October Solutions
Grades 6-9
Maine Math, Science and Engineering Talent Search

1. **(The Mystery Clock)** The electricity in Josh's house goes off and the only clock in the house is electric so it stops. The power comes on later but his family doesn't know the time. He lives close to Mark's family who do not lose power and so Josh devises a clever plan to reset the clock in his house by going to Mark's house, checking their clocks, and then returning home to reset the clock that is running but has the wrong time. (No, they don't have telephones or Internet connections.) How does Josh manage to do this ? The only assumption we make is that the time it takes for Josh to walk to Mark's house is the same as it takes to walk back home.

Solution Before Josh leaves his house he sets the clock with the wrong time to some arbitrary time, say 5:00. Then after he arrives at Mark's house, he looks at the clock in Mark's house, suppose is 7:15. Josh stays awhile and then leaves for home, and suppose the clock reads 8:00. He then arrives back home, say his own clock shows 6:15. Now, his own clock (which doesn't show the correct time) shows that he has been gone for 1 hr and 15 minutes. Josh also knows that he stays at Mark's house for 45 minutes. This means that the total time it takes Josh to walk to and from Mark's house is 30 minutes, which means that it takes him 15 minutes to walk one way. If Josh then adds this time to the 8:00, which was the correct time when he left Mark's house, it should be exactly 8:15 when he returns home.

In general, suppose Josh sets his clock to x hrs before leaving home and upon arrival at Mark's house, he sees the correct time is y hours. When Josh leaves Mark's house he observes the correct time is now z hours, hence he has spent $z-y$ hours at Mark's house. If now we call w the time showing on the (still unset) clock in Josh's house when he returns home, we have the total time it takes Josh to walk to and from Mark's house to be $(w-x)-(z-y)$ hours, and so the time it talks to talk one way is $\frac{(w-x)-(z-y)}{2}$ hours. Hence, adding this time to z, the correct time when Josh left Mark's house, we have the correct time to be

$$\text{correct time } = z + \frac{(w-x)-(z-y)}{2} \text{ hours}$$

2. **(Patterns)** In each of the following cases, what do the given numbers have in common ?

(a) 77, 49, 36, 18, 8
(b) 7, 3, 15, 19, 11
(c) 8, 3, 0, 24, 15
(d) What is the common theme in the following squares ?

1	2
3	9

3	2
3	15

7	29
3	108

Solution

(a) each consecutive number is the product of the digits of the previous number

(b) each number is of the form $4k+3,$ where k is one of the numbers 0, $1,\ 2,$ ...

(c) each number is of the form $n^2-1,$ where n is one of the numbers $1,\ 2,\ 3,$ ....

(d) each square is of the form

a	b
c	d

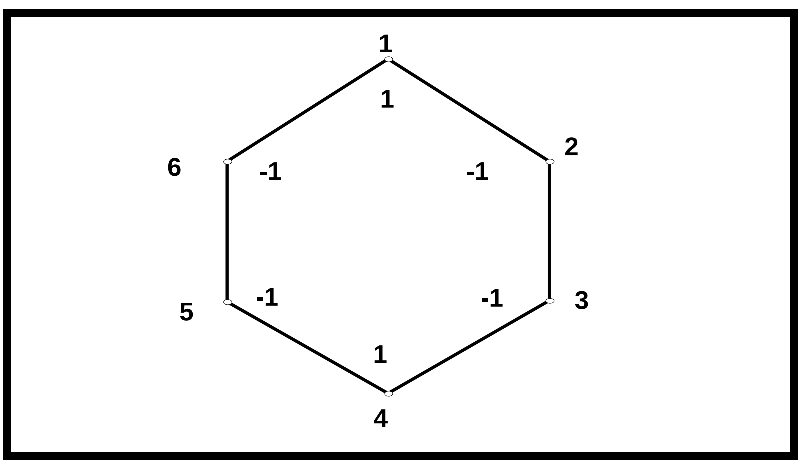
where $(a+b)\cdot c = d$

3. **(Sequence of 1s and -1s)** Given the first five numbers 1, -1, -1, 1, -1 in a sequence of numbers, you are told:

- the 6th number is the product of the first two numbers,
- the 7th is the product of the 2nd and 3rd numbers,
- the 8th is the product of the 3rd and 4th numbers,
- and so on

Find the 1999th number.

