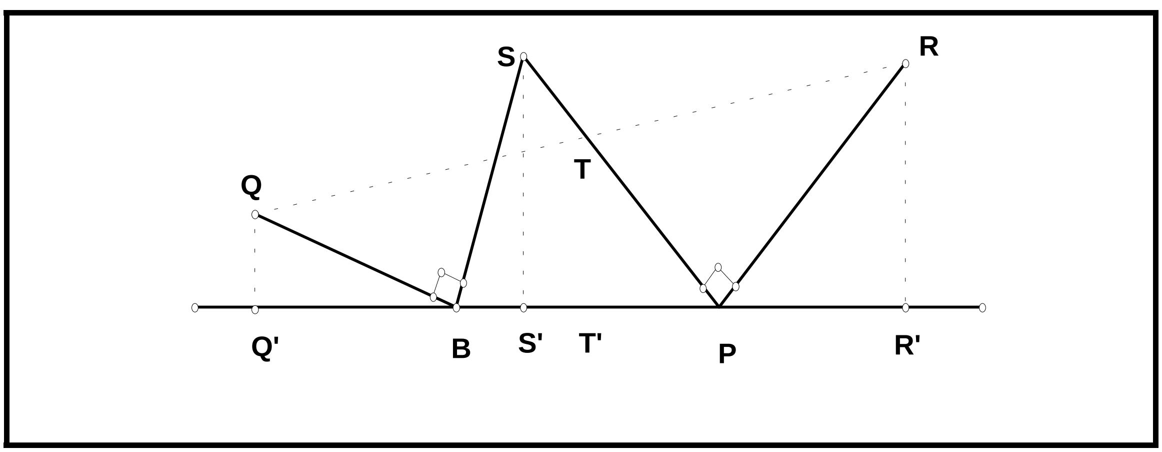
October Problems

Problem 1 (Hidden Treasure)

On a remote desert island illustrated below two pirates, Roger and Biff, are burying looted treasure. To start, they drive a stake into the ground at a point S which lies near the only two trees on the island; a palm tree located at P, and a birch tree located at B, which lies directly west of the palm tree. After driving the stake in the ground, Roger walks off paces to the birch tree (B), whereupon he turns right 90° and walks the same distance SB to a point Q. (In other words, $SB = BQ$.) After Roger reaches Q, then Biff starts at S and walks off paces to P, after which he turns *left* 90 degrees and walks an equal distance SP, reaching the point R. (In other words, $SP = PR$.) Now, Roger and Biff advance towards each other and bury the treasure halfway between them at a point T. (In other words, T is halfway between R and Q.)

A few years later, Roger and Biff return to dig up the treasure and discover that although the two trees were still there, the stake S was gone. They thought they would never find the treasure until Biff, who had once taken a course in high school math, said he had a plan for finding the treasure even though they didn't know the location of the stake S. Can you think of how Biff might be able to do this ?



Drawing of Pirate Problem

Solution

We assume (for the moment) that the position of the stake S is known and hence can find the points Q, R, and the treasure point T. When we finish, we will be surprised to see that the location of the treasure T does not depend on where we placed the stake S, but only on the points B and P ! We begin by dropping perpendiculars from Q, R, S, and T to the line segment BP. Doing this and using basic geometric arguments, we find the pair of congruent triangles $\Delta(QQ'B) \cong \Delta(BS'S)$, $\Delta(SS'P) \cong \Delta(PR'R)$, and hence we have the following distances the same: $BS' = QQ'$, $S'P = RR'$, $Q'B = SS' = R'B$. Using the first two relations $BS' = QQ'$ and $S'P = RR'$, the distance between the birch and pine trees can be written as

$$\boldsymbol{BP = BS' + S'P = QQ' + RR'}$$

But since the midline TT' of the right trapezoid $QQ'RR'$ has length

$$\boldsymbol{TT' = \frac{QQ' + RR'}{2}}$$

we have $TT' = BP/2$. But the midline TT' of the trapezoid $QQ'RR'$ is also the perpendicular bisector of B and P since $Q'B = R'B$, and so the pirates can find the treasure even if the location P of the stake is no longer known by finding the perpendicular bisector of BP, and then walking along this bisector towards the riverbank a distance of $BP/2$.

