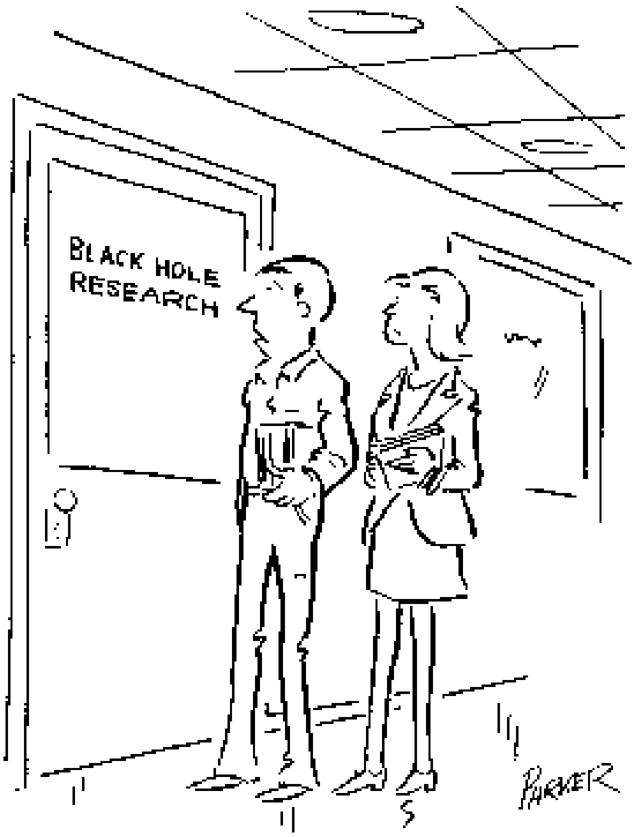
Pair Production of Charged Black Holes

L. A. Brussaard



All I know is no one who goes in there ever comes out.

Supervisor: Prof. dr. G. 't Hooft

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Introduction

In Newtonian physics one can imagine the existence of a dark star. This happens when the escape velocity on the surface of the star exceeds the velocity of light. The dark star is not yet a black hole, as in Newtonian physics one can go faster than light and thus one will still be able to escape from the star. Near the surface, the star still looks bright, the light just can not escape to infinity (where the star looks dark.)

To be able to describe a black hole we have to use general relativity. A black hole can be formed by, for instance, the gravitational collapse of a star. During this collapse there will be formed a so called horizon. At the horizon the escape velocity is equal to the velocity of light. Therefore one can not send a signal from the inside of the horizon to the outside. Everything inside the horizon will eventually fall in the singularity, which is in the center of the black hole. The horizon is a sphere with radius r_s, which is usually referred to as the Schwarzschild radius.

Black holes appear in various sizes. A black hole that was formed by the collapse of a star with the mass of our sun has a Schwarzschild radius of approximately 3 km. The Schwarzschild radius of a collapsed star cluster is for example $r_s(10^9 M_\odot) \sim 20$A.U. A primordial black hole (which might have been formed by the collapse of inhomogeneities in the hot early stages of the universe) has a Schwarzschild radius of for instance $R_s(10^{15}\mathrm{g}) \sim 10^{-13}$ cm. The smallest possible black hole has a mass equal to the Planck mass (10^{-5}g) and a Schwarzschild radius equal to the Planck length (10^{-33}cm.) All black holes are described by only three parameters, their mass, charge and angular momentum.

In classical general relativity, a black hole is a stable object. Its properties can be calculated using ordinary laws of physics. Quantum mechanically, however, there arise some problems. A black hole slowly evaporates by Hawking radiation. This radiation is thermal, so the black hole possesses a continuous spectrum. Eventually the black hole disappears completely, taking with it all the information carried in by the infalling matter which originally formed the black hole. So it is impossible to predict a unique final quantum state for the system. This is the so called information paradox.

There are several suggestions to solve this problem. One of these solutions is to assume that the black hole does not disappear, but leaves a stable Planck mass remnant behind. To store all the information of the initial black hole, there must be an infinite number of remnants, one for each state of the black hole. This leads to severe phenomenological difficulties. There is a finite (although small) probability to produce a remnant via pair production. After summing over all species, the total remnant production rate will diverge.

However, when it can be shown that the production rate is finite, this is a nice option to solve the paradox. But whatever the solution will be, it will affect our views on the laws of physics at the Planck scale.

Throughout this thesis, we will put the constants c, G and $\hbar$ to one. The metric signature we will use is $(-1, 1, 1, 1)$.

Chapter 1

Thermodynamics, information paradox and metrics

1.1 Black hole thermodynamics

During the 1960's and 1970's many fascinating aspects of classical black hole physics were studied. One of the main results of these investigations was a set of four laws of black hole mechanics. These laws have a strong analogy with thermodynamics. I will discuss them shortly here. For a review of black hole thermodynamics see [1, 2, 3, 4, 5].

By its very definition, a classical black hole cannot emit anything, so it seems at first futile to attempt to associate a nonzero temperature with it. On the other hand, there must be some relationship between the change in the mass of a black hole and the change in its horizon area.

Let us first look at a rotating black hole. Penrose suggested a classical process by which one could exploit the existence of the ergo-region to extract the rotational energy of such a black hole. Particle 1 is sent into the ergo-region, where it breaks up into particles 2 and 3. This is arranged so that particle 3 has a negative energy and falls across the horizon

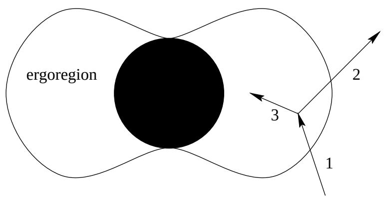

Figure 1.1: Particles 1, 2 and 3 in Penroses energy-extraction process.

while particle 2 escapes to infinity with energy greater than the initial energy of particle 1, so total energy is conserved (figure 1.1).

The extracted energy must come at the expense of the rotational energy of the black hole, so particle 3 must have an angular momentum opposite to that of the black hole.

The most efficient energy extraction process would be one for which the ratio of energy to angular momentum extracted is maximized.

To determine this most efficient process, define a time-translation (at infinity) Killing[1] field, ξ, and an axial rotation Killing field, ψ. Then the conserved quantities for a particle with a four-momentum vector p are the energy, $E = p \cdot \xi$, and the angular momentum, $L = -p \cdot \psi$. The minus sign is due the fact that ψ is space-like at infinity, while ξ is time-like. On the horizon both ψ and ξ are space-like. The horizon is generated by null geodesics and there must be a linear combination $\chi \equiv \xi + \Omega\psi$, that is a future pointing null Killing vector generating translations along the horizon generators. Ω is the angular velocity of the horizon. When particle 3 crosses the horizon, the future pointing vectors p_3 and χ must satisfy $0 \leq p_3 \cdot \chi = E_3 - \Omega L_3$. From this follows that $L_3 \leq \frac{1}{\Omega} E_3 < 0$, thus particle 3 must carry an angular momentum opposite of that of the black hole. The most efficient process occurs when p_3 is a null vector, tangent to the horizon generators. This has the implication that, when the particle enters the black hole, it does not affect the area of the event horizon. For the most efficient process one has $\Delta M = \Omega \Delta J$, where $\Delta M = E_3$ is the change in mass and $\Delta J = L_3$ is the change in angular momentum of the black hole.