Solution We find it convenient to write the first six numbers alongside the vertices of a hexagon as shown below.



If we compute a few more numbers, we find

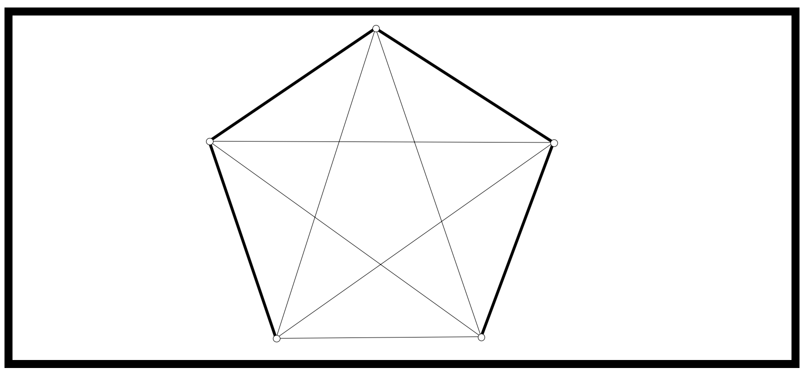
7th term $=$ term 2 $\times$ term 3 $=$ 1
8th term $=$ term 3 $\times$ term 4 $=$ -1
9th term $=$ term 4 $\times$ term 5 $=$ -1
10th term $=$ term 5 $\times$ term 6 $=$ 1
11th term $=$ term 6 $\times$ term 7 $=$ -1

12th term = term 7 $\times$ term 8 = -1
13th term = term 8 $\times$ term 9 = 1
14th term = term 9 $\times$ term 10 = -1
15th term = term 10 $\times$ term 11 = -1
16th term = term 11 $\times$ term 12 =1

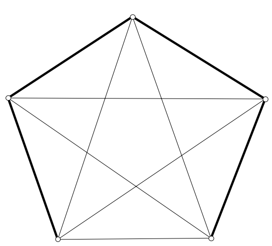
Hence, we see the general pattern. The terms of the form 7, 10, 13, 16, ... $3k+1$ are +1 and the other terms are -1. Since the 1999 th term has the form 1999 $= 3 \times 666 + 1$ it is +1.

4. **(Calendar Problem)** A certain month has three Tuesdays, all of which are on an even date (like 2, 4, 6, ...). What is the date of the last Friday of this month ?
Solution The only possible dates for the three Tuesdays are 2, 16, and 30. Hence the last Friday of the month is on the 26th.

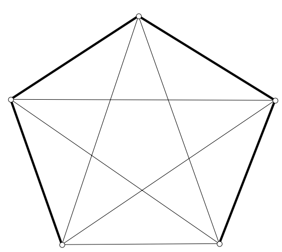
5. **(Counting Triangles)** Below we have drawn a regular pentagon and connected all the vertices. How many triangles can you find in the picture ?



Solution Counting the different kinds of triangles, we find



10 of these



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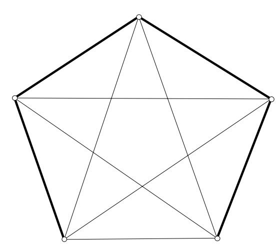

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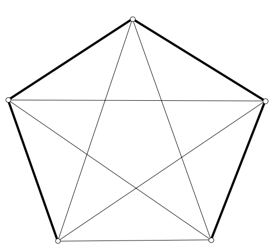

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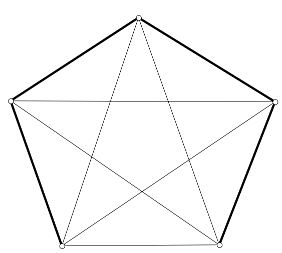

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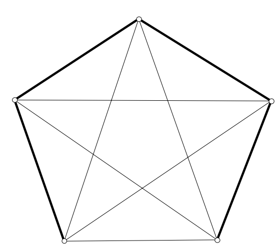


5 of these

Hence, there are a total of 35 triangles.