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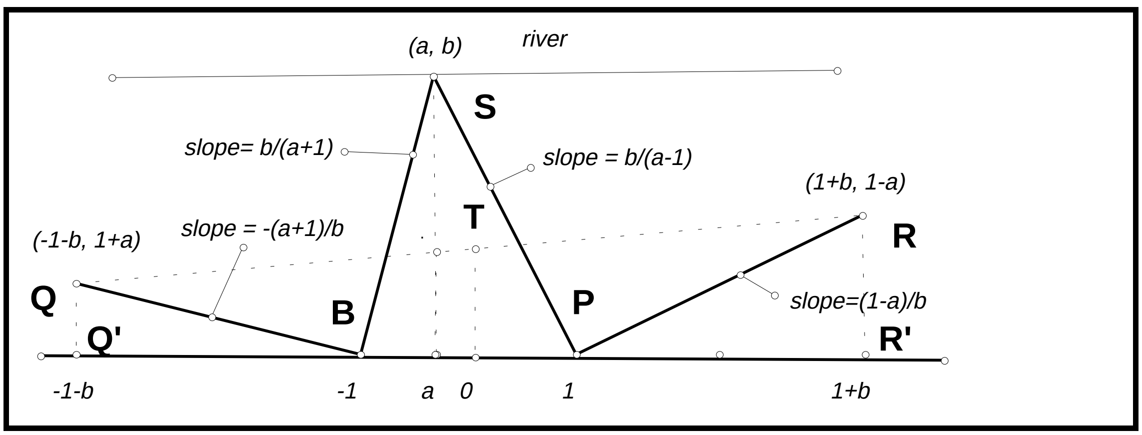
Analytic Solution (Real Analysis):

We can also solve this problem analytically by introducing the Cartesian plane, where we place the x-axis through the points B and P with $B =$ (-1, 0) and $P = (1, 0)$. See the diagram below.



Using basic analytic geometry the points $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{T}$ can be found

If we now place the stake at an arbitrary point $S = (a,\, b)$, we can use basic analytic geometry to find the slopes and equations of the lines in the figure. (Remember that slopes of perpendicular lines are negative reciprocals.) Doing this, you will discover that Q and R are located at $Q = (\text{-}1\text{-}b,\, 1 + a)$, $R = (1 + b, 1\text{-}a)$. Hence, to find the treasure T, we then take the average of these two points, which gets the midpoint $T = (Q + R)/2 = (0,\, 1)$. Again, T is located on the perpendicular bisector of

B = (-1, 0) and P = (1, 0), and the distance along the bisector is the distance from B or P to the bisector.

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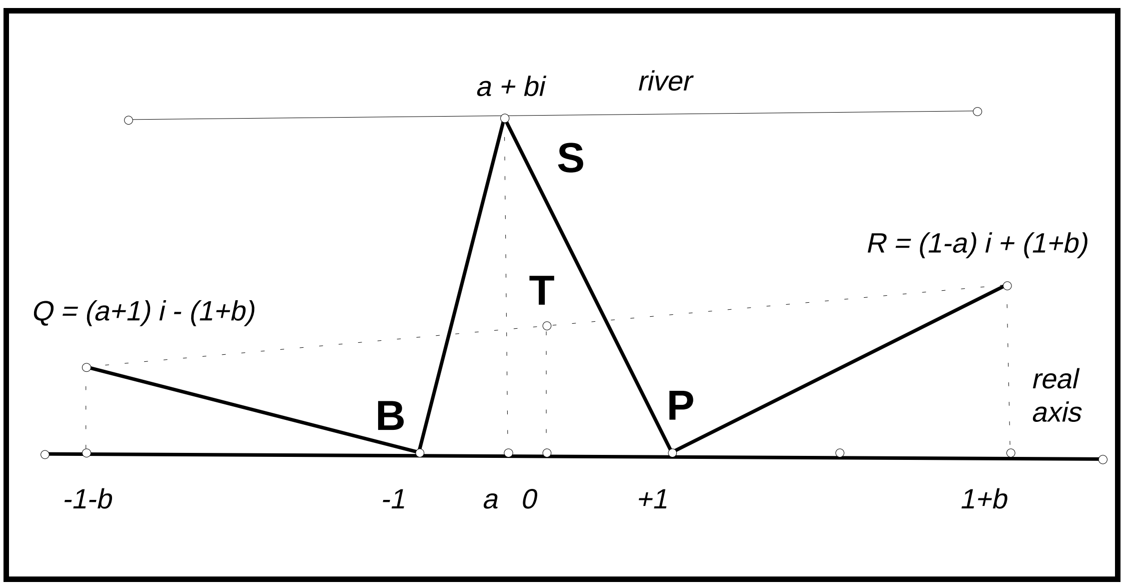
Analytic Solution (Complex Analysis):

