September Solutions : Grades 6-12
Maine Math and Science Talent Search

1. **Gone Fishing** Andrew, Billy, and Tom have gone fishing and each caught a different number of fish and suppose you hear that:

Andrew:	"I caught the most fish." "Tom caught the least number of fish."
Billy:	"I caught the most number of fish." "I caught more than Andrew and Tom put together."
Tom:	"I caught the most number of fish." "Billy caught half the number of fish as I did."

The only thing you know for certain is that out of the six statements exactly three are true and three are false. Who caught the most number of fish ? Who caught the least number of fish ?

Solution Of the boy's first statement, exactly one is true and two are false. Hence, two of their second statements must be true and one false. Now, the second statement of Andrew and Tom are contradictory, and so

1. Billy's second statement must be true
2. Hence, Billy's first statement is true also which means Andrew and Tom's first statements are false.
3. Hence, Andrew and Tom's first statements are false
4. Hence, Andrew's second statement is true.

We summarize this below by writing Ts and Fs next to the respective statements saying whether they are true or false:

Andrew:	"I caught the most fish." (F) "Tom caught the least number of fish." (T)
Billy:	"I caught the most number of fish." (T) "I caught more than Andrew and Tom put together." (T)
Tom:	"I caught the most number of fish." (F) "Billy caught half the number of fish as I did." (F)

2. **The Pitcher's Age** Three former professional baseball players, a catcher, a pitcher, and a shortstop are telling their old manager about the good old days. They recall:

   1. The age of the catcher is an odd integer and the age of the others are both even integers.

2. The shortstop is four years older than the pitcher.
3. When the catcher was as old pitcher is now, he was twice as old as the pitcher was then.
4. The least common multiple of the three player's ages is the year the manager was born.

In what year was the manager born ?

Solution From 1 we know the pitcher's age is an even integer and so it is convenient to denote it by $2x$. From 2, we then have that shortstop's age is $2x+4$. Now, if you read fact 3 carefully, it says that the catcher's age is 1.5 times the age of the pitcher. In other words, the catcher's age is $3x$. This is clear since when the catcher was $2x$ years old (same as the pitcher is now), the pitcher was x years old (half the age as the catcher). Hence the ages of the players are

Pitcher	**Catcher**	**Shortstop**
$2x$	$3x$	$2x+4$

Now finally, fact 4 states the manager was born in the year of the least common multiple of the ages of the three players, or the least common multiple of $2x,\ 3x,\ 2x+4$. We now make a table showing x and the least common multiple of $2x,\ 3x,\ 2x+4$ for different values of x. If we do this we will see the only possible value of x which gives a feasible year of birth of the manager is $x=17$ which means the pitcher, catcher, and shortstop are 34, 51, and 38 respectfully, and the manager was born in the year 1938 (which means he is 61 this year).

$\boldsymbol{x}$	$\boldsymbol{(2x,\ 3x,\ 2x+4)}$	**Least common multiple of** $\boldsymbol{(2x,\ 3x,\ 2x+4)}$	
1	$(2,\ 3, 6)$	6	
2	$(4,\ 6,\ 8)$	24	
3	$(6,\ 9,\ 10)$	90	
.	...	...	
17	$(34,\ 51,\ 38)$	1938	$\leftarrow$ only feasible date of birth
.	...	...	
24	$(48,\ 72,\ 52)$	1872	
25	$(50,\ 75,\ 54)$	2025	
.	...	...	

Note: If you are really good with numbers, then you can convince yourself that the least common multiple of the three numbers $2x,\ 3x,\ 2x+4$, which we write as LCM $(2x,\ 3x,\ 2x+4)$ can be expressed as

$$\text{LCM }(2x,\ 3x,\ 2x+4)\ =\begin{cases} 3x\,(x+4) & x \text{ even} \\ 3x\,(2x+4) & x \text{ odd and } x+2 \text{ is not a multiple of 3} \\ x\,(2x+4) & x \text{ odd and } x+2 \text{ is a multiple of 3} \end{cases}$$

and so you can compute the least common multiples for any x every easily.

3. **Mystery Number** Find a 4-digit number which is a perfect square with the property that the first two digits is one more than the last two digits.