Now we take a charged black hole. If a black hole is electrically charged, one can extract energy from it by neutralizing it. One can drop a particle into the black hole with a charge opposite to that of the black hole. This will change the mass and the charge of the black hole by the amounts $\Delta M = q\Phi$ and $\Delta Q = q$, where Φ is the potential difference between the horizon and infinity and q is the charge of the particle. To get the most efficient process to extract energy in this way, one should drop the charge in just outside the horizon. As in the case of the Penrose process, this process will not change the area of the horizon. So in the this case one gets $\Delta M = \Phi \Delta Q$.

We have calculated that, when the extraction processes are most efficient, the change in mass is given by $dM = \Omega dJ + \Phi dQ$. This expresses changes of the energy of the black hole in reversible processes like work done on a thermodynamic system or a change in the number of particles. It is like the first law of thermodynamics but with the heat flow term, $dQ = TdS$, missing. When we calculate the terms ΩdJ and ΦdQ, we get the missing term [2].

It turns out that the missing term is given by $\kappa dA/8\pi G$, where κ is the surface gravity of the horizon. The first law for a rotating charged black hole thus takes the form:

$$dM = \frac{\kappa}{8\pi G} dA + \Omega dJ + \Phi dQ \tag{1.1}$$

with M the mass, A the horizon area, κ the surface gravity of the horizon, J and Q the angular momentum and charge of the black hole, Ω and Φ the angular velocity and electric potential of the horizon. Note that this formula looks the same as the first law in thermodynamics, which is:

$$dE = TdS - PdV + \Omega dJ \tag{1.2}$$

[1]If the local one-parameter group of diffeomorphisms ϕ_t generated by a vector field K is a group of isometries (i.e. for each t, the transformation ϕ_t is an isometry) we call the vector field K a Killing vector field. This field satisfies Killing's equation: $K_{a;b} + K_{b;a} = 0$ [6].

with E the energy, T the temperature, S the entropy, P the pressure, V the volume of the system and ΩdJ some work done on the system. It suggests that κ plays the role of a temperature. This gives us the analog of the zeroth law, which states that κ is constant over the horizon (in thermodynamics temperature is uniform over a body in thermal equilibrium.)

The second law of black hole physics is Hawking's area theorem, which states that the area of the horizon can not decrease in time; $\Delta A \geq 0$. Moreover when two black holes collide and merge together to form a single black hole, the area of the horizon of this final black hole is *larger* than the sum of the areas of the horizons of the two initial black holes. This property led Bekenstein in 1972 to suggest that the area might be connected with the entropy, which measures the degree of disorder of a system or one's lack of knowledge of it. He pointed out that the 'no-hair theorem[2]' implied that a very large amount of information about a star was irretrievably lost when it collapsed to form a black hole. And he claimed that the area of the horizon was a measure of this unobservable information which could be regarded as the entropy of the black hole. Thus this law is similar with the second law in thermodynamics ($\Delta S \geq 0$.) Later Bekenstein generalized this to what is now known as the generalized second law, which tells us that the total entropy, i.e. the entropy of the black hole and that of the system outside the black hole, will never decrease.

It can be shown [3, 7] that κ vanishes only for extreme[3] black holes. But explicit calculations show that the closer one gets to an extreme black hole the harder it is to get a further step closer, in a similar manner to the third law of thermodynamics. Thus the third law of black hole physics states that it is impossible to achieve $\kappa = 0$ by a physical process.

But in spite of this beautiful analogy there were still some difficulties. One of the problems was how one could attribute a finite entropy to a black hole. This would imply that it should have a finite temperature and should be able to remain in equilibrium with thermal radiation at the same temperature. However this seemed impossible because the black hole would absorb some of the radiation but it would not be able to emit anything in return.

This problem was solved in 1974 when, after he applied quantum mechanics, Hawking discovered that black holes thermally radiate at a temperature [8]

$$\begin{aligned} T &= \frac{\kappa}{2\pi} \\ &= \frac{\hbar}{8\pi M} \end{aligned} \tag{1.3}$$

This radiation can be seen as follows. In quantum field theory the vacuum is populated by virtual pairs of particles and anti-particles. The gravitational field of the black hole can promote one member of a pair into a real outgoing particle while the other partner

[2]A conjecture by Wheeler by which he meant that a black hole rapidly settles down to a stationary state which was determined by only three parameters (mass, angular momentum and electric charge) and was otherwise independent of the details of the body that collapsed to form the black hole.

[3]Extreme means that the parameters of the black hole satisfy the condition $M^2 = Q^2 + \frac{J^2}{M^2}$.

falls into the black hole with a net negative energy as measured at infinity. (Each pair of particles produced in this way is in a pure state. The one that falls in the black hole carries information in the form of correlations with the outgoing particle.) Equivalently, one can think of the member of the pair that fell into the black hole (say the particle) as an anti-particle traveling backwards in time and coming out of the black hole. When it reached the point at which the particle-anti-particle pair first appeared, it would be scattered by the gravitational field into an anti-particle traveling forwards in time and going out to infinity. Thus we can think of the emitted radiation as having come from inside the black hole and having quantum mechanically tunneled through the potential barrier around the black hole created by the gravitational field, a barrier that could not be surmounted classically. By the 'no-hair theorem' an external observers knowledge of the region from which the radiation came is limited to its total mass, angular momentum and electric charge. For this reason the black hole emits with equal probability every configuration of particles with the same total energy, angular momentum and charge (though not all these configurations escape to infinity with equal probability because the potential barrier around the black hole depends on the energies and angular momenta of individual particles and may reflect them back into the black hole.) The emitted particles have a thermal spectrum. For more information see for example: [3, 9, 10].

1.2 Entropy and the information paradox

We now have a beautiful connection between gravity, quantum mechanics and thermodynamics. But there are still some problems that are not solved yet.

First, let us look at the Bekenstein-Hawking entropy (in Planck units):

$$\begin{aligned} S_{BH} &= \frac{1}{4}A \\ &= 4\pi M^2 \end{aligned} \tag{1.4}$$

with A the area of the black hole horizon and M the mass of the black hole. If we describe a system in terms of a micro-canonical ensemble, then the entropy is directly related to the total number of quantum states that can describe the system. Then the number of quantum states of a black hole is given by

$$N = e^{S_{BH}} \tag{1.5}$$

These micro-states might be either 'internal states' inside the black hole or 'horizon states' somehow associated with the degrees of freedom of (or near) the black hole horizon, or both [11]. It is not clear yet what the interpretation is of (1.5). If we try to calculate S_{BH} directly by counting physical degrees of freedom near the horizon, the answer goes to infinity. Counting all black hole states gives us also problems.

The generalized second law was tested successfully. But two questions arose. What mechanism insures that the generalized entropy grows in any situations and are there any exceptions to the law? These are not trivial questions.

