October Solutions
Grades 6-12
Maine, Maine, Science and Engineering Talent Search

1. **(The Mystery on Elm Street)** On one side of the street on a block of Elm street there are a row of houses whose addresses increase by two as you go from house to house. If the sum of the addresses is 117 on this block, what is the address of the 5th house in the block ?

Solution If we call a_1 the address of the first house in the block, and denote by n the number of houses on the block, then the address of the last house on the block will be $a_n = a_1 + 2(n-1)$. What's more, we are given the sum of the n address to be

$$s_n = \frac{(a_1+a_n)\,n}{2} = \frac{[2a_1+2(n-1)]\,n}{2} = (a_1 + n - 1)\,n = 117$$

and so we have the relation between the address of the first house a_1 on the block and the number n of houses as $a_1 = \frac{117}{n} - n + 1$. But what do we know about n? We know it is at least 5 and that it must divide 117 (from the relation between a_1 and n). But if we write 117 in terms of its prime factors, we have $117 = 3^2 \cdot 13$, and thus 117 as prime factors 1, 3, 9, 13, 39, and 117. If we plug these values into $a_1 = \frac{117}{n} - n + 1$ for n, we see

n	$a_1 = \frac{117}{n} - n + 1$	
1	$a_1 = \frac{117}{1} - 1 + 1 = 117$	
3	$a_1 = \frac{117}{3} - 3 + 1 = 37$	
9	$\mathbf{a_1 = \frac{117}{9} - 9 + 1 = 5}$	**← only valid canditate**
13	$a_1 = \frac{117}{13} - 13 + 1 = -3$	
39	$a_1 = \frac{117}{39} - 39 + 1 = -35$	
117	$a_1 = \frac{117}{117} - 117 + 1 = -117$	

The last three entries are no good since a_1 is negative (house numbers are never negative). The first two are also invalid since they result in the sum of the address to be greater than 117 (remember we have at least 5 houses). Hence, we have $n = 9,\ a_1 = 5$, which means the address of the first house is $a_1 = 5$, and there are $n =$ houses on the block. In other words, the addresses of the houses are 5, 7, 9, 11, 13, 15, 17, 19, 21. Hence, the address of the 5th house is 13. You can sum these addresses to see that they sum to 117.

2. **(A Little Trigonometry)** Find the angle A if

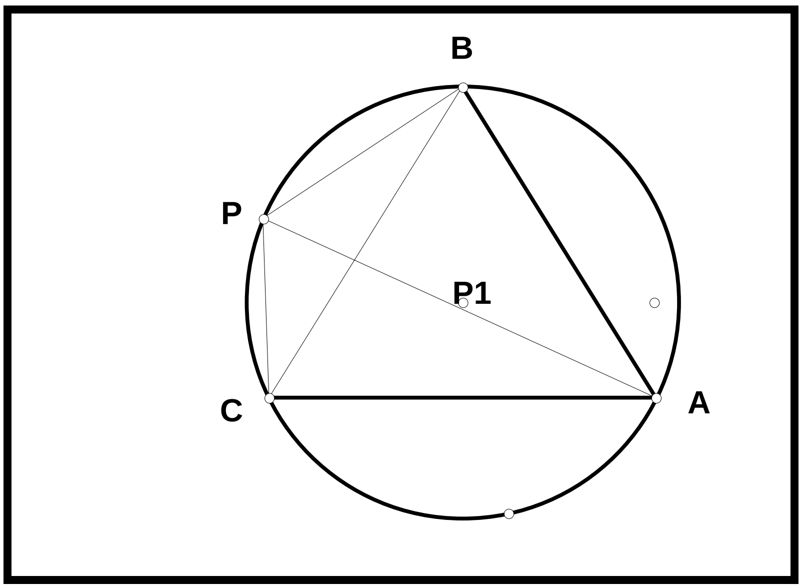
$$tan\, 54° \cdot tan\,(45° - A) = 1$$

Do not use a calculator to find your answer.
Solution From the given equation we have

$$tan\,(45° - A) = \frac{1}{tan\, 54°} = cot\, 54° = tan\, 36°$$

Setting arguments of tan equal to each other, we have $45° - A = 36°$, or $A = 9°$.

3. **(Equilateral Triangle)** Consider the equilateral triangle ABC drawn below where we have drawn the circumscribed circle about the triangle. We have also drawn a point at random on the circle, labeled P . Show $PA = PB + PC$.