Solution If we denote by x be the number formed from the last two digits, then $x+1$ is the number formed from the first two digits. And since we know the 4-digit number is a perfect square, we can write it as $100\,(x+1)\;+\;x=\;y^2$, which simplifies to

$$(y-10)(y+10)=\;101x$$

But since 101 is a prime number, we know that either $y-10$ or $y+10$ is divisible by 101, and furthermore since y^2 is a 4-digit number, we know that y must be at most a 2-digit number. Hence $y+10=101$, which gives $y=91$, and hence the mystery number is $y^2=91^2=\;8281$.

4. **Radical Equation** Solve

$$(2x+3)^2+(3x-5)^2-\;\sqrt{13x^2-18x+20}\;=\;44$$

Solution We can rewrite this equation as

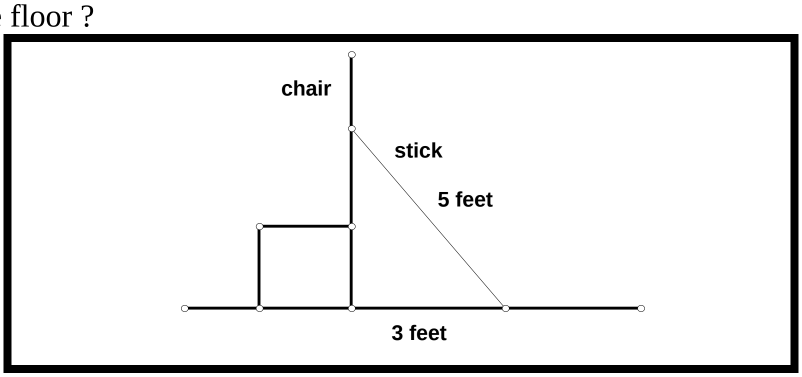
$$13x^2-18x-10-\sqrt{13x^2-18x+20}\;=\;0$$

and by letting $y=\;\sqrt{13x^2-18x+20}$, we get $y^2-y-30=0,$ which has roots

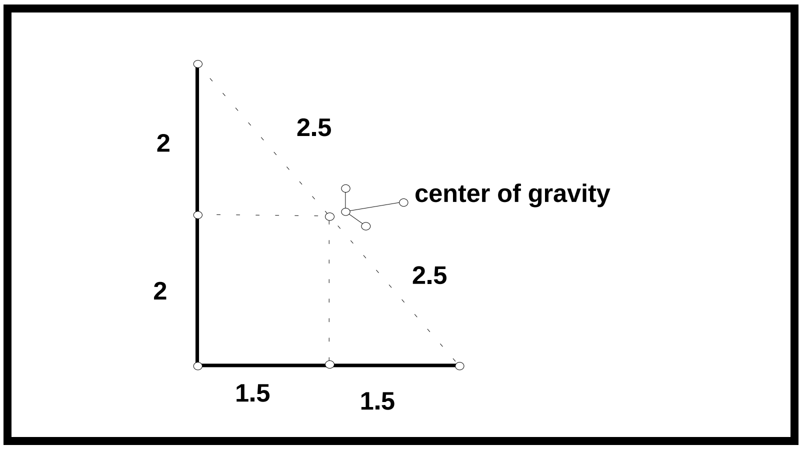
$$y_1\;=\;-5 \Rightarrow \text{ no solutions}$$

$$y_2=\;6\;\Rightarrow\;x_1\;=\;2,\;x_2\;=\;-\tfrac{8}{13}$$

5. **Stick Problem** A 5 foot long stick leans against a chair in such a way that the base of the stick is 3 feet from the chair. Suddenly the chair is yanked away from the stick so the stick falls to the floor. Assuming the floor is perfectly smooth and offers no friction to the stick, what is the final position of the two ends of the stick ? In other words, how much further to the left and right will the two ends of the stick be when the stick is on the floor ?



Solution If the surface of the floor offers no friction to the stick then the center of gravity of the stick will fall straight down, causing the stick to rotate around the center of gravity. Hence, the end point of the stick that was originally against the chair will end up 1 foot further to the left, and the right end of the stick on the floor will end up 1 foot to the right from where it was originally.



6. **Numbers Game** You are given 5 numbers and if you add all possible combinations of them two at a time you will get the numbers 0, 2, 4, 4, 6, 8, 9, 11, 13, and 15. What are the numbers ?