For more information see references [9, 10, 11, 12].

Hawking's result gives us also another great problem. It is now possible that pure states evolve into mixed states, which is quantum mechanically forbidden. In other words, even if we know precisely the initial quantum state, we cannot predict with certainty what the final quantum state will be. We can only assign probabilities to various alternatives. This is what is called the information paradox.

To see how a pure state evolves into a mixed state, let us begin with a black hole in a quantum mechanical pure state. The black hole will start to evaporate by Hawking radiation. In section 1.1 we saw that the radiation comes from pair creation of two particles out from the vacuum. The quantum state of the Hawking radiation by itself lacks the quantum correlations with the lost quanta which are part of the original pure vacuum state with which we begun. Hence the Hawking radiation all by itself is in a mixed state. It happens to be a nearly mixed state, and so the radiation is thermal. If the black hole truly disappears by evaporation, one is left with a thermal (mixed) state of radiation with nothing to correlate with in order to reconstruct the pure state. Hawking concluded from this that black hole evaporation catalyzes unitarity violation, that quantum mechanics is not fully correct in the presence of black hole horizons, and that contrary to the venerable rules, a pure state can become mixed [13].

Mainly, there are three possible solutions to solve this paradox. I will briefly discuss them.

First, the information can be simply lost, eaten by the singularity. This means there is a fundamental non-unitarity which is in conflict with energy conservation.

Another scenario where the information is lost, is that the information escapes to a 'baby universe' down a wormhole. This baby universe is causally disconnected from our own universe. Energy is now conserved and there is no apparent loss of unitarity, wormholes just shift the coupling constants. But this is not a very useful solution. In practice there is no difference with the previous solution.

The second solution is re-emission of the information. The information can come out with the Hawking radiation. If this is true, than the information must be bleached off the matter when it falls through the horizon. It is hard to imagine any reasonable way to achieve this. To a freely falling observer the horizon is not a special place.

If bleaching of the information does not occur, then a macroscopic violation of causality seems to be required to transport the information from the collapsing body to the outgoing radiation. This too is a very unlikely solution.

The re-emission can also happen at a final burst when the black hole reaches the Planck scale. Causality now may fail and the information can escape from inside. The information is encoded in correlations between the thermal quanta emitted earlier and the quanta emitted at this point. The remaining energy of the black hole at the Planck scale is $\sim M_{pl}$. The initial energy of the black hole is M_0, so an amount of information of $\frac{M_0^2}{M_{pl}^2}$ must be emitted. This is possible, for example, by radiating many very soft photons, one at a time. But this must take a time $\sim \frac{M_0^4}{M_{pl}^4 t_{pl}}$, which is a very long time. Therefore we can assume these Planck scale remnants to be stable.

This brings us to the final solution where the information is retained in remnants. At first sight, this is a nice solution to the paradox. But this is not the case. When we start with an arbitrary black hole, an arbitrary large amount of information can be stored in the remnant. So this remnant should have an infinite number of internal states or there should be an infinite number of species. The problem now is that the infinite number of remnant species would appear to lead to infinite total remnant production rate in various physical processes.

For more information see references [5, 14, 15, 16, 17, 18].

1.3 Some metrics

In this section I will give a short review of some solutions of Einstein's equations. For more details I refer to [3, 6, 7] and other textbooks about this subject.

To describe a space-time we use a pair ($\mathcal{M}$,g) where $\mathcal{M}$ is a connected four-dimensional Hausdorff C^∞ manifold and g is a Lorentz metric on $\mathcal{M}$. We can use the metric tensor to write an 'infinitesimal' length as:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{1.6}$$

To specify the metric we have to search a solution to Einstein's equations. These equations are given by:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{1.7}$$

with $R_{\alpha\beta}$ the Ricci tensor, R the total curvature, Λ the cosmological constant and $T_{\alpha\beta}$ the energy-momentum tensor. Some solutions of these equations can be calculated exactly. I will mention the solutions that are generally known. First we will take the cosmological constant to be zero.

We can take the energy-momentum tensor $T_{\alpha\beta}$ zero. The simplest solution for such an empty space-time is Minkowski space-time. This is a manifold R^4 with a flat Lorentz metric η. In terms of the natural coordinates (x^0, x^1, x^2, x^3) on R^4, the Minkowski metric can be expressed in the form:

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{1.8}$$

In figure 1.2, the Penrose diagram (see Appendix A) of Minkowski space is sketched.

The solution for a spherically symmetric empty space-time outside a spherically symmetric massive body is the Schwarzschild metric. This metric can be given in the form:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1.9}$$

with M the mass of the body and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. There are two singularities, one in $r = 0$ and one in $r = r_s = 2M$. The first is a true singularity. The latter is a consequence of the coordinates we used. This singularity is at the Schwarzschild radius r_s and acts as

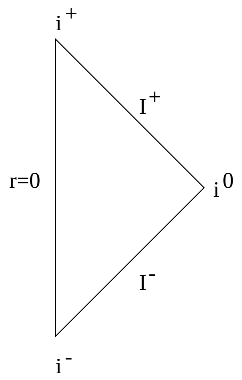

Figure 1.2: Penrose diagram of Minkowski space-time.

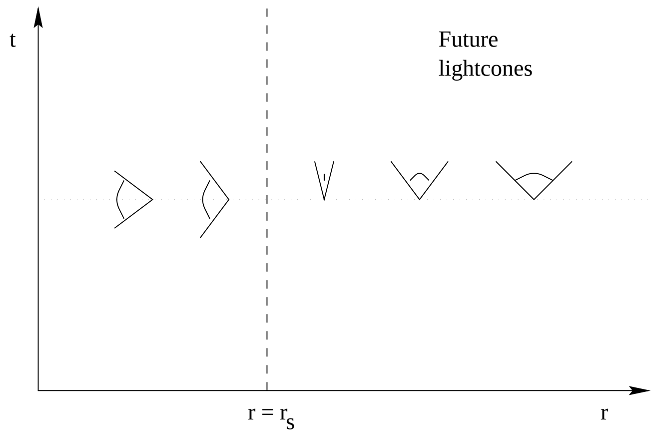

Figure 1.3: Schwarzschild space-time.

