Maine Math, Science and Engineering Talent Search
Round 6 Solutions/Grades 6-12

1. **(The Case of the Mailroom Foul-up)** After a typist had written 5 letters and addressed the corresponding envelopes, someone in the mailing room carelessly inserted the letters in the envelopes at random, one letter per envelope. What is the probability that exactly 4 of the 5 letters were inserted in the proper envelopes ?

Solution It is impossible for *exactly* 4 of the 5 letters to be placed in correct envelope since if 4 are placed correctly then so is the last letter. Hence, the probability is zero.

2. **(Candy Problem)** Mary goes to a candy counter and buys p pieces of candy for d dollars, where we assume p and d are integers. When Mary is leaving the store owner tells Mary if she buys 10 more pieces the total cost will be \$2, and that she will save 80 cents per dozen in the process. How many pieces of candy (p) did Mary originally buy and what was the original price (d)?

Solution Since the original price d for all the candy is an integer less than \$2, we know $d = \$1$. Also if c_0 is the original cost for each piece of candy, we have $c_0 p = d = 1$, and hence the cost per piece is $c_0 = 1/p$. Now, we are told that if Mary buys 10 more pieces of candy the cost is \$2, and hence if c_1 is the new price per piece, we have $c_1\,(p + 10) = 2$, which means the new cost per piece will be $c_1 = 2/(p + 10)$. In addition, the store owner tells Mary that if she buys these 10 extra pieces she will save 80 cents per dozen in the process, and hence

$$12\,(c_0 - c_1) \equiv 12\left(\frac{1}{p} - \frac{2}{p+10}\right) = 0.80$$

which is equivalent to $p^2 + 25p - 150 = 0$, which has one positive solution $p = 5$.

As a check, Mary goes to the store wanting to buy $p = 5$ pieces of candy for $d = \$1$, at a cost per piece of $c_0 = 1/p = \$0.20$, which means she was paying (\$0.20)(12) $=$ \$2.40 per dozen. The store owner told Mary that if she buys $p_0 + 10 = 15$ pieces, the total cost will be \$2, which means the cost per piece is now $2/(p_0 + 10) = 2/15$ dollars, or (2/15)(12) $=$ \$1.60 per dozen. Hence, if Mary takes up the store owner's offer, she will save \$0.80 per dozen as advertised.

3. **(Kissing Problem)** Every person on earth has kissed a certain number of persons. Show that the number of persons who has kissed 1, 3, 5, ... other persons is either 0, 2, 4, 6, ... In other words the number of persons who has kissed an odd number of times is always an even integer.

Solution Before anyone kisses anyone the number of persons who has kissed 1, 3, ... persons is 0, and hence we initially have an even number of odd kissers (those how have kissed an odd number of times 1, 3, 5, ...). The first kiss between any two people produces two "odd kissers". After that each kiss will be either between (a) two odd kissers, (b) two even kissers, or (c) an odd and even kisser. Each even-even kiss will *increase* the number of odd kissers by two (keeping the odd kissers an even number). Each odd-odd kiss will *decrease* the number of odd kissers by two (keeping the odd kissers an even number). And finally an even-odd kiss will not change the number of odd kissers since the odd kisser becomes and even kisser, and the even kisser becomes an odd kisser. Therefore the number of odd kissers will always be even.

To understand more thoroughly, draw 5 people, and start them kissing. You will see the claim is true, no matter who kisses whom how many times.

4. **(Printing Machines)** We have three machines. The first machine accepts a card with the numbers a and b on it and returns a card with the numbers $a+1$ and $b+1$. The second machine accepts a card with numbers a and b on it and returns a card with the numbers $a/2$ and $b/2$. The third machine accepts two cards with numbers a, b and b, c respectively and returns a card with numbers a and c. Is it possible for any of machines to produce a card with numbers 1 and 1988, if the first machine is given a card with numbers 5 and 1988. We assume each machine is able to accept cards previously formed.

