Maine Math, Science and Engineering Talent Search
Round 6 Solutions/Grades 6-9

1. **(Jogging Problem)** John and Mary are jogging next to each other around a quarter mile oval track (circular ends and a straight straight-away) where Mary runs 3 feet outside of John. How much further must Mary run each lap than John ?

Solution The only time Mary runs further than John is at the two ends of the track, so if we collapse the ends into a circle, the extra distance Mary must run is the circumference of the circle Mary travels minus the circumference of the circle John travels. Calling the circumferences of Mary and John's circles as R_M and R_J respectively, we have

$$\begin{aligned}\text{Mary's circle} - \text{John's circle} &= 2\pi R_m - 2\pi R_J \\ &= 2\pi\,(\mathrm{R_M} - R_J) \\ &= 2\pi\,(3) = 6\pi \\ &\approx 6\,(3.14) \\ &= 18.84\,\text{feet}\end{aligned}$$

In other words Mary must run $2\pi\ \Delta\mathrm{R} = 2\pi\,(3) \approx 18.84$ feet further each lap, where $\Delta\mathrm{R} = 3$ feet is the distance Mary is outside of John. If she had been $\Delta R = 4$ feet outside, she would have had to run $2\pi\ \Delta\mathrm{R} = 2\pi\,(4) = 8\pi \approx 25.12$ feet further each lap.

2. **(General Term of a Sequence)** Find a formula for the nth term in the sequence $-5,\ 3,\ -5,\ 3,\ -5,\ 3,\ -5,\ 3, \ldots$.

Solution The midpoint between -5 and 3 is the average

$$\tfrac{1}{2}\,(-5+3) = -1$$

and the distance of this point to both 3 and -5 is 4. Hence, the general expression for the nth term is $a_n = -1 + (-1)^n\,4,\ n = 1, 2, \ldots$ which gives $-5,\ 3,\ -5,\ 3,\ \ldots.$.

3. **(Lawn Mowing Problem)** John can mow a lawn in 2 hours and Mary can mow the same lawn in 3 hours. How long will it take both of them together to mow the lawn ?

Solution Since John can mow the lawn in 2 hours, we know that in 1 hour he will have 1/2 the lawn mowed. By the same argument in one hour Mary will have mowed 1/3 of the lawn. Hence together in one hour they can mow 1/2 + 1/3 $= 2/3$ of the lawn. Hence, together it will take them 1/(2/3) = 3/2 = 1 hr 30 minutes to mow the lawn.

In general if John can mow the lawn in J hours and Mary can mow it in M hours, then in one hour John will have mowed 1/J th of the lawn and Mary 1/M th of the lawn. Hence together the can mow (1/J + 1/M) th of the lawn. Taking the reciprocal, we have

$$\text{hours it takes mow the lawn together} = \frac{1}{1/J + 1/M}$$

4. **(Chessboard Problem)** Can an ordinary 8 $\times$ 8 square chessboard be covered with 1 $\times$ 2 dominoes so that only the squares on the opposite corners are uncovered ?

Solution Since each domino covers two squares we know that exactly 32 dominoes will cover the board of 64 squares. We also know that the opposite corners of a chess board have the same color, hence by not covering the opposite corners we will have covered 32 squares of one color and 30 squares of the other color. But each domino covers a square of each color, hence it is impossible to cover the board leaving two opposite corners uncovered.

5. **(Three Piles of Stones)** Divide a pile of 555 stones of masses 1 oz., 2, oz., 3., oz, ... 555. oz. into three piles of equal mass.

Solution We subdivide the stones into two parts; the 370 lighter ones (1 oz., 2 oz, ... 370 oz) in one group, and the 185 heavier ones (371 oz, 372 oz, ... 555) oz in the second group. We then proceed by selecting stones for the 3 piles in the following order:

	Pile 1		Pile 2		Pile 3
Mass	1, 2, 555 $= 558$	$\rightarrow$	3, 4, 554 $= 561$	$\rightarrow$	5, 6, 553 $= 564$

Note the masses of the piles increase by 3. To offset this, we now pick the next selection of stones for the piles in backwards order, getting

	Pile 3	→	Pile 2	→	Pile 1
Mass	7, 8, 552 = 567		9, 10, 551 = 570		11, 12, 550 = 573