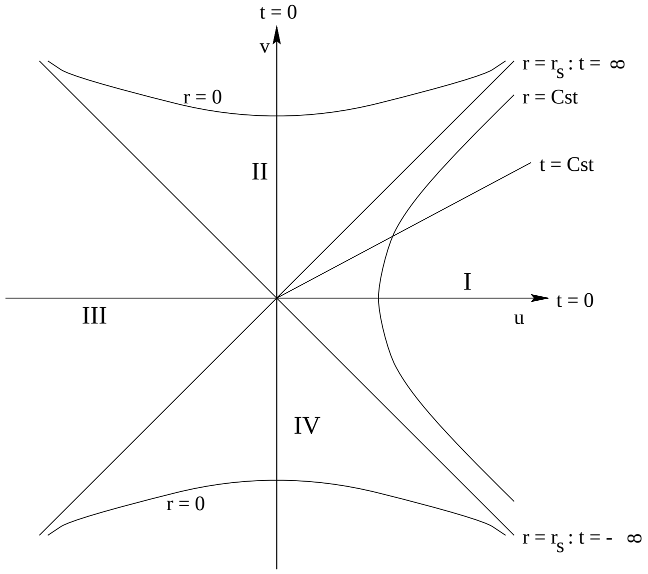

Figure 1.4: Kruskal diagram of Schwarzschild space-time.

a horizon. From figure 1.3 we see that you can *only* cross the horizon from the outside ($r > r_s$) to the inside ($r < r_s$). To avoid this coordinate singularity we can use for example Kruskal-Szekeres coordinates. These are defined as:

$$I \qquad \left.\begin{array}{l} u = \left\{\frac{r}{r_s} - 1\right\}^{\frac{1}{2}} \exp\left(\frac{r}{2r_s}\right) \cosh \frac{t}{2r_s} \\ v = \left\{\frac{r}{r_s} - 1\right\}^{\frac{1}{2}} \exp\left(\frac{r}{2r_s}\right) \sinh \frac{t}{2r_s} \end{array}\right\} \quad r > r_s \tag{1.10}$$

$$II \qquad \left.\begin{array}{l} u = \left\{1 - \frac{r}{r_s}\right\}^{\frac{1}{2}} \exp\left(\frac{r}{2r_s}\right) \sinh \frac{t}{2r_s} \\ v = \left\{1 - \frac{r}{r_s}\right\}^{\frac{1}{2}} \exp\left(\frac{r}{2r_s}\right) \cosh \frac{t}{2r_s} \end{array}\right\} \quad r < r_s \tag{1.11}$$

The coordinates θ and ϕ do not change. The inverse transformation is:

$$\begin{array}{c} \left\{\frac{r}{r_s} - 1\right\} \exp\left(\frac{r}{r_s}\right) = u^2 - v^2 \\ \frac{t}{r_s} = \log\left|\frac{u+v}{u-v}\right| \end{array} \tag{1.12}$$

In these coordinates the metric takes the form:

$$ds^2 = \frac{4r_s^3}{r} \exp\left(-\frac{r}{r_s}\right)(du^2 - dv^2) + r^2 d\Omega^2 \tag{1.13}$$

With these coordinates we can draw a Kruskal diagram (figure 1.4.) This diagram can be divided in four domains. The Schwarzschild coordinates are mapped to the domains I ($r > r_s$) and II ($r < r_s$). The domains III and IV are the time reverse of the domains I and II respectively. These correspond to what is usually referred to as a white hole, while the domains I and II correspond to the black hole.

The metric which describes the space-time outside a spherically symmetric charged body carrying an electric charge is the Reissner-Nordström metric:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1.14}$$

with M the mass and Q the charge of the body. There is again a true singularity in $r = 0$ and there are two horizons:

$$r_\pm = M \pm (M^2 - Q^2)^{\frac{1}{2}} \tag{1.15}$$

For $Q < M$ the singularity at $r = 0$ is hidden behind these horizons. For $Q > M$ there are no real roots and the solution has a naked singularity. For $Q = M$ (the extremal solution) it seems that there is one horizon, but when you calculate the distance between the the horizons you will see that this is not the case. It is possible to choose other coordinates where the horizons do not seem to coincide.

If $Q < M$ the black hole will evaporate until extremality ($Q = M$). (This is only possible when there are no charged particles, otherwise extremality will never be reached as stated earlier.) When we throw matter into the black hole it will again start to evaporate until it reaches extremality again. We can repeat this process as many times as we like. If we assume that information is neither destroyed nor re-emitted, than the extremal black hole must have an infinite number of internal states. As already mentioned in the previous section a remnant with an infinite number of internal states would lead to an infinite total remnant production rate. When the production rate of Reissner-Nordström black holes is finite, this possibly can serve as a guide to the formulation of theories of infinitely degenerate remnants with finite production. This might also extend to neutral remnants.

Finally we will discus the de Sitter space-time. The space-time metric of constant curvature is locally characterized by the condition:

$$R_{abcd} = \frac{1}{12} R(g_{ac}g_{bd} - g_{ad}g_{bc}) \tag{1.16}$$

The Einstein tensor then can be written as

$$R_{ab} - \frac{1}{2} R g_{ab} = -\frac{1}{4} R g_{ab} \tag{1.17}$$

One can therefore regard these spaces as solutions of the field equations for an empty space with $\Lambda = \frac{1}{4}R$. When $R = 0$ we get back the Minkowski space-time. When $R > 0$ we get what is called the de Sitter space-time. This space-time has topology $R^1 \times S^3$. The de Sitter space-time can be visualized as a hyperboloid

$$-v^2 + w^2 + x^2 + y^2 + z^2 = \alpha^2 \tag{1.18}$$

in flat five-dimensional space R^5 with metric

$$-dv^2 + dw^2 + dx^2 + dy^2 + dz^2 = ds^2 \tag{1.19}$$

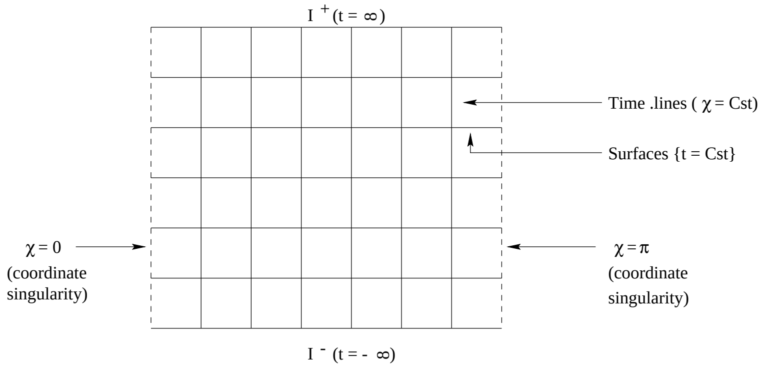

Figure 1.5: Penrose diagram of de Sitter space-time.

When we now introduce the coordinates (t, χ, θ, ϕ) on the hyperboloid

$$\begin{aligned} \alpha \sinh(\alpha^{-1}t) &= v && (1.20)\\ \alpha \cosh(\alpha^{-1}t) \cos\chi &= w && (1.21)\\ \alpha \cosh(\alpha^{-1}t) \sin\chi \cos\theta &= x && (1.22)\\ \alpha \cosh(\alpha^{-1}t) \sin\chi \sin\theta \cos\phi &= y && (1.23)\\ \alpha \cosh(\alpha^{-1}t) \sin\chi \sin\theta \sin\phi &= z && (1.24) \end{aligned}$$

the metric becomes

$$ds^2 = -dt^2 + \alpha^2 \cosh^2(\alpha^{-1}t)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)] \qquad (1.25)$$

There are coordinate singularities for $\chi = 0$, $\chi = \pi$ and $\theta = 0$, $\theta = \pi$. The coordinates are restricted to the areas $-\infty < t < \infty$, $0 \le \chi \le \pi$, $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$. To study the infinities in de Sitter space-time, define

$$t' = 2\arctan(\exp\alpha^{-1}t) - \frac{1}{2}\pi \qquad (1.26)$$

with $-\frac{1}{2} < t' < \frac{1}{2}\pi$. Then the metric becomes

$$ds^2 = \alpha^2 \cosh^2(\alpha^{-1}t)[-dt'^2 + d\chi^2 + \sin^2\chi d\Omega^2] \qquad (1.27)$$

This gives us the Penrose diagram which is sketched in figure 1.5.