Solution We first consider the case when the distance PC is less than PB. If we measure the distance PB along the line PA reaching the point P_1 (that is $PB = PP_1$), we form the equilaterial triangle PP_1B. We also have congruent triangles $\triangle AP_1B \cong \triangle CPB$ since they have two sides and an angle in common. Hence, $AP_1 = PC$, and so $AP_1 + PP_1 = PC + PB$, which means the problem is solved.

We now must consider the other situation when P is drawn such that PB = PC. In this case $\triangle$ APB $\cong$ $\triangle$ APC , and hence PB $\cdot$ PC $= \frac{1}{2}$ PA, Thus PA $=$ PB + PC.

Note: If you really know your geometry, you can resort to the theorem of *Ptolomaios* which states

$$PB \cdot AC + PC \cdot AB = PA \cdot BC$$

from this the conclusion follows quite readily.

4. **(School Dance)** In a school dance 430 couples danced. One girl dances with 12 boys, a second girl dances with 13 boys, a third girl dances with 14 boys, and so on until the girl who danced with the most boys danced with every boy at the dance. How many boys and girls were at this dance ?

Solution Let n be the number of girls at the dance. Hence, we are given: first girl dances with 12 $= 1 + 11$ boys, second girl dances with 13 $= 2 + 11$ boys, ... and the last or the nth girl dances with $n + 11$ boys. Hence, we know that the number of boys at the dance is $n + 11$ (11 more boys than girls). And since 430 couples danced, we have

$$(1 + 11) + (2 + 11) + (3 + 11) + ... + (n + 11) = 430$$

which is an arithmetic sequence with first term 12 and last term $n + 11$. Hence, we have

$$\frac{n}{2}(12 + n + 11) = 430$$

or the quadratic

$$n^2 + 23n - 860 = 0$$

which has solutions $n_1 = 20$, $n_2 = -43$. Since the negative solution is meaningless, we conclude the dance has $n = 20$ girls and $n + 11 = 31$ boys.

5. **(Gold, Silver, and Copper)** We have three alloys of gold, silver, and copper, with proportion of gold ÷ silver ÷ copper as follows: 1 ÷ 3 ÷ 5 for the first alloy, 3 ÷ 5 ÷ 1 for the second alloy, and 5 ÷ 1 ÷ 3 for the third. How would you make 351 grams of a fourth alloy with proportion 7 ÷ 9 ÷ 11 using the given alloys?

Solution If 351 grams of the produced alloy should be 7 parts gold, the amount of gold in alloy should be

$$\frac{7}{7+9+11} \cdot 351 = 91 \text{ grams gold}$$

Likewise the amount of silver and copper in the produced allow should be

$$\frac{9}{7+9+11} \cdot 351 = 117 \text{ grams silver}$$

$$\frac{11}{7+9+11} \cdot 351 = 143 \text{ grams copper}$$

(Note, they add up to 391 grams.)

Now, let us assume we want to mix the 3 given alloys by mixing x grams of the first, y grams of the second, and z grams of the third. Then, in order to get 91 grams of gold, 117 grams of silver, and 143 grams of copper, x, y, and z must satisfy

$$1 \cdot \frac{x}{9} + 3 \cdot \frac{y}{9} + 5 \cdot \frac{z}{9} = 91 \text{ grams gold}$$

$$3 \cdot \frac{x}{9} + 5 \cdot \frac{y}{9} + 1 \cdot \frac{z}{9} = 117 \text{ grams silver}$$

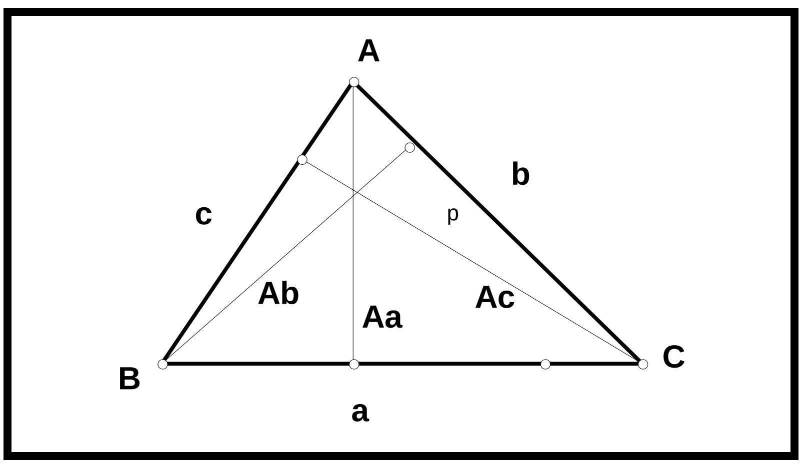
$$5 \cdot \tfrac{x}{9} + 1 \cdot \tfrac{y}{9} + 3 \cdot \tfrac{z}{9} = 143 \text{ grams copper}$$