Solution Let us call the five numbers $x,\ y,\ z,\ v,\ w$ and assume we have named them so they are in increasing size. In other words x is the smallest, then y, then z, then v, and finally w is the largest. The first thing we observe is that if we add all 10 sums

$$\begin{aligned} x+y,\ x+z,\ x+v,\ x+w \\ y+z,\ y+v,\ y+w \\ z+v,\ z+w \\ v+w \end{aligned}$$

we get 4 $(x+y+z+v+w) = 72$, which implies $x+y+z+v+w = 18$. We also know that the smallest sum is $x + y = 0$, and the largest sum is $v+w = 15$, and so combining these two equations with the first equation gives $z = 3$. But the second smallest sum $x+z$ is 2, and so $x = -1$ (and hence $y = 1$). Also, the second from the largest sum of $z+ w$ is 13 and so we have $w = 10$. Finally, using the equation $v+w = 15,$ we get $v = 5$. Hence, the five numbers are -1, 1, 3, 5, and 10.

7. **Trucks and Containers** You are given the assignment of transporting several containers. You are not told how *many* containers but you are told that none of the containers weighs more than 1 ton, and that the combined weight of all of the containers is exactly 10 tons. You are given trucks to move the containers, and each truck holds a maximum of 3 tons. How many trucks are required to haul all the containers ?

Solution First of all, let's establish the fact that we can *always* carry all the containers with 5 trucks since any truck can carry containers whose total weight is at least 2 tons.

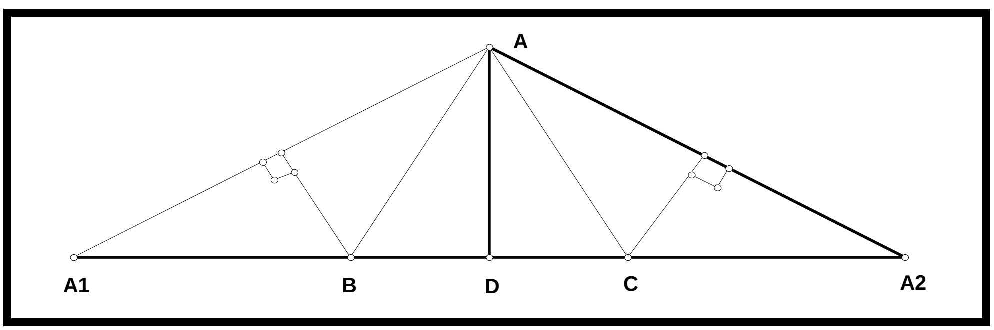
(You see why this is true ?) The question then is, can we get by with four trucks? The answer is no since if we happened to have 13 containers, each weighing 10/13 tons, then the maximum number of containers we could haul in each truck would be 3 since 4 (10/13) > 3 which mean 4 containers is too much. That means we would need a 5th truck to haul the last container.

8. **The Case of the 4-digit Number** We start with any four consecutive digits (positive integers) and form the 4-digit number consisting of these four numbers. (For example, if the numbers were 5, 6, 7, 8, then the 4-digit number would be 5678.) Now, exchange the order of the first and second digits in this 4-digit number and show that the resulting 4-digit number is divisible by 11.

Solution We denote the four original digits as $x,\ x+1,\ x+2,\ x+3$, and form the 4-digit number $1000x+100\,(x+1)\ +\ 10\,(x+2)\ +\ x+3$. Exchanging the first two digits gives $1000\,(x+1)\ +\ 100x\ +\ 10\,(x+2)\ +\ x+3\ =\ 1111x+1023$. But since both 1111 and 1023 are divisible by 11, so is $1111x+2023$.

9. **Constructing an Isosceles Triangle** Construct an isosceles triangle, if the perimeter and altitude are given.

Solution Before showing the construction, consider the isosceles triangle ABC drawn below, and the line A_1BCA_2, where the length of $A_1B\ =$ length AB, length $A_2C =$ length AC.



We are now given the perimeter of the triangle, which is A_1A_2, and the altitude, which is AD. So we draw a line of length A_1A_2 and find its perpendicular bisector of length AD. We then form the triangle AA_1A_2. Finally, the perpendicular bisectors of A_1A and A_2A determine the points B and A, respectively.

10. **The Clock Mystery** Suppose you are told the two hands of the a clock are perpendicular to each other. What are the possible times?

