Maine Math and Science Talent Search
Round 5 Solutions/Grades 6-12

1. **Maze Problem** Is it possible to go from Cell A to Cell B in the diagram below in such a way that you pass through each cell exactly once ? If so how ?

	A				
					B

↓	A	→	→	→	↓
→	→	↑	↓	←	←
↓	←	←	←	→	↓
→	→	→	→	↑	↓
↓	←	←	←	←	←
→	→	→	→	→	B

Solution There are several solutions. One of the simplest is above at the right.

2. **100 Problem** It is possible to write the number 8 using each of the digits 1, 3, 4, 6, and 7 exactly once as $8 = 7 + \frac{1}{3} + \frac{4}{6}$. Write 100 using each of the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 exactly once.

Solution Yes, write 100 $= 50 + 49 + \frac{1}{2} + \frac{38}{76}$

3. **Mystery Numbers** Find the smallest positive integer (whole number) whose half is the square of some integer and whose third is the cube of some integer.

Solution Calling this unknown integer I, we know that I/2 and I/3 are also integers since we are given they are the squares and cubes of integers. Since I/2 is an integer we know that I/2 $=$ M^2 for some integer M, and I/3 $= N^3$ for some integer N. Hence, I = $2M^2$ and I = $3N^3$ and thus $2M^2 = 3N^3$. Since $2M^2$ clearly contains a factor of 2 in it so does N^3 and N must also. Hence we know N = 2, 4, 6, .. But $3N^3$ clearly has a factor of 3 in it and thus so does M^2 and likewise so does M. Hence M = 3, 6, 9, ... Using this as a guide we try a few numbers, keeping in mind M must be larger than N in order that $2M^2 = 3N^3$.

N	**M**	**N^3**	**M^2**	**$3N^3$**	**$2M^2$**	
2	3	8	9	24	18	← M is too small (increase it)

2	6	8	36	24	72	← M is too big (try new N)
4	6	64	36	192	72	← M is too small (increase it)
4	9	64	81	192	162	← M is too small (increase it)
4	12	64	144	192	288	← M is too big (try new N)
6	9	216	81	648	162	← M is too small (increase it)
6	12	216	144	648	288	← M is too small (increase it)
6	15	216	225	648	450	← M is too small (increase it)
6	**18**	**216**	**324**	**648**	**648**	**← M is just right**

Hence, we have N $=$ 6, M $= 18$, and I = 648. We check this by observing that one half of 648, or $\frac{1}{2} \cdot 648 = 324$ is the square of 18, and one third of 648, or $\frac{1}{3} \cdot 648 = 216$ is the cube of 6.

4. **Maximum Months Problems** What is the maximum number of months in a year with 5 Sundays?

Solution Since each month has either 28, 29, 30, or 31 the number of Sundays in each month will be 4 or 5. Also, since there are either either $365 = 52 \times 7 + 1$, or $366 = 52 \times 7 + 2$ days in a year, depending on whether it is a leap year, every year has either 52 or 53 Sundays. Now you can't have 6 months with 5 Sundays since this would mean the total number of Sundays in a year would be 6 × 5 = 30 Sundays plus 6 × 4 = 24 Sundays equals 54 Sundays, which is impossible. Hence the maximum number of months with 5 Sundays is not more than five. To determine which months have the 5 Sundays we could make up calendars in which Jan 1 occurs on each of the 7 days Sun, Mon, Tue, ... Sat and see when the Sundays occur. You will discover that in a non leap year (28 days in February) the only time you will get 5 months with 5 Sundays is when Jan 1 occurs on a Sunday. On leap years (29 days in Februrary) the only time you get five months with 5 Sundays is when Jan 1 occurs on a Saturday. You can figure out which months they are yourself.

5. **Three Students Problem** Three students are buying snacks for their friends. One student buys 4 sandwiches, a coke, and 10 donuts for \$10. Another student buys 3 sandwiches, a coke, and 7 doughnuts for \$7.50. The third student buys a sandwich, a coke, and a doughnut. How much did the third student pay ?