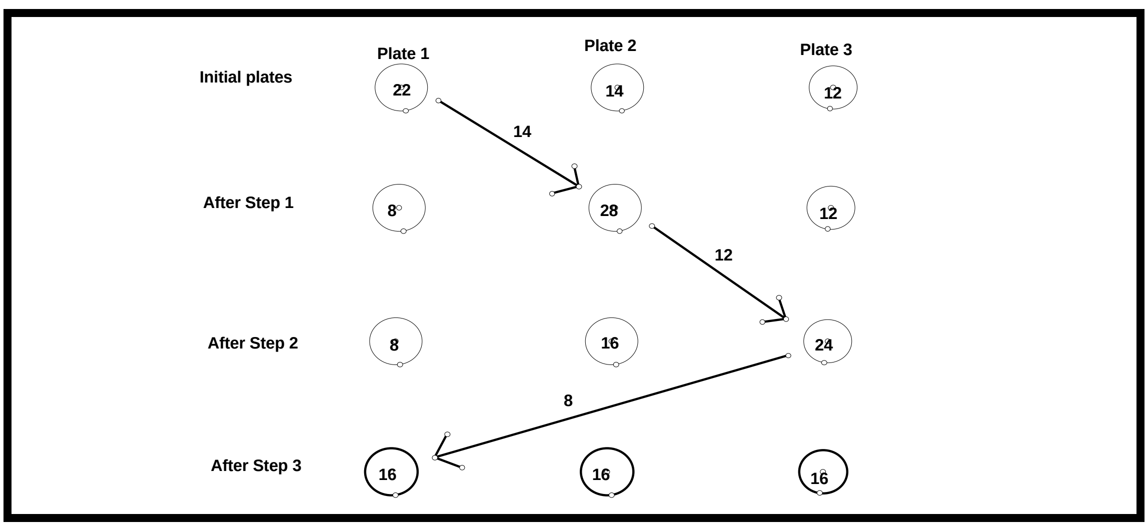
6. **(Rook Problem)** How many ways can you place a black and white rook on a chessboard so that they cannot land on each other in a series of moves?

Solution The idea is to multiply the number of places we can put one rook times the number of places we can put the other rook without having then land on each other. Starting the one rook, say the white rook, we can place it in 64 places. Then for each placement of the white rook, we count 15 squares where the black rook *can* reach the white rook, and so there are $64 - 15 = 49$ squares where the black rook *cannot* reach the white rook. Hence, the total number of ways the two rooks can be placed so that they cannot land on each other is $64 \times 49 = 3136$ ways.

7. **(Cards and Plates)** You are given three plates with 22 cards on the 1st plate, 14 on the 2nd plate, and 12 on the 3rd plate. In three steps we want the same number of cards on each plate, where on each step we the number of cards we can move from one plate to another plate is the number of cards on the destination plate. How can you do this ?

Solution Since there are a total of 48 cards, we should end up with 16 cards on each plate. The following diagram shows how we can accomplish the goal in 3 steps. Starting with 22, 14, and 12 cards on the respective plates, we carry out:

Step 1: Move 14 cards from plate 1 to plate 2 (ending up with 8, 28, 12 cards)
Step 2: Move 12 cards from plate 2 to plate 3 (ending up with 8, 16, 24 cards)
Step 3: Move 8 cards from plate 3 to plate 1 (ending up with 16, 16, 16 cards)



8. **(Tournament Problem)** Fifteen teams participate in a round-robin tournament in which every team competes with every other team. In each game the winner gets 3 points, the loser gets 1 point, and in case of a tie both teams receive 2 points. Suppose at the end of this tournament each team had a different number of points and the last-place team had 21 points. Show that the winning team had at least one tie.

Solution First of all, every team plays 14 games and since there are 15 teams, we suspect the total number of games played by all teams is $15 \times 14 = 210$. But this number is twice as many as the correct number since we are counting both A plays B and B plays A, which is really one game. In other words, the total number of games played by all the teams is (15)(14)/2 $=$ 105.

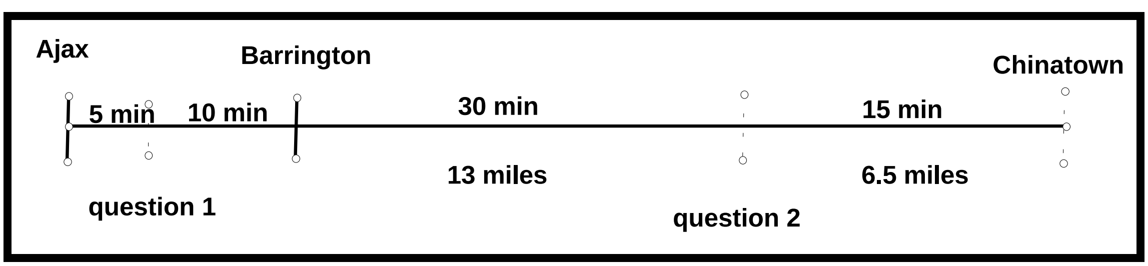
Now, since a total of 4 points is given out in each game between the two teams, the total number of points given out to all the teams in the season is 4 $\times$ 105 = 420. Now the last-place team gets 21 of these points and each team receives a different number of points, we have that the 15 teams had at least