Solution This problem reeks of the concept of invariance; something that does not change no matter what step in the operation. The idea is to associate something with the first two numbers 5 and 19 that will not change from step to step, but something that is *not* possessed by the desired final pair 1 and 1988. The $64 dollar question is; WHAT IS IT? Well, if we consider a few feasible steps the machines might take:

machine 1 (5, 19) → (6, 20) (difference a multiple of 7)
machine 2 (6, 20) → (3, 10) (difference a multiple of 7)
17 steps of machine 3 (3, 10) → (20, 27) (difference a multiple of 7)
machine 3: (6, 20) and (20, 27) → (6, 27) (difference a multiple of 7)

But we see that the starting pair 5, 19 has a difference of -14, a multiple of 7, and it appears that machine, although changing the pairs will always have a difference that is a multiple of 7. We have

machine 1: $(a,\, b) \to (a+1,\, b+1)$

will not change the difference between the numbers, and

machine 2 : $(a,\, b),\ \to (a/2,\, b/2)$

which only accepts even numbers, has a new difference $(a-b)/2$ which is also a multiple of 7, and

machine 3: $(a,\, b),\ (b,\, c)\ \rightarrow (a,\, c)$

produces a pair with difference $a-c = (a-b)+(b-c)$, and thus has a difference which is a multiple of 7 if $a-b,\ b-c$ are multiples of 7. But the final pair 1, 1988 has a difference of 1987 which is not a multiple of 7. Hence, it is impossible to arrive at 1, 1988.

5. **(Mr. Smith's Age)** Mr. Smith lived the first 1/4th of his life as a child, the next 1/5th his life as a youth, the next 1/3 rd his life in his middle years, and the last 13 years until the present time. What is Mr. Smith's age ?

Solution Calling

$c =$ number of years Mr. Smith was a child
$y =$ number of years Mr. Smith was a youth
$m =$ number of years Mr. Smith spent in his middle years
$a =$ Mr. Smith's present age

we have $c = a/4$, $y = a/5$, $m = a/3$, $c+y+m+13 = a$, which is a system of 4 equations and 4 unknowns. We solve by substituting the first 3 equations into the last equation getting the equation $a/4 + a/5 + a/3 + 13 = a$, which has the solution $a = 60$. Hence, $c = 15$, $y = 12$, $m = 20$ and so Mr. Smith lived 15 years as a child, 12 years as a youth, 20 years in his middle years and is presently 60 years old.

6. **(Sally's Party)** Sally has prepared 84 cookies for a party and want every person at the party to have the same number of cookies (she never eats cookies at her own party). What are the possibilities for the number of persons she can invite to the party?

Solution Calling $N =$ number of persons at the party, $c =$ number of cookies each person eats, we must have $Nc = 84$, and so we look for all positive integer solutions of this equation. By making a simple table we see

N	**c**		**N**	**c**		**N**	**c**
84	1		14	6		4	21
42	2		12	7		3	28
28	3		7	12		2	42
21	4		6	14		1	84

In other words there are 12 different possibilities, ranging from 84 guests eating 1 cookie each to 1 guest eating 84 cookies.

7. **(Number Problem)** Find a multiple of 17 that consists of all 9s. That is like 99, 999, 9999, 99999, ... Don't use a calculator.

Solution Since any integer whose digits are all nines can be written in the form 9999 .... 9 = $10^n - 1$ for some integer n, we seek an integer n such that $\frac{10^n-1}{17}$ is also an integer. We observe (by long division) that $\frac{1}{17} = 0.588235294117647$... after which the decimals *repeat*. Hence we have $\frac{10^{16}}{17} = 588235294117647\,.\,0588235294177\ldots$ Subtracting, the decimal form of 1/17 above from the decimal form of $10^{16}/17$, we get

$\frac{10^{16}-1}{17} = 588235294117647$ or 588235294117647)(17) = 9999 ..... 9 (16 digits)

8. **(Interesting Number System)** An operation ☐ is defined on the positive integers so that if x and y are positive integers so is $x \,\square\, y$. Assuming that ☐ also satisfies the conditions

$$\begin{aligned}
&(1) \quad (x+y)\,\square\,1 = (x\,\square\,1) + (y\,\square\,1)\\
&(2) \quad x\,\square\,(y+z) = (x\,\square\,y)\cdot(x\,\square\,z)\\
&(3) \quad (x+y)\,\square\,2 = (x\,\square\,2) + 4\,(xy\,\square\,1) + (y\,\square\,2)
\end{aligned}$$

then find 7 ☐ 9 and justify your answer.