The de Sitter metric can also be written in terms of a Killing vector $\frac{\partial}{\partial t}$ and the cosmological constant Λ [7]

$$ds^2 = -\left(1 - \frac{1}{3}\Lambda r^2\right) dt^2 + \left(1 - \frac{1}{3}\Lambda r^2\right)^{-1} dr^2 + r^2\, d\Omega^2 \qquad (1.28)$$

There is a horizon at $r = (3\Lambda^{-1})^{\frac{1}{2}}$.

Black hole solutions, which are asymptotically de Sitter instead of asymptotically flat, can also be obtained. The simplest of these is the Schwarzschild-de Sitter solution, with metric [3]

$$ds^2 = -V\,dt^2 + V^{-1}\,dr^2 + r^2\,d\Omega^2 \tag{1.29}$$

and

$$V = 1 - 2Mr^{-1} - \frac{1}{3}\Lambda r^2 \tag{1.30}$$

Chapter 2

Pair production of black holes in an electro-magnetic background

2.1 Why look at the pair production of black holes?

In the last chapter we saw that (long-lived) remnants were a solution to the information paradox. But it is possible that an arbitrarily large amount of information has been dumped into the black hole. Therefore there would be an infinite number of distinct remnants to account for all that information. Each of these distinct remnants has a finite probability to be pair produced in a suitable background field. So there would be an infinite pair production rate of remnants (see references [19, 20, 21], references therein and references in section 1.2.) There are various mechanisms suggested which would suppress the infinite production and yield a finite answer. For example Banks et al. [22] offered a picture where the information that resides in a black hole remnant is contained in a long, narrow throat that is attached onto space-time. Production of remnants is heavily suppressed because it is necessary to add a large volume (the volume of the throat) to the background space-time, and this process requires a large Euclidean action. But the several mechanisms are the subject of extensive contention, so an explicit calculation of the production rate is essential to further consideration of the remnants proposal.

The study of black hole pair production is also of considerable interest for the exploration of quantum gravity. It represents a truly quantum gravitational process, being classically completely forbidden, just like black hole evaporation. At the same time, it is easy to achieve a physical understanding of what is happening. There is a strong analogy with the creation of particle-anti-particle pairs in quantum field theory. This can be seen from the fact that the black hole pair creation rate reduces to the particle-anti-particle production rate in the limit of small black holes.

If we trust this analogy to particle physics, the pair creation rate should depend on the number of accessible states for the black hole, so we can find out how many states the black hole should have by studying pair creation (although we can't find out what those states are.)

It is often said that black hole pair creation changes the topology of space [20, 23, 24, 25]. The configuration of two black holes would have a different spatial topology than the vacuum. Then it is not possible to continuously deform the one in the other. However, this is not as trivial as it looks. One can avoid the singularities and then the topology is not different from that of the vacuum. Moreover, when we look at the four dimensional space-time, the creation of a pair of black holes does not change the topology, because it is just a part of the space-time.

2.2 Properties of the theory

I will consider the pair production of charged Reissner-Nordström black holes in an electromagnetic background. The evaporation of such a black hole stops when its mass M approaches the (absolute) value of its charge q. Thus, for a sufficiently large charge we can study the black hole when it is (much) larger than the Planck mass, where any semi-classical description of the black hole becomes impossible.

But a strongly charged black hole may discharge by creation of pairs of charged particles. We want to circumvent this problem by isolating these effects from the effects of quantum gravity. To do this one may consider models without charged particles. The electric and magnetic fields are than assumed to be produced by flow from the singularity. At the classical level there is no difference between a Reissner-Nordström black hole with charged particles and a black hole with a spherically symmetric field originating from the singularity. In both cases we will have the same theory of black hole evaporation, but in the last case we do not need to be concerned with extra complications, such as particle production in strong electric fields, charge quantization, etc.

This idea proves to be especially productive if one can find a way of embedding the original bosonic theory in a super-symmetric theory. In such a case one has the same description of the classical properties of the black hole and of its evaporation, but higher order quantum corrections are much better under control. For example, an extreme Reissner-Nordström black hole with mass $M = q$ may be embedded in $N = 2$ super-gravity [26]. All higher order quantum corrections to the effective action of $N = 2$ super-gravity in the field of the extreme black hole could be shown to vanish if the theory had no anomalies [27]. In particular, the effective action would have no imaginary part, which would mean no particle creation in the field of an extreme Reissner-Nordström black hole. These and some other properties of super-symmetric black holes indicate that they may prove to be a unique laboratory for investigation of black hole physics and quantum gravity in general. Indeed, until we started studying super-symmetric black holes, we had no example of a nontrivial Lorentzian four-dimensional background where all quantum gravity corrections to the effective action vanished.

The theory of $N = 2$ super-gravity, however, has one-loop anomalies. Therefore the formal proof of the absence of quantum corrections for the classical extreme Reissner-Nordström black hole is not sufficient. One is led to try to find a super-symmetric embedding of charged dilaton black holes [28, 29, 30] in dimensionally reduced string theory or

$d = 4$, $N = 4$ super-gravity, where the anomalies can be canceled.

The pair creation process of the two oppositely charged black holes is classically forbidden. Therefore the creation of a pair of black holes is studied in the path integral approach to quantum gravity. We therefore have to find a suitable instanton (see appendix B) which describes the transition from the given initial data to final data. The instanton is a saddle point for both the Euclidean and Lorentzian functional integrals, and it interpolates between the initial and final states of a classically forbidden transition.

We thus want an action which describes both the Einstein-Maxwell theory (as we saw in the section 2.1 we expect in the limit of small black holes a pair production rate of particle-anti-particle pairs in an electro-magnetic field) and the embedded charged dilaton black hole in a dimensionally reduced string theory or $d = 4$, $N = 4$ super-gravity. This theory describes the interaction of two $U(1)$ gauge fields and a complex scalar. The real part of this complex scalar is the dilaton, the imaginary part the axion. We will put the axion to a constant, so we only look for solutions which depend only on the dilaton. The theory with the two $U(1)$ gauge fields and the dilaton will be referred to as the $U(1)^2$ theory.

When the coupling constant a, which governs the strength of the coupling of the dilaton to the vector fields, is zero, we get the standard Einstein-Maxwell theory. If we take $a = \sqrt{3}$, the theory corresponds to standard Kaluza-Klein theory (creation of magnetic monopoles.) We will take $a = 1$, because this corresponds to the description of low-energy dynamics of string theory.