Solution The purchases of the first two students says

$$(1)\ 4s + c + 10d = \$10, \qquad (2)\ 3s + c + 7d = \$7.50$$

If we now compute 3 times the second equation minus 2 times the first equation, we get

$$\begin{aligned} 3\,(3s + c + 7d) - 2(4s + c + 10d) &= s + c + d \\ &= 3\,(\$10) - 2\,(\$7.50) \\ &= \$15 \end{aligned}$$

6. **Santa's Bonuses** Santa is giving \$50 apiece to a group of students who did their homework all year but discovers that the last student only got \$45. In order to be fair, he decides to give each student \$45 and keep the remaining money, which is \$90, for next Christmas. How much money did Santa have at the start ?

Solution By taking \$5 back from each student Santa gets \$90 for next year, which means there are 90/5 = 18 students that originally got \$50. Hence, the total number of students is 19, which means the total amount of money Santa had was $18 \times \$50 + \$45 = \$945$.

7. **Going to School** Every day Mary walks to work and her mother gives her a ride home, the total time being an hour and a half. Sometimes when it rains her mother takes her both ways, this time the total time is 30 minutes. How long does it take Mary to walk to school ?

Solution If it takes 1 hr 30 min the first way (walking to school, riding home) then it takes 3 hours to make two round trips this way. Hence, time for a round trip walking $+$ time for a round trip riding $= 3$ hours. But we know the time for a round trip riding is 30 minutes so it takes $3\,\text{hrs} - 30$ min = 2 hrs 30 min to make a round trip walking. Hence, it takes Mary 1 hr and 15 minutes to walk to school.

8. **Math Magic** There are 9 cards on the table of a single suit from the ace to the nine. You tell two people to select a card but don't show their cards to you. You tell the first person to double the number of his card, add 1, multiply by 5 and pass the result to the second person. Then you tell the second person to add his card number to the number given him, multiply by 2, add 1, multiply by 5 and tell you the result. Suppose the number told by the second person is 83. What are the numbers picked by the 2 people ?

Solution If the numbers picked by the first and second persons were a and b, then the first person computes in succession: $a,\ 2a,\ 2a+1,\ 5\,(2a+1) = 10a+5$. The second person then computes in succession:

$$b + 10a + 5$$
$$2\,(b + 10a + 5) = 2b + 20a + 10$$
$$2b + 20a + 11$$
$$5\,(2b + 20a + 11) = 10b + 100a + 55$$

after which this person gives you this number. To determine a and b we subtract 55 from the given number and divide by 10, getting $10a + b$, which means that the number drawn by the first person is the tens digit of the given number, and the number drawn by the second person is the units digit of the given number. Hence, if the second person gives you the number 83, this means the first person drew an 8 and the second person drew a 3.

The interesting thing about this problem is that if you had any number of people (9 or less), with each person doing the same thing, you could find the number each person drew just as easily. For example, if there were 4 people and the last person gave you the final computed number, and after you subtract 5555 (4 fives in this case) and dividing by 10, you got 3521, you can conclude that the first person got the 3, the second person the 5, the third person the 2, and the last person the ace.

9. **Doing Arithmetic in Your Head** Devise a strategy so that you can square 75 in your head.

Solution The number 75 can be written $75 = (10a + 5)$, where $a = 7$. Hence

$$\begin{aligned} 75^2 &= (10a+5)^2 \\ &= 100a^2 + 100a + 25 \\ &= 100\,a\,(a+1) + 25 \\ &= 100\,(7)\,(8) + 25 \\ &= 5625 \end{aligned}$$

To find 85^2 we have $a = 8,$ and so $85^2 = 100(8)(9) + 25 = 7225$.

10. **Interesting Properties** Find two numbers whose difference and quotient is each 10.

Solution Calling the numbers x and y we have the two equations $x - y = 10$, $x/y = 10$. Solving for y in the second equation we get $x = 10y$, and then plugging this value into the first equation gives $10y - y = 10$, or $9y = 10$, or $y = 10/9$. Plugging this value into either of the two equations give $x = 100/9$. In general the solution of $x - y = a$, $x/y = a$ is $x = a^2/(a-1)$, $y = a/(a-1)$.