Rewriting these equations, we find

$$\begin{aligned} x + 3y + 5z &= 819 \\ 3x + 5y + z &= 1053 \\ 5x + y + 3z &= 1287 \end{aligned}$$

which has the solution $x = 195,\ y = 78,\ z = 78.$

6. **(Triangle Problem)** We are given a triangle with vertices $A, B,$ and C; opposite sides $a,\ b,$ and c; and altitudes $A_a,\ A_b,$ and A_c (An altitude is a line from a vertex to the opposite side which is perpendicular to the opposite side.) Suppose we add the altitudes two at a time $A_a + A_b : A_b + A_c : A_a + A_c$ getting the proportion $5 \div 7 \div 8$. Find the proportion $a \div b \div c$ of the sides of the triangle.



Solution We can write

$$\begin{aligned} A_a + A_b &= 5x \qquad (1) \\ A_b + A_c &= 7x \qquad (2) \\ A_a + A_c &= 8x \qquad (3) \end{aligned}$$

where x is a positive number. We now perform some basic algebra on these equations. We first add the three equations and divide by 2, getting

$$\tfrac{1}{2}\,[\,(1) + (2) + (3)] = A_a + A_b + A_c = 10x \qquad (4)$$

and

$$\begin{aligned} (4) - (1) &= A_a = 3x \\ (4) - (1) &= A_b = 2x \\ (4) - (3) &= A_c = 5x \end{aligned}$$

We also know that if we multiply each altitude by the corresponding side, we get twice the area of the triangle, and thus

$$a \cdot A_a = b \cdot A_b = c \cdot A_c = 2 \cdot \text{area of the triangle}$$

and hence $3ax = 2bx = 5cx = 2 \cdot$ area. Hence

$$a \div b \div c = \frac{2\,area}{3x} \div \frac{2\,area}{2x} \div \frac{2\,area}{5x} = \frac{2}{3} \div \frac{2}{2} \div \frac{2}{5} = 10 \div 15 \div 6.$$

7. **(How Old is Joe ?)** In 1999 Joe's age equals the sum of the digits of the year he was born. In what year was Joe born? (Do not guess but reason!)

Solution Clearly Joe was born sometime in the 20th century since if the first two digits were 18xx, then the largest number would be 1899, whose sum is 27, but Joe would not be 27 in 1999 if he were born in 1899. (The same argument holds if he were born in the 1700s, 1600s, ...) In other words, Joe was born sometime in the 1900s. If we call d the decade Joe was born, and y the digit in the 1s place of the year he was born, we seek the values of d and y in the equation

$$\underbrace{(1900 + 10d + y)}_{\text{year Joe was born}} + \underbrace{(1 + 9 + d + y)}_{\text{sum of the digits of Joe's birth year}} = 1999$$

This equation simplifies to $11d + 2y = 89$, where d and y are one of the values $0, 1, 2, \ldots 9$. A few trial and errors shows the only solution to be $d = 7,\ y = 6$. Hence, Joe was born in 1976. Note, this year checks since Joe will be $1 + 9 + 7 + 6 = 23$ in the year 1999.

8. **(What Do You Know About Remainders?)** What are the remainders if each of the numbers 65^{6n}, 65^{6n+1}, 65^{6n+2}, 65^{6n+3} if they are divided by 9 ?