The action for the $U(1)^2$ theory exists in two versions. The first one is usually referred to as the $SO(4)$ version [31, 32, 33] and the second one as the $SU(4)$ version [34]. The spectrum of the $SO(4)$ version consists of the graviton, four Majorana spin $\frac{3}{2}$ fields and four Majorana spin $\frac{1}{2}$ fields, six vector fields, a scalar field and a pseudo-scalar field. That of the $SU(4)$ version consists of the graviton, four Majorana spin $\frac{3}{2}$ fields, four spin $\frac{1}{2}$ fields, three vector fields, three axial vector fields, a scalar field and a pseudo-scalar field. For the $SU(4)$ theory, the action is invariant under a rigid $SU(4) \otimes SU(1,1)$ symmetry (which makes this theory simpler.) In both versions, the vector fields transform under a $SO(4) = SU(2) \otimes SU(2)$ group. In the $SO(4)$ version, both factors contain three vector fields, while in the $SU(4)$ version one factor contains three vector fields and the other factor three axial vector fields. In [34] it is shown that these two theories are equivalent. We will consider $U(1) \otimes U(1)$ solutions, i.e. solutions with one non-trivial factor in each subgroup, and we will put the axion to a constant. The remaining bosonic part of the action in the two versions is given by [20, 25]

$$I_{SO(4)} = \frac{1}{16\pi} \int_M d^4x \sqrt{-g} \left(-R + 2\partial^\mu \varphi \partial_\mu \varphi - \left(e^{2\varphi} \tilde{F}^2 + e^{-2\varphi} G^2 \right) \right) \tag{2.1}$$

and

$$I_{SU(4)} = \frac{1}{16\pi} \int_M d^4x \sqrt{-g} \left(-R + 2\partial^\mu \varphi \partial_\mu \varphi - e^{-2\varphi} \left(F^2 + G^2 \right) \right) \tag{2.2}$$

where R is the scalar curvature of the metric, g is the metric, φ is the dilaton and

$$\begin{aligned}\tilde{F}_{\mu\nu} &= \partial_\mu \tilde{B}_\nu - \partial_\nu \tilde{B}_\mu \\ F_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \\ G_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu\end{aligned}$$

are the $U(1)$ gauge fields. The equations of motion for (2.1) are given by

$$\nabla_\mu \left(e^{2\varphi} \tilde{F}^{\mu\nu} \right) = 0 \tag{2.3}$$

$$\nabla_\mu \left(e^{-2\varphi} G^{\mu\nu} \right) = 0 \tag{2.4}$$

$$\nabla^2 \varphi + \frac{1}{2} e^{2\varphi} \tilde{F}^2 - \frac{1}{2} e^{-2\varphi} G^2 = 0 \tag{2.5}$$

$$\begin{aligned} R_{\mu\nu} + 2\partial_\mu \varphi \partial_\nu \varphi - e^{2\varphi} \left(2\tilde{F}_{\mu\lambda} \tilde{F}_{\nu\delta} g^{\lambda\delta} - \frac{1}{2} g_{\mu\nu} \tilde{F}^2 \right) & \\ - e^{-2\varphi} \left(2G_{\mu\lambda} G_{\nu\delta} g^{\lambda\delta} - \frac{1}{2} g_{\mu\nu} G^2 \right) &= 0 \end{aligned} \tag{2.6}$$

Those for (2.2) are given by

$$\nabla_\mu \left(e^{-2\varphi} F^{\mu\nu} \right) = 0 \tag{2.7}$$

$$\nabla_\mu \left(e^{-2\varphi} G^{\mu\nu} \right) = 0 \tag{2.8}$$

$$\nabla^2 \varphi - \frac{1}{2} e^{-2\varphi} (F^2 + G^2) = 0 \tag{2.9}$$

$$\begin{aligned} R_{\mu\nu} + 2\partial_\mu \varphi \partial_\nu \varphi - e^{-2\varphi} \left(2F_{\mu\lambda} F_{\nu\delta} g^{\lambda\delta} - \frac{1}{2} g_{\mu\nu} F^2 \right) & \\ - e^{-2\varphi} \left(2G_{\mu\lambda} G_{\nu\delta} g^{\lambda\delta} - \frac{1}{2} g_{\mu\nu} G^2 \right) &= 0 \end{aligned} \tag{2.10}$$

(For the variation of the metric see for example ref. [35].)

When we take $\tilde{F}_{\mu\nu} = \frac{1}{2} e^{-2\varphi} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$, it can be seen that the equations of motion are equivalent. The actions are not invariant under this duality. This can be seen by looking at the Einstein-Maxwell action. This action is $F^2 = B^2 - E^2$ (with B and E the magnetic and electric field respectively) and it changes sign when the magnetic field is replaced by the electric field and vice versa.

One interesting feature of the $U(1)^2$ theory is that it has two distinct duality symmetries. The equations of motion of this theory are invariant under a duality transformation

$$\begin{array}{lcl} F_{\mu\nu} & \rightarrow & \tilde{F}_{\mu\nu} \equiv \frac{1}{2} e^{-2\varphi} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \\ G_{\mu\nu} & \rightarrow & \tilde{G}_{\mu\nu} \equiv \frac{1}{2} e^{-2\varphi} \epsilon_{\mu\nu\rho\sigma} G^{\rho\sigma} \\ \varphi & \rightarrow & -\varphi \end{array} \tag{2.11}$$

which is analogous to the ordinary electric-magnetic duality transformation of Einstein-Maxwell theory. The equations of motion and the action (2.2) are also invariant under the interchange of the two gauge fields, $F_{\mu\nu} \leftrightarrow G_{\mu\nu}$. If we combine these, we find that the equations of motion and the action (2.1) are invariant under the duality

$$\begin{array}{lll} F_{\mu\nu} & \rightarrow & \tilde{G}_{\mu\nu} \\ G_{\mu\nu} & \rightarrow & \tilde{F}_{\mu\nu} \\ \varphi & \rightarrow & -\varphi \end{array} \tag{2.12}$$

If we think of $\tilde{F}$ as the field variable rather than F, this duality just interchanges the two gauge fields and reverses the sign of the dilaton.

On the Einstein-Maxwell solutions, for which $\tilde{F}_{\mu\nu} = G_{\mu\nu}$ and $\varphi = 0$, (2.12) is a trivial transformation. In general we will consider solutions for which the transformation (2.12) just corresponds to an interchange of the parameters of the solution. These solutions will be said to have a manifest duality symmetry. The action (2.1) is invariant under this manifest duality symmetry.