$$21 + 22 + 23 + \ldots + 35 = 420 \text{ points}$$

which is the exact number of points given out to the 15 teams. Hence, each team, from bottom to top, receives 21, 22, 23, ... 35 points! Let us assume the winning team has now ties. If we now call x the number of games the top wins, and 14 $-$ x the number of games the top team looses, then we have the relationship $3x + (14 - x) \cdot 1 = 35$ points. But the solution of this equation is the fraction $x = 10.5$, which of course is impossible! Hence, we cannot assume that the winning team had all wins and losses and so it must have at least one tie.

9. **(Bus Problem)** Ian takes the bus from Ajax to Chinatown through Barrington. Five minutes after his departure he asks the bus driver how far they were from Ajax. The bus driver said they were twice as far from Barrington as from Ajax. Thirteen miles after Barrington, Ian asks the bus driver again how far they were from Chinatown and the bus driver says they were twice as far from Barrington as from Chinatown. Fifteen minutes later they arrived in Chinatown. How far is it from Ajax to Chinatown ?

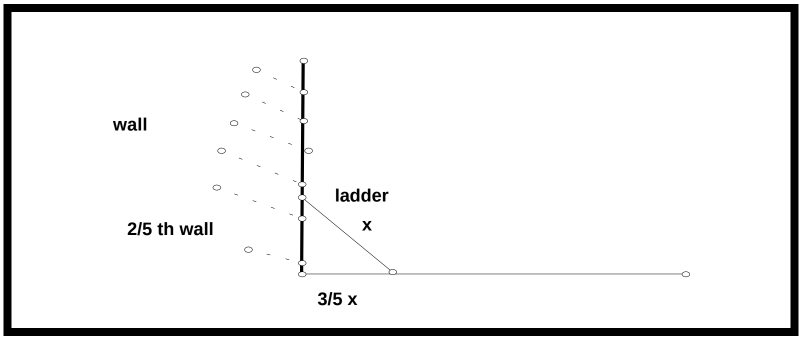
Solution Based on the first answer from the bus driver, it took 15 minutes to get from Ajax to Barrington. Based on the second and third answers, it took the bus 15 minutes to drive 13/2 $=$ 6.5 miles. Hence, as we illustrate in the diagram below, the distance from Barrington to Chinatown is $13 + 6.5 = 19.5$ miles, which would take $15 + 30 = 45$ minutes.



It takes us 15 minutes to go to Alax to Barrington, so the distance between them is also 6.5 miles. Thus, the total distance is 6.5 + 13 + 6.5 = 26 miles.

10. **(Another Ladder Problem)** A ladder is 2 $\frac{2}{3}$ feet shorter than the height of a wall. If the ladder leans against the wall in such a way that when the base of the ladder is 3/5 th the length of the ladder from the wall, then the top of the ladder is 2/5 th the height of the wall. What is the length of the ladder ?

Solution Let x be the length of the ladder. We then have that the height of the wall is $x + 2\,\frac{2}{3}$ feet. We have drawn the ladder in the diagram below showing it leaning against the wall as described in the problem. Since the ladder is 2/5 th the way up the wall, we have that the ladder reaches a height of $\frac{2}{5}\,(x + 2\,\frac{2}{3}) = \frac{2}{5}\,(\frac{3x+8}{3})$ up the wall.



Hence, using the Pythagorean theorem, we have

$$\left(\frac{6x+16}{15}\right)^2 + \left(\frac{3}{5}x\right)^2 = x^2$$

which after performing a little algebra, reduces to

$$\left(\frac{6x+16}{15}\right)^2 = \left(\frac{4}{5}x\right)^2$$

which since the quantities inside the parenthesis are positive, we can write

$$\frac{6x+16}{15} = \frac{4}{5}x$$

Solving for x, gives $x = \frac{8}{3} = 2\,\frac{2}{3}$ feet.