Solution

First of all we know that the minute hand turns 360° every hour and the hour hand turns 30° every hour. Or in terms of minutes, the minute hand turns 6° every minute and the hour hand turns 0.5° every minute. The problem is to find those times (lets say in minutes) when the angle between the two hands is either 90°, 3 (90°) $=\ 270°$,

$5\,(90°)\; = 450°,\; 7\,(90°)\; = 630°,\; ...\,(2k+1)\,90°$. Hence, if we call x the number of minutes past midnight, we seek those values of x that satisfy

$$6x\; -\; 0.5x\; =\; (2k+1)\,90° \qquad k\; =\; 0,\; 1,\; 2,\; ....$$

or

$$5.5x\; =\; (2k+1)\,90°$$

Solving this equation over the course of one day, gives

$k=0$	$5.5x = \; 1(\,90)\; = \; 90$	$\Rightarrow\; x\; =\; 0\,hr\,16\,\frac{4}{11}$ minute
$k=1$	$5.5x\; =\; 3\,(90) = 270$	$\Rightarrow\; x\; =\; 0\,hr\,49\,\frac{1}{11}$ minute
$k=2$	$5.5x\; =\; 5\,(90)\; = 450$	$\Rightarrow\; x = 1\,hr\,21\,\frac{9}{11}$ minute
$k=3$	$5.5x\; =\; 7\,(90)\; = 630$	$\Rightarrow\; x = 1\,hr\,54\,\frac{6}{11}$ minute
$k=4$	$5.5x\; =\; 9\,(90) = \; 810$	$\Rightarrow\; x = 2\,hr\,27\,\frac{3}{11}$ minute
$k=5$	$5.5x\; =\; 11\,(90)\; =\; 990$	$\Rightarrow\; x = 3\,hr$
$k=6$	$5.5x\; =\; 13\,(90)\; =\; ...$	$\Rightarrow\; x = 3\,hr\,32\,\frac{8}{11}$ minute
$k=7$	$5.5x\; =\; 15\,(90)$	$\Rightarrow\; x = 4\,hr\,5\,\frac{5}{11}$ minute
$k=8$	$5.5x\; =\; 17\,(90)$	$\Rightarrow\; x = 4\,hr\,38\,\frac{2}{11}$ minute
$k=9$	$5.5x\; =\; 19\,(90)$	$\Rightarrow\; x = 5\,hr\,10\,\frac{10}{11}$ minute
$k=10$	$5.5x\; =\; 21\,(90)$	$\Rightarrow\; x = 5\,hr\,43\,\frac{7}{11}$ minute
$k=11$	$5.5x\; =\; 23\,(90)$	$\Rightarrow\; x = 6\,hr\,16\,\frac{4}{11}$ minute
$k=12$	$5.5x\; =\; 25\,(90)$	$\Rightarrow\; x = 6\,hr\,49\,\frac{1}{11}$ minute
$k=13$	$5.5x\; =\; 27\,(90)$	$\Rightarrow\; x = 7\,hr\,21\,\frac{9}{11}$ minute
$k=14$	$5.5x\; =\; 29\,(90)$	$\Rightarrow\; x = 7\,hr\,54\,\frac{6}{11}$ minute
$k=15$	$5.5x\; =\; 31\,(90)$	$\Rightarrow\; x = 8\,hr\,27\,\frac{3}{11}$ minute
$k=16$	$5.5x\; =\; 33\,(90)$	$\Rightarrow\; x = 9\,hr$
$k=17$	$5.5x\; =\; 35\,(90)$	$\Rightarrow\; x = 9\,hr\,32\,\frac{8}{11}$ minute
$k=18$	$5.5x\; =\; 37\,(90)$	$\Rightarrow\; x = 10\,hr\,5\,\frac{5}{11}$ minute
$k=19$	$5.5x\; =\; 39\,(90)$	$\Rightarrow\; x = 10\,hr\,38\,\frac{2}{11}$ minute
$k=20$	$5.5x\; =\; 41\,(90)$	$\Rightarrow\; x = 11\,hr\,10\,\frac{10}{11}$ minute
$k=21$	$5.5x\; =\; 43\,(90)$	$\Rightarrow\; x\; = 11\,hr\,43\,\frac{7}{11}$ minute