The charged black hole solutions of the $U(1)^2$ theory are [21]

$$ds^2 = -\lambda\, dt^2 + \lambda^{-1}\, dr^2 + R^2\, d\Omega \tag{2.13}$$

$$e^{2\varphi} = e^{2\varphi_0} \frac{r+\Sigma}{r-\Sigma} \tag{2.14}$$

$$\tilde{F} = Q e^{-\varphi_0} \sin\theta d\theta \wedge d\phi \tag{2.15}$$

$$G = P e^{\varphi_0} \sin\theta d\theta \wedge d\phi \tag{2.16}$$

where

$$\lambda = \frac{(r-r_+)(r-r_-)}{R^2} \tag{2.17}$$

$$R^2 = r^2 - \Sigma^2 \tag{2.18}$$

$$r_\pm = M \pm \sqrt{M^2 + \Sigma^2 - P^2 - Q^2} \tag{2.19}$$

$$\Sigma = \frac{P^2 - Q^2}{2M} \tag{2.20}$$

There is a curvature singularity at $r = |\Sigma|$. The physical degrees of freedom are P, Q, M and φ_0. M is the mass of the black hole, and $e^{-\varphi_0}Q$ and $e^{\varphi_0}P$ are its gauge charges. Note that both the gauge fields are magnetic, when we write the solutions in this way. One can also obtain a solution with two electric fields, but we will restrict the attention to the magnetic case. Further we could keep the asymptotic value of the dilaton φ_0 as a free parameter, but we will fix it by requiring that the dilaton match to an appropriate background value at infinity. The solution has a manifest duality, as the solution is unchanged when

$$\begin{array}{lll} \tilde{F} & \leftrightarrow & G \\ \varphi & \leftrightarrow & -\varphi \end{array} \tag{2.21}$$

and

$$\begin{array}{lll} Q & \leftrightarrow & P \\ \Sigma & \leftrightarrow & -\Sigma \\ \varphi_0 & \leftrightarrow & -\varphi_0 \end{array} \tag{2.22}$$

corresponding to the general symmetry (2.11). The parameters are constrained to $M \geq M_{ext} = \frac{1}{\sqrt{2}}(|P| + |Q|)$ by positivity bounds arising from super-symmetry [20]. The black holes have one unbroken $N = 1$ super-symmetry in the extremal limit if both gauge fields are nonzero, and two if one of the gauge fields is zero. The temperature of these black holes goes smoothly to zero as extremality is approached.

In the following section I will use the generalization of the so called C metric [36] to this theory. The C metric was rediscovered among others by Ehlers and Kundt [37] (they examined some different classes of static vacuum fields, labeled by A through C, where the terminology used here comes from.) The generalization of this C metric is given by [21]

$$\begin{aligned} ds^2 &= \frac{1}{A^2(x-y)^2}\Big\{F(x)\left[G(y)\,dt^2 - G^{-1}(y)\,dy^2\right] \\ &+ F(y)\left[G^{-1}(x)\,dx^2 + G(x)\,d\phi^2\right]\Big\} \qquad (2.23) \\ e^{-2\varphi} &= e^{-2\varphi_0}\left(\frac{1+\Sigma A y}{1-\Sigma A y}\right)\left(\frac{1-\Sigma A x}{1+\Sigma A x}\right) \qquad (2.24) \\ \tilde{F}_{yt} &= \frac{\alpha e^{-\varphi_0}}{(1+\Sigma A y)^2} \qquad (2.25) \\ G_{x\phi} &= \frac{\beta e^{\varphi_0}}{(1-\Sigma A x)^2} \qquad (2.26) \end{aligned}$$

where,

$$\begin{aligned} F(\xi) &= 1 - \Sigma^2 A^2 \xi^2 \qquad (2.27) \\ G(\xi) &= \frac{(1-\xi^2 - r_+ A\xi^3)(1 + r_- A\xi)}{(1-\Sigma^2 A^2 \xi^2)} \qquad (2.28) \end{aligned}$$

and

$$\begin{aligned} \alpha^2 &= \frac{1}{2}(r_+ - \Sigma)(r_- - \Sigma) + \frac{1}{2}A^2\Sigma^3(r_- - \Sigma) \\ &= Q^2 + \frac{1}{2}A^2\Sigma^3(r_- - \Sigma) \qquad (2.29) \end{aligned}$$

$$\begin{aligned} \beta^2 &= \frac{1}{2}(r_+ + \Sigma)(r_- + \Sigma) - \frac{1}{2}A^2\Sigma^3(r_- + \Sigma) \\ &= P^2 - \frac{1}{2}A^2\Sigma^3(r_- + \Sigma) \qquad (2.30) \end{aligned}$$

The manifest duality symmetry is also given by (2.21) and (2.22). The coordinates x and ϕ are angular coordinates. In figure 2.1 are the coordinates x and y indicated. The solution describes two black holes, although the coordinates used here contain only one of them. But an extension can be made like the Kruskal-extension of the Schwarzschild metric (see section 1.3.) This means that instead of one black hole, we are actually dealing with two

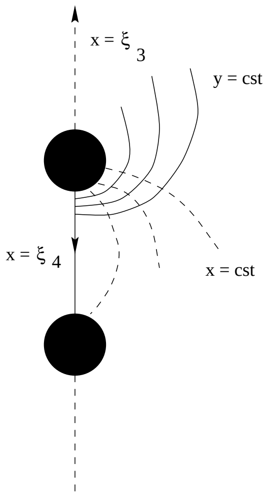

Figure 2.1: The two black holes, with the x and y coordinates indicated. The coordinates actually describe just one of the two black holes.

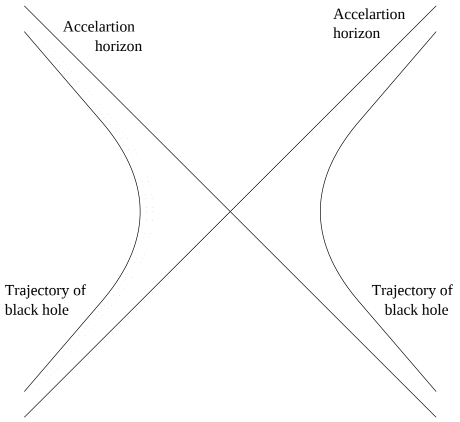

Figure 2.2: Qualitative picture of the C metric extended to include both black holes world lines. The dotted lines indicate the two inner horizons, which lie (like tubes) around each of the black holes.

(see figure 2.2.) In order to make the field continuous across the interface, the second black hole must carry an opposite charge. The possibility of a mutual interaction will not arise, since nowhere do their fields overlap.

For $r_+A < \frac{2}{3\sqrt{3}}$, the function $G(\xi)$ has four real roots. We denote these roots in ascending order by ξ_1, ξ_2, ξ_3 and ξ_4. We may restrict the parameters so that $\xi_1 = -\frac{1}{r_-A}$ and $\xi_1 < \xi_2 \leq \xi_3 < \xi_4$. The surface $y = \xi_0 \equiv \frac{-1}{|\Sigma|A}$ is singular, it is analogous to the singular surface at $r = |\Sigma|$ in the black hole solutions (2.13). As $r_- \geq |\Sigma|$, $\xi_1 \geq \xi_0$. The surfaces $y = \xi_1$, $y = \xi_2$ are the black hole horizons, and $y = \xi_3$ is the acceleration horizon. The coordinate x is restricted to the range $\xi_3 \leq x \leq \xi_4$ in which $G(x)$ is positive, so that the metric has the appropriate signature. At $x = \{\xi_3, \xi_4\}$, the norm of $\frac{\partial}{\partial \phi}$ vanishes, so these points are interpreted as the poles of two-spheres around the black hole. The axis $x = \xi_3$ points along the symmetry axis towards spatial infinity. The axis $x = \xi_4$ points towards the other black hole (the coordinates used, cover the region of space-time where one of the black holes is.) There is a divergence in the metric at $x = y$, because of the factor $(x-y)^{-2}$, which is interpreted as the point at infinity, so y is restricted to the range $\xi_0 < y < x$. Spatial infinity is reached when $y = x = \xi_3$ at fixed t, and null or time-like infinity when $y = x \neq \xi_3$.