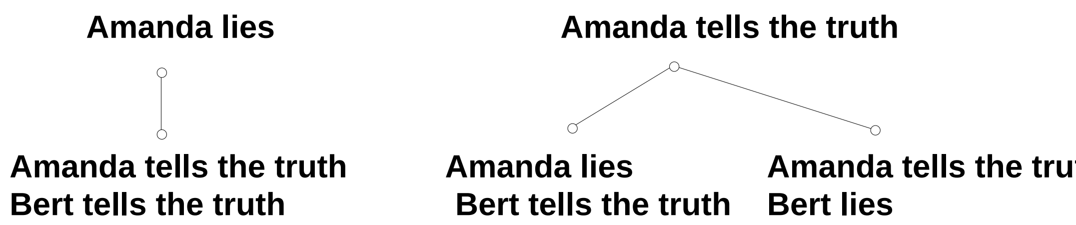
so that now, the three piles have a common total mass of 1131 oz. If we continue this process, taking larger and larger stones from the first group and a lighter stone from the heavier group, we get

Pile 1	Pile 2	Pile 3
13, 14, 549	15, 16, 548	17, 18, 547
23, 24, 545	21, 22, 544	19, 20, 546
....	....	...
359, 360, 376	361, 362, 375	363, 364, 374
369, 370, 371	367, 368, 372	365, 366, 373

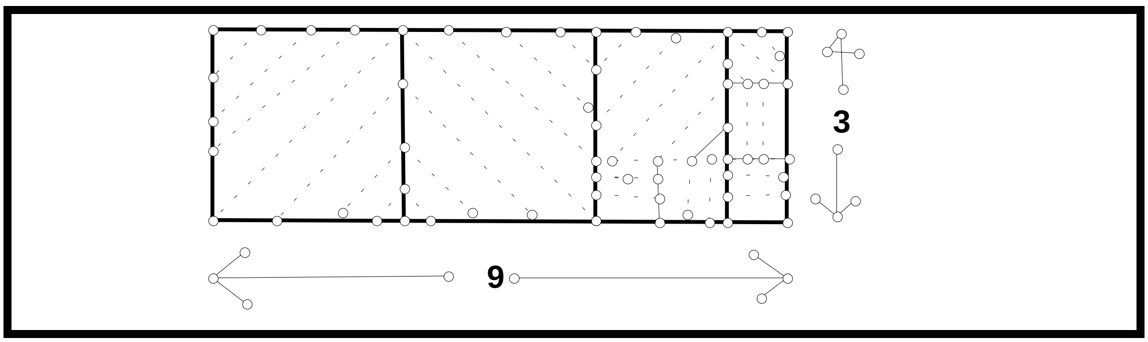
The sum of the masses in every two rows of this process will be a constant; the common weight of the piles in the last two rows being 2205. Since the total mass of the stones is $1 + 2 + 3 + ... + 555 = 555\,(556)/2 =$ 154,290 oz, each pile will have a mass of 154,290/3 = 51,430 oz.

6. **(Knights and Knaves)** In a town there are two kinds of people; those who always tell the truth and those who always lie. Amanda and Bert live in this town. Amanda says to Bert, "At least one of us is a liar." Is Amanda telling the truth or lying ? What kind of a person is Bert ?

Solution The diagram helps explain the various possibilities. Let us assume for the moment Amanda is a liar. Then it follows that that both she and Bert always tell the truth. But this can't happen since we are assuming Amanda is lying. Hence, Amanda must be telling the truth (look at the right part of the diagram below), which means that either Amanda is a liar and Bert tells the truth or that Amanda tells the truth and Bert a liar. But since we are assuming Amanda is telling the truth, we know she must be telling the truth and Bert the liar. Hence, we conclude Amanda is the constant truth-teller and Bert is the constant liar.



7. **(Subdividing a Rectangle)** Cut a 3" $\times$ 9" rectangle into 8 squares.
Solution The 3" $\times$ 9" rectangle below is cut into 8 squares; two 3" $\times$ 3" squares, one 2" $\times$ 2" square, and five small 1" $\times$ 1" squares.