If you are familiar with the geometry of complex numbers, there is a nice way to solve this problem without resorting to geometry or slopes of lines and so on. This is based on the fact that performing arithmetic on complex numbers acts to move points in the complex plane. In particular, multiplication of any complex number $a+bi$ by the imaginary number i rotates the number around the origin in the complex plane in the counterclockwise direction 90 degrees; and multiplication by $-i$ rotates points around the origin in the clockwise direction 90 degrees.

Hence, we begin by placing the trees B and P at the points $B =$ -1, $P = +1$ in the complex plane as shown in the diagram below. This corresponds to the points (-1, 0) and (+1, 0) in the Cartesian plane.



Complex plane solution

We now place the stake at an arbitrary point $S = a+bi.$ Using complex arithmetic, the point Q is simply

$$\boldsymbol{Q = [\,(a+bi)\, +\, 1\,]\,i\, -\, 1\, =\, (a+1)i\, -\, (1\, +\, b)}$$

Why is this ? Well, the expression inside the brackets [...] essentially moves the origin of the complex plane to the new point $B =$ (-1, 0), after which points are multiplied by i, which rotates $a+bi$ to Q. But then we have to subtract -1 to get Q back in the original coordinates of the problem.

In the same way, we can find R by computing

$$\boldsymbol{R = [(a+bi) - 1](-i) + 1 = (1-a)i + (1+b)}$$

Note that we subtract -1 from $a+bi$, which places the origin of the complex plane at $P = (1, 0)$, and then multiply by -i which rotates points 90 degrees in the clockwise direction, and so $a+bi$ moves to R. Then, finally we add +1 to get R in terms of the original coordinates.

So, we can now find the treasure T by taking the average of these to complex numbers, or

$$\boldsymbol{T = \frac{Q+R}{2} = \frac{[(a+1)i - (1+b)] + [(1-a)i + (1+b)]}{2} = i}$$

Again, regardless of the original point $S = a+bi$, all remnants of S are canceled and we discover that the treasure can be determined by knowing only the points B and P. In particular, as before the treasure lies on the perpendicular bisector of B and P, with the distance from the line BP being $BP/2$. In fact, the treasure T will always be at the vertex of a right triangle whose hypotenuse connects the trees B and P.

It is interesting to note that if 25 people placed stakes in different positions, everyone's treasure would be buried in the same place using this strategy. In other words, it wasn't a very good strategy.

$\boldsymbol{Note}$**:** This problem goes back to a very interesting book, *One Two Three ... Infinity* by George Gamow. Viking Press (1947). This book is still in printing and makes very interesting reading.

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Problem 2 (Minimum Path on Tetrahedron)

The length of each edge of a tetrahedron (four triangular faces – six edges) is one unit long. An ant wants to make a path on the surface of the tetrahedron such that each of the four vertices can be reached from any other vertex. What is the shortest path the ant can make a path ?

