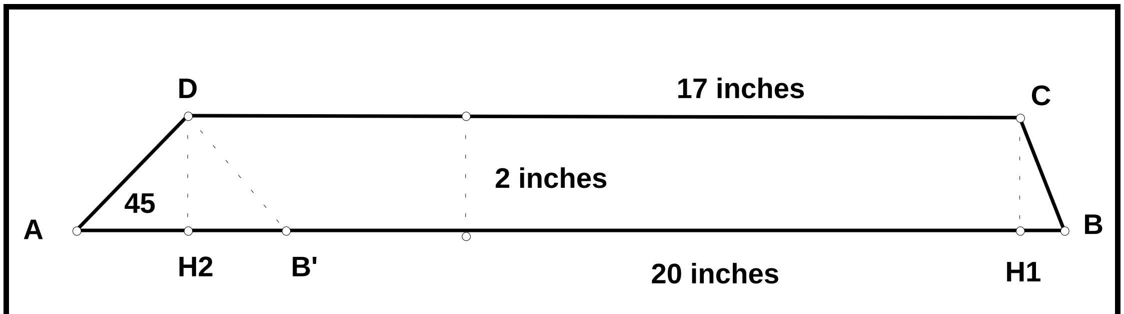
For example, suppose Bonnie takes 7 coins on her $n = 4$th turn, but leaves $r = 3$ coins for Clyde. We know immediately that Bonnie ends up with more coins. To check this, we know that Bonnie's total take is

Bonnie's take = 1 + 3 + 5 + 7 = 16
Clyde's take = 2 + 4 + 6 + 3 = 15

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΦΨΣΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 8 (Trapezoid Problem)

Consider the following trapezoid with parallel sides of 20 inches and 17 inches, and an altitude of 2 inches, where the angle at vertex A is 45 degrees.

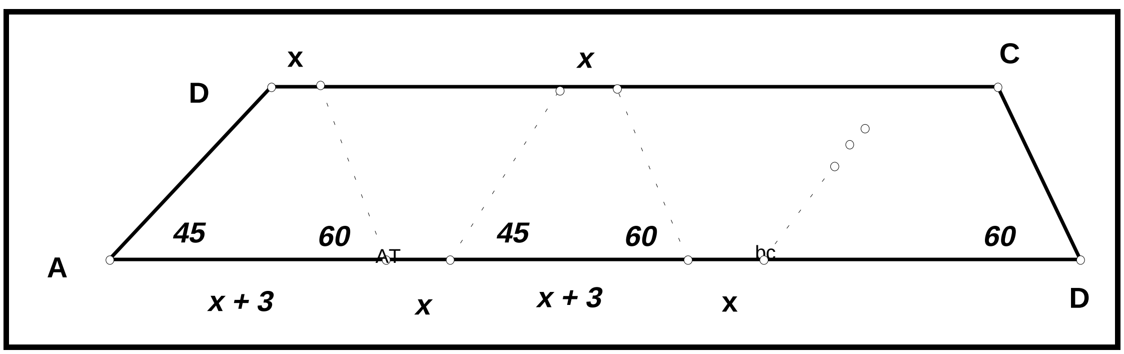


Trapezoid

Can you cut this trapezoid into smaller congruent figures ?

Solution

Since $\angle\ DAH_2 = 45°$ and the altitude of the trapezoid is 2, we have that $AD = AH_2 = 2$ inches. Hence, we can conclude that $H_1B = 1$, which makes the angle $\angle\ CBH_1 = 60°$. Hence, there appears no hope in subdividing the trapezoid into congruent *triangles* since $\triangle(ADH_2)$ has equal legs whereas $\triangle(ABH_1)$ is a 30-60 degree right triangle, and it is not difficult to show neither triangle can be broken into smaller congruent regions. Hence, we attempt to subdivide the trapezoid into smaller congruent trapezoids as illustrated in the following figure.