Solution By condition (1) we have

2 ☐ 1 = (1 ☐ 1) + (1 ☐ 1) $= 2a$ where we call $a = 1\,\square\,1$

3 ☐ 1 = (1 ☐ 1) + (2 ☐ 1) $= 3a$

... ...

n ☐ 1 = ((1 ☐ 1) + ($(n-1)$ ☐ 1) $= na$

Now by (2), we have

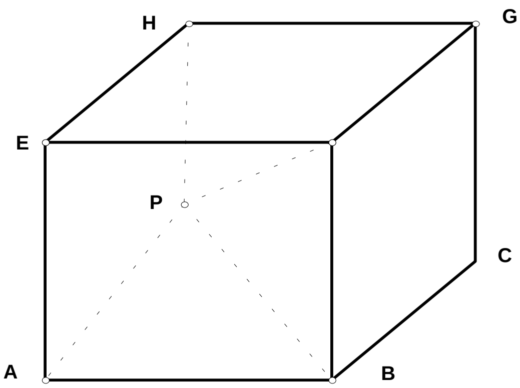
$$\begin{aligned}
n\,\square\,2 &= (n\,\square\,1)(n\,\square\,1) = (na)^2\\
n\,\square\,3 &= (n\,\square\,1)(n\,\square\,2) = (na)(na)^2 = (na)^3\\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
n\,\square\,m &= (n\,\square\,1)(n\,\square\,(m-1)) = (na)(na)^{m-1} = (na)^m
\end{aligned}$$

We can now find the value of a since we have already seen 2 ☐ 2 = 4 a^2, but from condition (3) with $x = y = 1$, we can also write

$$2 \,\square\, 2 \;=\; (1 \,\square\, 2) \;+\; 4\,(1 \,\square\, 1) \;+\; (1 \,\square\, 2) = \; a^2 \;+\; 4a \;+\; a^2$$

which has the nonzero solution $a = 2$. Hence, 1 $\square$ 1 = 2 and $n \,\square\, m \;=\; (2n)^m$. In particular, we have 7 $\square$ 9 = $(14)^9$ = 14,000,000,000.

9. **(Finding the Cube)** Let P be an interior point of a cube as shown in the figure below. We are given the distances $PA = \sqrt{50}$, $PC = \sqrt{70}$, $PH = \sqrt{90}$, $PF = \sqrt{110}$. Find the length of the edge of the cube.



Solution Let P = $(x,\, y,\, z)$ be the interior point and L the length of sides of the cube. Using the Pythagorean theorem, we have

(1) $x^2 + y^2 + z^2 = 50$

(2) $(L-x)^2 + (L-y)^2 + z^2 = 70$

(3) $(L-x)^2 + y^2 + (L-z)^2 = 90$

(4) $x^2 + (L-y)^2 + (L-z)^2 = \; 110$

from which we want to solve for L. Multiplying out the squared terms in (2), (3), (4) and using (1), we arrive at

$$2L\,(x+y) \;=\; 2L^2 - 20, \;\; 2L\,(x+z) \;=\; 2L^2 - 40, \quad 2L\,(y+z) \;=\; 2L^2 - 60$$

or $x \;=\; L/2$, $y \;=\; L/2 \;-\; 10/L$, $z \;=\; L/2 \;-\; 20/L$. Substituting these values of x, y, z into (1) we find