Solution

Let $\triangle(ABC)$ and $\triangle(DBC)$ represent two adjacent faces of the tetrahedron unfolded onto the plane, and let Q be the middle of the line segment AB as shown in Figure 1.

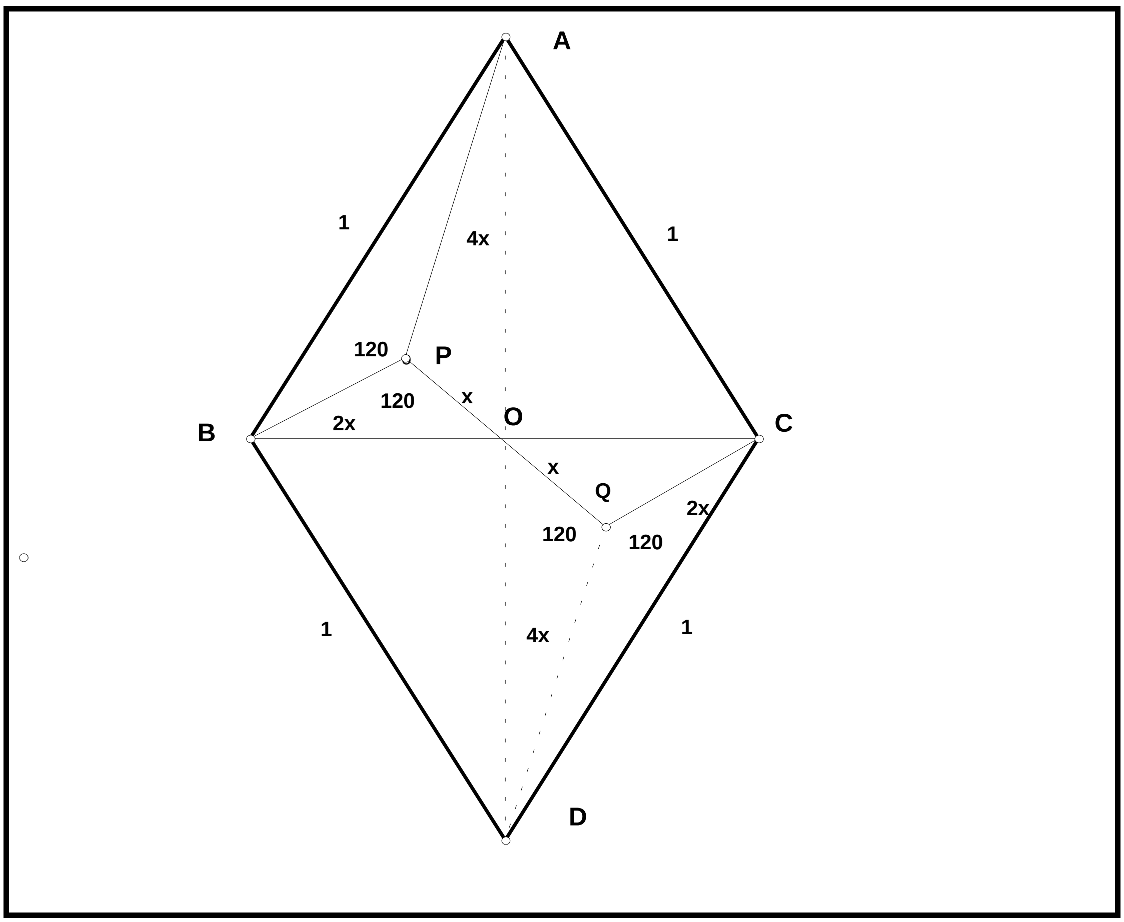

Figure 1: Shortest path network on the surface of a tetrahedron

We will consider the network of lines symmetric about the point 0, consisting of PA, PB, QC, QD, and PQ as shown in the figure. We will choose these lines in such a way that the roads at angles 120° at the points P and Q as illustrated in Figure 1.