Breaking the larger trapezoid into smaller congruent trapezoids

One thing is immediately clear at this point, if we *can* subdivide the trapezoid into smaller congruent trapezoids, the number of these trapezoids must be an odd integer

3, 5, 7, ... Letting n be the number of trapezoids, we can add the length that these trapezoids lie along the line AD for different numbers n, getting

number of trapezoids	**total length of AD = 20**
3	$2(x+3) + x = 20$
5	$3(x+3) + 2x = 20$
7	$4(x+3) + 3x = 20$
9	$5(x+3) + 4x = 20$
.	$\dots \quad \dots$
n	$\left(\frac{n+1}{2}\right)(x+3) + \left(\frac{n-1}{2}\right)x = 20$

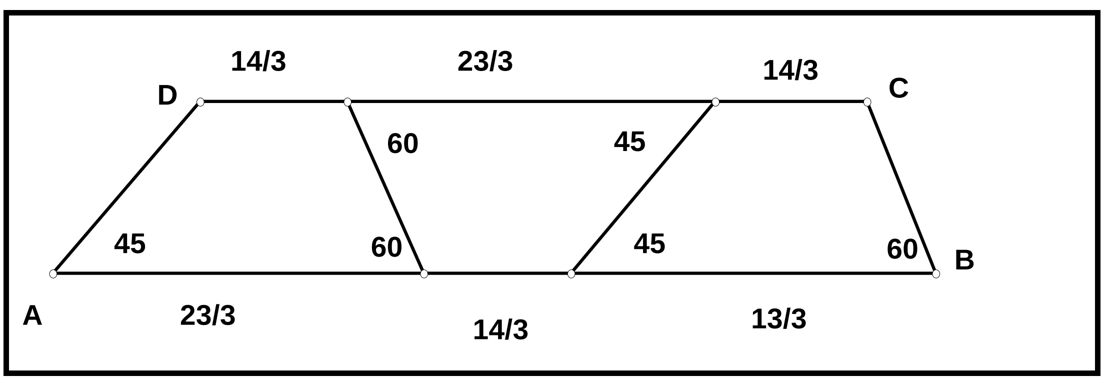
Solving this last equation for x in terms of the number of trapezoids $n,$ we get

$$x = \frac{37-3n}{2n} \qquad n = 3,\, 5,\, 7,\, ...$$

For $n = 3,\, 5,\, 7,\, ...$ the (only) positive values of x are

number of trapezoids n	3	5	7	9	11
size x in the trapezoid	$14/3$	$11/3$	$8/7$	$5/9$	$2/11$

Hence, we can subdivide the trapezoid into smaller congruent trapezoids with 3, 5, 7, 9, or 11 trapezoids. The subdivision into 3 trapezoids with size $x = 14/3$ is shown in the following diagram.



Three smaller congruent trapezoids

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΩΦΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 9 (Logarithms)

Find the positive real solutions of the equation

$$\boldsymbol{log_a\, x = x} \quad \text{where} \quad \boldsymbol{a = x^{log_4 x}}$$

Solution

Using the exponential form of the logarithm, we can write

$$a^x = x$$

or

$$\left(x^{log_4 x}\right)^x = x$$

from which we can write

$$x^{x\, log_4 x} = x$$

Setting the exponents of x in this equation equal, we have

$$x\, log_4 x = 1$$

or

$$log_4\, x^x = 1$$

and from the definition of the logarithm, we have $x^x = 4$ or $x = 2$.

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΦΩΦΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 10 (Good Samaritan Problem)

A baker, being a good Samaritan, bakes cookies for children. He figures the number of cookies he can bake *per hour* is proportional to the product of the number of hours of sleep he gets in a day times the number of hours he spends baking cookies. Assuming he is either sleeping or baking cookies, how many hours of sleep should the baker get each day in order to maximize the number of cookies he can bake in a day?

Solution

If the number of cookies N the baker can make *per hour* is proportional to the number t of hours the baker bakes times the number of hours he sleeps (he sleeps 24 - t hours a day), then

$$N = kt(24 - t) \qquad 0 \le t \le 24$$

where k is a constant of proportionality. Hence the *total* number T of cookies the baker makes every day is

$$T = tN = kt^2(24 - t) \text{ cookies}$$

To find the value of t that maximizes T, we find

$$T' = kt(48 - 3t) \qquad 0 \le t \le 24$$

which has extreme points (where $T' = 0$) at $t = 0, 16$. Since $T''(t) < 0$ on $0 \le t \le 24$, the point $t = 16$ is point where $T(t)$ attains its maximum value in the interval [0, 24]. Hence, the baker should work 16 hours a day and sleep 8 hours a day. Of course, you can't determine the number of cookies he makes per day using this strategy since we didn't tell you the constant of proportionality k.

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΦΨΦΩΛΓΠΣΦΛΩΠΘΞΥΞΠ