$$(L/2)^2 + (L/2 - 20/L)^2 + (L/2-10/L)^2 = \; 50$$

from which we get

$$3L^2/4 \;-\; 80 \;+\; 500/L^2 \;=\; 0$$

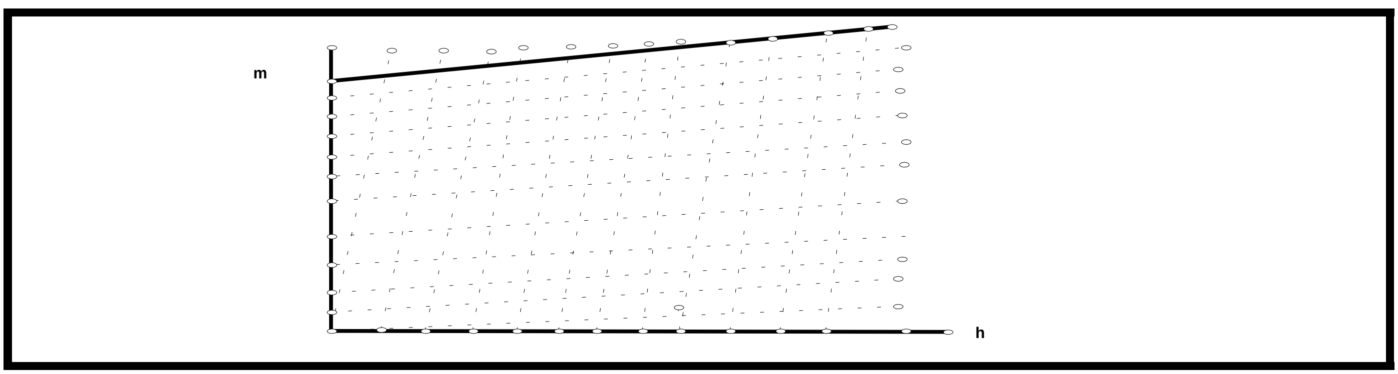
Introducing $w = L^2/4$, we get the quadratic expression $3w^2 - 80w + 125 = 0$, which as roots $25,\ 5/3$, and hence $L = 10,\ \sqrt{20/3}$. For $L = \sqrt{20/3}$ we have $y < 0$ and so P would not be an interior point. Hence, $L = 10$.

10. **(Clock Problem)** A watchmaker puts on the minute and hour hands of a clock but accidentally makes the two hands the same size so it is impossible to distinguish between the two. What the maximum error in time you can make with this clock ?

Solution If we call $m =$ coordinate on the watch of the minute hand, $h =$ coordinate of the watch face of the hour hand, then we have $m = 12\,[\,h\,]$ for $0 \le h < 12$, where $[\ ]$ represents the fractional part of the quantity. For example $[\,2.3\,] = .3$. The interesting thing is that even if the hands of the watch are the same it is possible to tell the except at 143 times during the day. The reason being

$$x \neq 12[\,y\,] \ \Rightarrow\ y = 12\,[\,x\,], \qquad y \neq 12[\,x\,] \ \Rightarrow\ x = 12[\,y\,]$$

In other words, if $h \neq 12[\,m\,]$, then we have $m = 12\,[\,h\,]$ and so we can determine the time. For example, if $h = 2.3$, then $m = 12\,[\,2.3\,] = 12\,(.3) = 3.6$ In other words the time is 3.6 (5) $= 18$ minutes past 2 (either AM or PM). The only times when we cannot determine the time is when both $m = 12\,[\,h\,]$ and $h = 12\,[\,m\,]$. If we plot these curves for $0 \le h < 12$, $0 \le m < 12$, we discover they intersect at 143 points as shown below, which represent the difference values of m and h. In other words, every $\frac{24 \cdot 60}{143} \approx$ 10.06993 minutes.



Note that the points of intersection lie on the lines $m = h + \frac{12}{13}k$, $k = 0,\ \pm 1,\ \pm 2,\ \ldots$ ± 6, and since the error in taking h as the hour hand and m as the minute hand (or visa-versa) is $|h - m|$, we have the error of $|\,h - m\,| = 12k/13$, which means the maximum error is $\frac{12 \times 6}{13} = \frac{72}{13} \approx 5$ hours, 32 minutes. This occurs at the intersection of the lines $m = 12(h - 2)$ and $m = \frac{1}{12}h + 8$, which occurs at the point $h = \frac{144 \cdot 8}{143 \cdot 3} \approx 2$ hrs, 40 min, and $m = \frac{1}{12}h + 8 \approx 8$ hrs 12 min, which if we make an error in the two hands this results in an error in time of 5 hrs, 32 minutes.