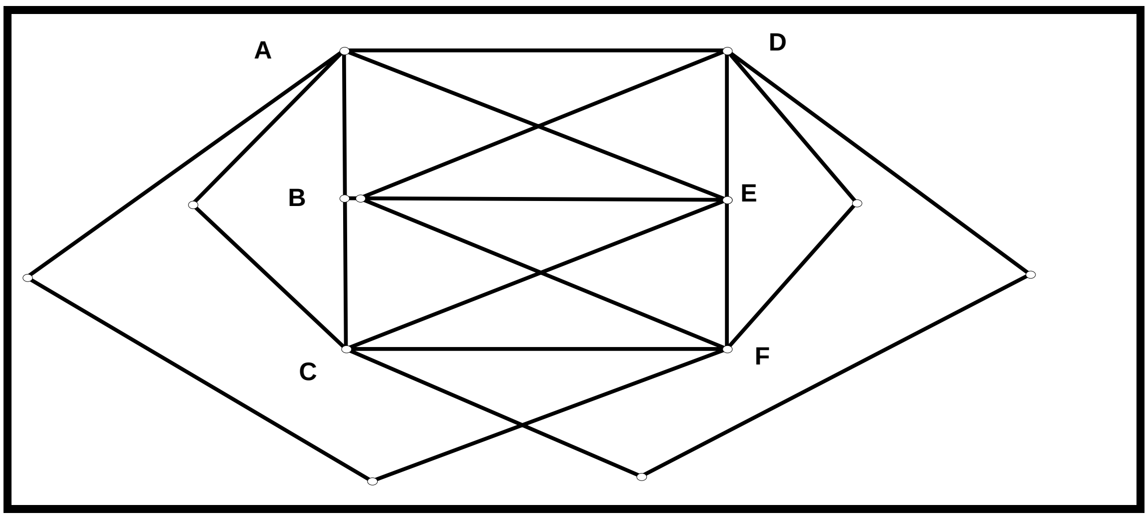


8. **(William Tell and the Dragon)** William Tell has two magic swords. One sword can cut off 21 heads of an evil dragon and the other sword can cut off 4 heads, but in the case of the second sword, the dragon grows back 1985 new heads! Can William Tell cut off all the heads of the dragon if the dragon originally has 100 heads ?

Solution This problem seems hard until we make a fundamental observation. No matter how many heads the dragon has, after William Tell uses either sword, the number of new heads will always have the same remainder after divided by 7! The reason being that the first sword cuts off 21 heads (thus decreases by a multiple of 7), and the second sword cuts off 4 heads but the dragon grows back 1985, thus a gain of $1985 - 4 = 1985 = 7 \times 283$, heads, another multiple of 7. Hence, if the dragon starts with 100 heads, (a remainder of 2 when divided by 7), the dragon will always have a remainder of 2 no matter how many times William Tell cuts off heads. Hence William Tell can never cut off all the heads of the dragon since 0 obviously does not have a remainder of 2 when divided by 7. The smallest number would be 2 heads. (Can you devise a strategy for getting down to 2 heads?)

9. **(The King's Fortress)** A king wants to build 6 fortresses and connect each pair of them with one road. Draw a plan that has only 3 intersections. (Each intersection is where exactly two roads intersect.)

Solution We have drawn 6 fortresses; A, B, C, D, E, and F below. Note that each pair of fortresses is connected by one road. There are exactly 3 intersections; on the roads AE and BD, another on the roads BF and CE, and another on the roads AF and DC.



10. **(Diving 1001 Stones into Piles of 3)** Initially there are 1001 stones on a table in a single pile. You are allowed to throw away one stone from this pile and divide the remaining stones into two, not necessarily equal, piles. You then continue this process; each step taking throwing 1 stone from any pile and dividing the remaining stones into two new, not necessarily equal, piles. Is it possible to eventually reach the situation where all the piles of stones have exactly 3 stones ?

Solution Initially there is 1 pile and 1001 stones. From then on

after the first step, there will be 2 piles and $1000 - 1 = 99$ stones.
after the second step, there will be 3 piles and $1000 - 2 = 998$ stones
after the 3rd step, there will be 4 piles and $1000 - 3 = 997$ stones
... ...
after the nth step with will be $n + 1$ piles and $1000 - n$ stones

If you observe carefully, you see that the number of piles $plus$ the number of stones will always be 1002. Hence, in order to have

1 pile of 3 stones we must have 1002 divisible by 1 + 3 = 4 (which it isn't)
2 piles of 3 stones each we must have 1002 divisible by 1 + 3 (2) = 7 (which it isn't)
3 piles of 3 stones each, we must have 1002 divisible by 1 + 3 (3) = 10 (which it isn't)
... ...
k piles with 3 stones each, we must have 1002 divisible by $k + 3k = 4k$ (which it isn't)

Hence, it is impossible to arrive at piles of 3 stones.