We start by considering the triangle $\triangle(AOB)$, where we have assumed the sides of the tetrahedron have length one: i.e. $AB = 1$, and hence we have $BO = 1/2$. We also know that ∠B = 60° and so from the triangle $\triangle(POB)$, if we let $\angle POB = \alpha$, then $\angle PBO$ = 60° - α. And since $\angle B$ = 60°, we have $\angle PBA = \alpha$ and so $\triangle(POB)$ and $\triangle(PBA)$ are similar triangles. Now, letting $PO = x$, we have $PB = 2x$ (since $AB/BO = 2/1$) and $PA = 4x$. We now use the law of cosines applied to the triangle $\triangle(POB)$ to get the relationship

$$\boldsymbol{x^2 + (2x)^2 - 2 \cdot x \cdot 2x \cdot cos\,(120°) = (1/2)^2}$$

which gives or $x = 1/(2\sqrt{7})$. Hence, the length of the network in $\triangle(AOB)$ is $x + 2x + 4x = 7x$, and the length of the entire network on the surface of the tetrahedron connecting the four vertices is $4x = \sqrt{7} \approx$ 2.645751311... .

Note: We really haven't *proven* that this is the shortest network, but it is the shortest we have found. If you can find a shorter network, we would be happy to know about it.

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Problem 3 (Algebra Problem)

Solve

$$\mathbf{1 + 2\sqrt{x} - \sqrt[3]{x} - 2\sqrt[6]{x} = 0}$$

Problems

If we introduce the new variable $y = \sqrt[6]{x}$, this leads to the cubic equation in $y : 1 + 2y^3 - y^2 - 2y = 0.$ Factoring this equation, we get $(2y - 1)(y^2 - 1) = 0,$ which has three solutions $y = 1/2, \ +1,$ -1. Substituting back in terms of $x,$ we find

$$\begin{aligned} y &= \tfrac{1}{2} \Rightarrow \sqrt[6]{x} = \tfrac{1}{2} \Rightarrow x = \tfrac{1}{64} \\ y &= \text{-1} \Rightarrow \sqrt[6]{x} = \text{-1} \quad \text{(has no real root)} \\ y &= 1 \Rightarrow \sqrt[6]{x} = 1 \quad \Rightarrow x = 1 \end{aligned}$$

Hence, the original equation has real roots of 1/64 and 1.

Note: You may ask the question, does the equation have *complex* roots or have we found *all* solutions?

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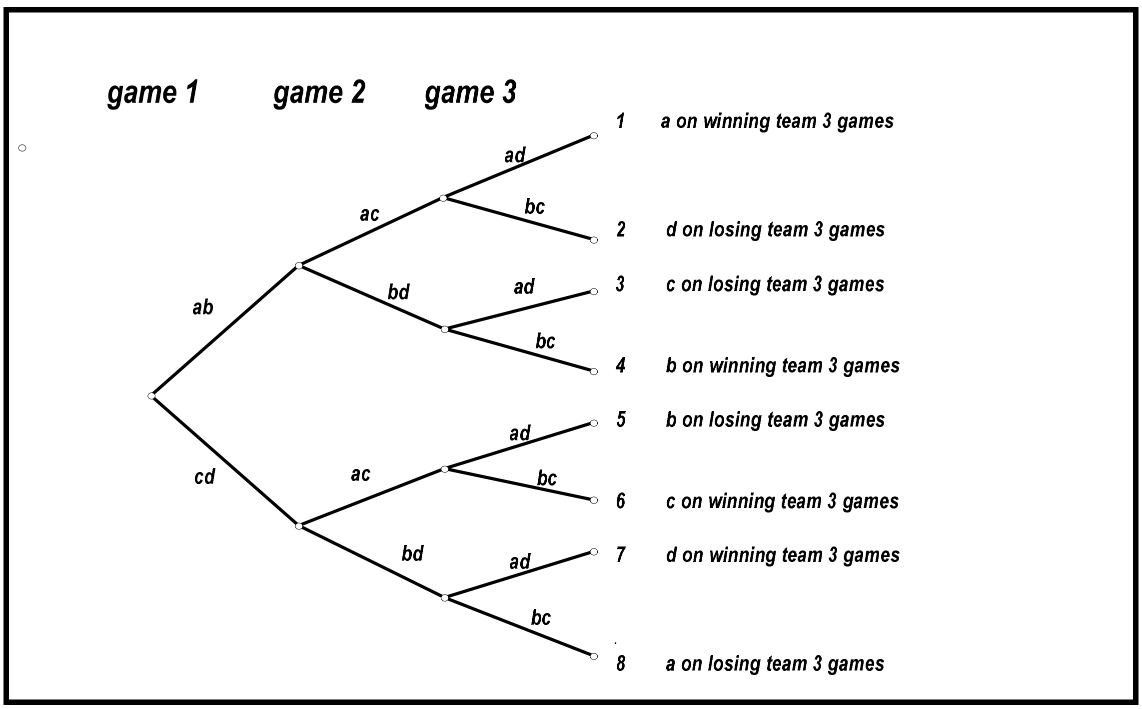
Problem 4 (Tennis Anyone ?)