Solution Applying the binomial theorem

$$(a+b)^n = a^n + na^{n-1}b + \ldots + nab^{n-1} + b^n$$

we can write

$$\begin{aligned}
65^{6n} &= (63+2)^{6n} &&= 63^{6n} + 6n\,(63)^{6n-1} \cdot 2 + \ldots + 2^{6n} \\
65^{6n+1} &= (63+2)^{6n+1} &&= 63^{6n+1} + (6n+1)\,(63)^{6n} \cdot 2 + \ldots + 2^{6n+1} \\
65^{6n+2} &= (63+2)^{6n+2} &&= 63^{6n+2} + (6n+2)\,(63)^{6n+1} \cdot 2 + \ldots + 2^{6n+2} \\
65^{6n+3} &= (63+2)^{6n+3} &&= 63^{6n+3} + (6n+3)\,(63)^{6n+2} \cdot 2 + \ldots + 2^{6n+3}
\end{aligned}$$

Hence, we see that every term in each equation has the factor $(63)^n = (7 \cdot 9)^n = 7^n 9^n$. Hence, we only need to examine the remainders of $2^{6n}, 2^{6n+1}, 2^{6n+2}, 2^{6n+3}$ when divided by 9. But

$$\begin{aligned}
2^{6n} &= 64^n = (63+1)^n \text{ has a remainder of 1 when divided by 9}\\
2^{6n+1} &= 2\cdot 64^n = 2(63+1)^n \text{ has a remainder of 2 when divided by 9}\\
2^{6n+2} &= 4\cdot 64^n = 4(63+1)^n \text{ has a remainder of 4 when divided by 9}\\
2^{6n+3} &= 8\cdot 64^n = 8(63+1)^n \text{ has a remainder of 8 when divided by 9}
\end{aligned}$$

Hence, the remainders of 65^{6n}, 65^{6n+1}, 65^{6n+2}, 65^{6n+3} when divided by 9 are, respectively, 1, 2, 4, and 8.

9. **(Dividing Polynomials)** Factor $x^4+1 = (x^2+px+q)(x^2+rx+s)$

Solution We divide x^2+1 by x^2+px+q, getting

$$\begin{array}{rl}
 & x^2 - px + p^2 - q \\
x^2+px+q \,\big) & x^4 + 1 \\
 & x^4 + px^3 + qx^2 \\
 & ---------- \\
 & -px^3 - qx^2 + 1 \\
 & -px^3 - p^2x^2 - pqx \\
 & --------------- \\
 & (p^2-q^2)\,x^2 + pqx + 1 \\
 & (p^2-q^2)x^2 + p\,(p^2-q)x + q\,(p^2-q) \\
 & ------------------------- \\
\text{Remainder } = & p\,(2q-p^2)\,x - q\,(p^2-q) + 1
\end{array}$$

Hence, we have found two factors

$$x^4+1 = (x^2+px+q)(x^2-px+p^2-q)$$

provided p and q are chosen so the remainder $p\,(2q-p^2)\,x - q\,(p^2-q) + 1$ is zero. But this requires

$$p\,(2q-p^2) = 0 \qquad (1)$$

$$q\,(p^2-q) + 1 = 0 \qquad (2)$$

From (1), we have $p=0$ and $p = \sqrt{2q}$. Plugging theses values into (2), we get $q^2 = -1$ and $q^2 = 1$, respectively. Hence, we have three pairs of values of p and q:

$$p=0,\ q = \sqrt{-1} = i$$

$$p = \sqrt{2},\ q = 1$$

$$p = \sqrt{-2} = i\sqrt{2},\ q = -1$$

Hence, we have the factorizations

$$x^4 + 1 = (x^2+i)(x^2-i)$$

$$x^4+1 = (x^2 + \sqrt{2}\,x + 1)(x^2 - \sqrt{2}\,x + 1)$$

$$x^4+1 = (x^2 + \sqrt{2}\,i\,x - 1)(x^2 - \sqrt{2}\,ix - 1)$$

10. **(Interesting Distance Problem)** Ann and Bob walk back and forth between two points A and B. Ann walks from A to B and back to A, and Bob walks from B to A and back to B. Although they person walks at a different speed, each person maintains a constant speed. They start at the same time and meet 700 feet from A. After each person reaches the opposite point, they turn around immediately and walk back to their point of origin, this time meeting each other 400 feet from B. How far is it from A to B ?
Solution Ann leaves point A and walks 700 feet before meeting Bob. At that point both Ann and Bob have walked together the distance from A to B. Ann then continues to B and starts back where she meets Bob the second time 400 feet from B. At this time And and Bob together have walked *three* time the distance from A to B. But since Ann walks at a constant speed, she has walked a total of 3×700 feet $=$ 2100 feet. Hence, the distance from A to B is 2100 $- 400 = 1700$ feet. We illustrate their walks in the diagram below.