Anne, Brent, Cindy, and Dylan are playing tennis doubles (two versus two). First, Anne and Brent play against Cindy and Dylan; next Anne and Cindy play against Brent and Dylan; finally Anne and Dylan play against Brent and Cindy. Show that for every possible outcome of the three games, there is one player that is either on the winning team every game or on the losing team every game.

Solution

If we enumerate the different outcomes of the three games, we have a total of eight outcomes illustrated by the tree below. The labels on the edges of the tree indicate the winner of each game. For example, outcome 1 means ab (Anne and Brent) win game 1, ac (Anne and Cindy) win game 2, and ad (Anne and Dylan) win game three. Likewise, outcome 4 means ab win game 1, bd wins game 2, and bc win game 3.



Tree diagram showing the winning team of each game

If you look carefully, you will see that for every outcome, there is one player that will be either on the winning or losing team every game.

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Problem 5 (Interesting Algebra Problem)

The numbers $x,\ y, z$, and k satisfy the equation

$$\frac{7}{x+y} = \frac{k}{x+z} = \frac{11}{z-y}$$

What is the value of k ?

Solution

We use the interesting (but not too-well known) algebraic fact that

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{b} = \frac{a+c}{b+d} \qquad (1)$$

For example

$$\frac{1}{2} = \frac{5}{10} \Rightarrow \frac{1}{2} = \frac{1+5}{2+10}$$

To prove this statement, we start with the identity $ab = ab,$ and write

	$ab = ab$	(identity)
$\therefore$	$ab + ad = ab + bc$	(since $ad = bc$ by assumption)
$\therefore$	$a(b + d) = b(a + c)$	(factoring)
$\therefore$	$\frac{a}{b} = \frac{a+c}{b+d}$	(simple algebra)

In our problem, we are given

$$\frac{7}{x+y} = \frac{k}{x+z} = \frac{11}{z-y}$$

which we can write as

$$\frac{7}{x+y} = \frac{11}{z-y} \qquad (2\text{-}a)$$

$$\frac{k}{x+z} = \frac{7}{x+y} \qquad (2\text{-}b)$$

If we apply our basic algebraic law (1) to (2-a), we have

$$\frac{7}{x+y} = \frac{7+11}{(x+y)+(z-y)} = \frac{18}{x+z}$$

We can now write $(2\text{-}b)$ as

$$\frac{k}{x+z} = \frac{7}{x+y} = \frac{18}{x+z}$$

Hence, the numerators must be equal, giving $k = 18$.

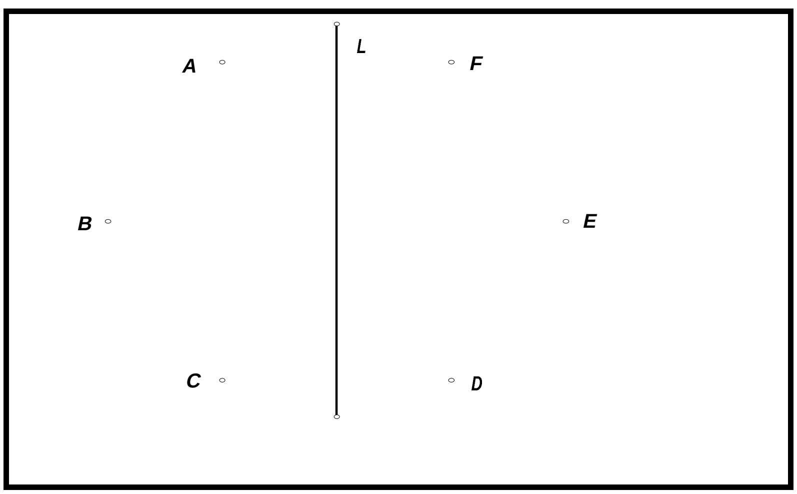
ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΦΨΣΠΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 6 (Six-Point Problem)

We start with 6 points in the plane such that no 3 of them are on the same line. Two players play a game where they take turns connecting any two points with a line. Once two points are connected, they cannot be joined a second time. The players continue to play until one player is forced to connect points resulting in the formation of a triangle. This player is the loser. Show that the person who makes the opening move will always will if he/she plays skillfully.

Solution

Let's suppose you are the player who makes the opening move. We show you how you can always win. The first thing you do is redraw the game as a hexagon as shown in the diagram below. (You can always redraw the points as a hexagon no matter how they were originally drawn.)



Typical Six-Point Game

You now draw an imaginary line L that separates any three points from the others. In the drawing above, we have drawn the line L so it separates A, B, C from D, E, F. For this given imaginary line L, you begin by drawing the *middle* line BE. After that, you simply respond to your opponent's move by drawing the *mirror* line. That is, if your opponent draws the line AF, you draw CD; if your opponent draws AB, you draw DE; if your opponent draws AD, you draw CF, and so on. You can convince yourself after playing a few games that you will never be the one to draw the first triangle. You could enumerate all the possible games in a diagram and show that in each case you will win.

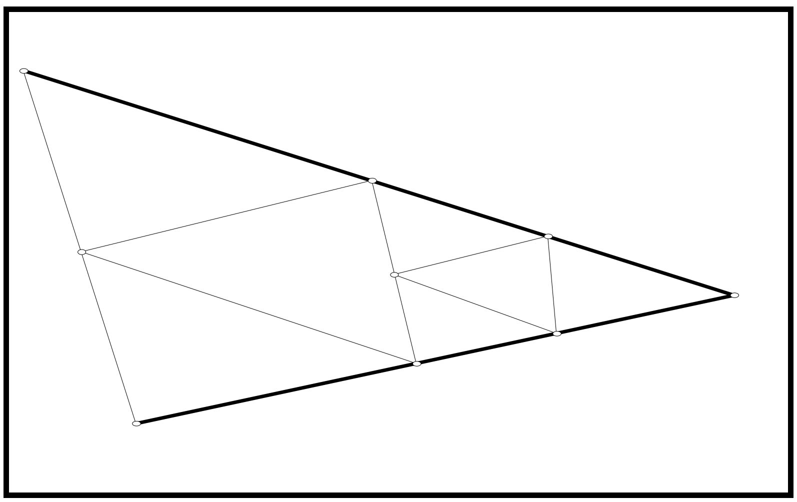
ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΦΨΦΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 7 (Triangle Problem)

Any triangle can be divided into similar subtriangles. Show that it is possible to subdivide any triangle into 1999 similar triangles

Solution

A well-known property from geometry states that by drawing the three midlines of a triangle (lines connecting the center points of adjacent sides), we obtain four *subtriangles*, each similar to the original triangle. See the drawing below. (The similarity is not difficult to verify and comes from the congruence of the corresponding angles.)



Finding four similar subtriangles from one triangle

If we now repeat this procedure for any of the subtriangles (doesn't matter which one), we get three more subtriangles. If we repeat this procedure on *any* subtriangle, we obtain three new subtriangles at each step. Hence, we have 1, 4, 7, 10, 13, ... $1+3k$, ... similar subtriangles. So, the equation we ask is: does the equation $1+3k=1999$ have a positive integer solution. Solving this equation, we find $k=666$. Hence, after 666 steps, we have 1999 similar triangles of various sizes.

Note: The size of these subtriangles depends on which subtriangle you decide to subdivide at each step ... there are all kinds of different sizes. You might play around by drawing different subdivisions yourself.

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΦΨΦΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 8 (Bridge Problem)

Four people come upon a bridge in the middle of the night. It is dark and they must use a flashlight to get across the bridge, but they only have one flashlight, and the bridge is so rickety that no more than two people an be on the bridge at the same time. One person (person A) can cross the bridge in 1 minute, the second person (person B) can cross the bridge in 2 minutes, the third person (person C) can cross the bridge in 4 minutes, and the last person (person D) can cross the bridge in 10 minutes. Can the four people cross the bridge to the other side in 18 minutes or less ? If so, how ?

Solution

The four people can cross the bridge in 18 minutes by using the following strategy:

- First person A crosses the bridge with person B (takes 2 minutes)
- then A returns (1 minute)
- then A crosses with C (4 minutes)
- then A returns (1 minute)
- finally A crosses with D (10 minutes)

Total: 18 minutes

The real question: Just because you can't find a faster way for the people to cross the bridge, have we really *proven* this is the best way ? Is possible to enumerate *all* the different ways to cross the bridge ?

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΛΦΨΦΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 9 (Algebra Problem)

Find all values of x that satisfy the equation

$$\boldsymbol{(x^2 - x - 1)^{(x^2-x-6)} = 1}$$

Solution

Using the general algebraic fact that if $a \neq 0$, then $a^b = 1$ implies $b = 0$, we have

$$x^2 - x - 6 = 0$$

which has solutions $x = 3,\ x =$ - 2.

ΦΛΨΘΠΩΘΣΠΔΥΞΣΦΠΨΣΦΩΠΣΓΔΓΩΦΩΛΓΠΣΦΛΩΠΘΞΥΞΠ

Problem 10 (Calculator Problem)

A calculator can find the reciprocal of a number as well as addition and subtraction. Describe how you would go about finding:

(a) the square of any number
(b) the product of two numbers

Solution:

(a) To find the square of any real number a $(a \neq 0)$, we use our calculator, which can add, subtract, and find reciprocals, to compute

$$A = \tfrac{1}{a} - \tfrac{1}{n+a} = \frac{n}{a^2 + an}$$

and

$$\tfrac{1}{A} = \frac{a^2 + an}{n} = \frac{a^2}{n} + a$$

where we can choose n as any positive integer as long as it is different from a. Using our calculator again, we compute the difference

$$\frac{a^2}{n} = \frac{a^2}{n} + a - a$$

and then add a^2/n with itself a total of n times, getting a^2, the square of the original number.

(b) To find the product of a and b, we observe the algebraic identity

$$ab = \left(a + \tfrac{b}{2}\right)^2 - a^2 - \frac{b^2}{4}$$

If you look carefully, every term on the right-hand side of the equation can be calculated using our ($+$, $-$, reciprocal) calculator since

(i) $\frac{b}{2} = \frac{1}{\frac{1}{b} + \frac{1}{b}}$

(ii) We can compute squares using the strategy in (a)

$(iii) \quad \frac{b^2}{4} = \left(\frac{b}{2}\right)^2$

Note: This is one of those problems that seems to have more entertainment value than practical value, until you consider the fact that it might be easier to make electronic circuitry for adding, subtracting, and taking reciprocals than for computing products. The same was true for the Radon transform earlier in this century until a medical researcher at Tuft's University discovered that it could be used for making medical images, such as *CAT* scans and *MRIs*.

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