As in the case with the ordinary C metric, it is not generally possible to choose the range of ϕ such that the metric (2.23) is regular at both $x = \xi_3$ and $x = \xi_4$. In order to see this, in a neighbourhood of each root define a new coordinate θ according to

$$\theta = \int_{\xi_i}^{x} \frac{1}{\sqrt{G(x')}}\, dx' \tag{2.31}$$

(with $i = 3$ or 4.) The angular part of the metric near one of the poles $x = \xi_i$ then has the form

$$dl^2 \approx \frac{F(y)}{A^2(\xi_i - y)^2} \left(d\theta^2 + \frac{1}{4}\lambda_i^2 \theta^2 d\phi^2 \right) \tag{2.32}$$

where $\lambda_i = |G'(\xi_i)|$. One can show that $\lambda_3 < \lambda_4$. Let the range of ϕ be $0 < \phi \leq \alpha$, then the deficit angles at the two poles δ_3, δ_4 are given by

$$\delta_3 = 2\pi - \frac{1}{2}\alpha\lambda_3 \tag{2.33}$$

and

$$\delta_4 = 2\pi - \frac{1}{2}\alpha\lambda_4 \tag{2.34}$$

We can remove the conical singularity at $x = \xi_4$ by choosing $\alpha = \frac{4\pi}{\lambda_4}$, but then there is a positive deficit angle running along the ξ_3 direction. Alternatively, we can choose $\alpha = \frac{4\pi}{\lambda_3}$ to remove the conical singularity at ξ_3. This means there is a negative deficit angle along the ξ_4 direction. For a general choice of α there will be conical singularities on both sides.

2.3 Melvin and Ernst solution

Schwinger calculated the rate of pair creation of charged particles in a background electric field [38, 39, 40]. The particles are produced at rest and subsequently run off to opposite ends of the electric field following hyperbolic space-time trajectories. We can see the pair

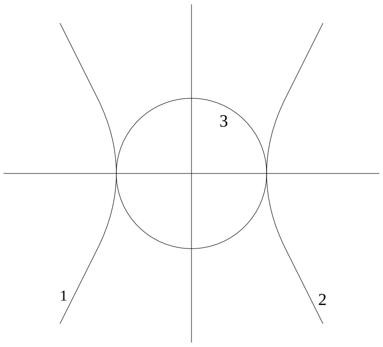

Figure 2.3: In the Schwinger model the hyperbolic trajectory 1 is that of the positron, the hyperbolic trajectory 2 that of the electron, both in real time and the circular orbit 3 that of the electron in Euclidean space. In the case of the Ernst solution, trajectories 1 and 2 are of the (oppositely) charged black holes and trajectory 3 is of the black hole in Euclidean space (the instanton.)

production as follows. The virtual loops that occur in empty space, can be made into real solutions by applying an external electric field. In the Euclidean space, there is a solution with, for instance, an electron moving on a circle in a uniform electric field. When we now analytically continue this solution from the positive definite Euclidean space to Lorentzian space, we obtain an electron and a positron accelerating away from each other, pulled apart by the electric field. The electron and positron are really the same particle. It tunnels through Euclidean space and emerges as a pair of real particles in Lorentzian space (see figure 2.3.)

There is a corresponding solution that represents the pair creation of charged black holes in an external electro-magnetic field. The Lorentzian space is in this case the universe discovered by Ernst [41]. It represents two charged black holes accelerating away from each other in a space-time that is asymptotic to the Melvin universe [42]. The Melvin universe is the solution of the Einstein-Maxwell equations that represents a uniform electro-magnetic field. The Ernst solution is the black hole analog of the electron-positron pair accelerating away from each other in Lorentzian space. Like the electron-positron solution, the Ernst solution can be analytically continued to a Euclidean solution. We just have to adjust the parameters of the solution, like the mass and charge of the black holes, so that the temperatures of the black holes and the acceleration horizons are the same. This allows us to remove the conical singularities and obtain a complete Euclidean solution of the Einstein-Maxwell equations. So, we get a picture as in figure 2.3, where at $\tau = it = 0$, the black holes are pair created by tunneling through Euclidean space and emerging as a

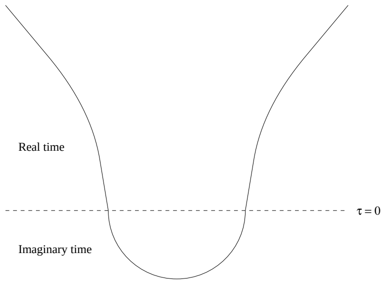

Figure 2.4: Under the dashed line is the Euclidean solution. Above the dashed line are the orbits of the black holes in the analytic continuation in Lorentzian space.

pair of real black holes in Ernst space. When we cut the Euclidean solution in half along $\tau = 0$ and join it to the upper half of the Lorentzian solution, we get a picture of the pair creation (see figure 2.4.)

The generalization of the Ernst solution of Einstein-Maxwell theory to the theory with the two gauge fields and the dilaton, the $U(1)^2$ Ernst solution, was obtained in [21]. Like the Ernst solution, it describes a pair of oppositely-charged black holes undergoing uniform acceleration under the influence of background electro-magnetic fields.

In [24] was shown that given an axisymmetric solution with $A_i = g_{i\phi} = 0$, where x^i are the other three coordinates, a new axisymmetric solution may be obtained by the transformation

$$g'_{ij} = \Lambda g_{ij} \tag{2.35}$$
$$g'_{\phi\phi} = \Lambda^{-1} g_{\phi\phi} \tag{2.36}$$
$$e^{-2\varphi'} = e^{-2\varphi}\Lambda \tag{2.37}$$
$$A'_\phi = -\frac{1}{B\Lambda}(1 + BA_\phi) \tag{2.38}$$
$$\Lambda = (1 + BA_\phi)^2 + \frac{1}{2}B^2 g_{\phi\phi} e^{2\varphi} \tag{2.39}$$

But in [24], there was only one $U(1)$ gauge field. Here, we have two $U(1)$ gauge fields, so we have to make an appropriate generalization of this transformation. In the string conformal gauge, $ds_T^2 = e^{2\varphi} ds^2$, the action of the $SU(4)$ version can be written as [21]

$$S = \int d^4x \sqrt{-g_T} e^{-2\varphi} [R_T + 4(\nabla\varphi)^2 - F^2 - G^2] \tag{2.40}$$

Now, write $F_{\mu\nu}$ and $G_{\mu\nu}$ in terms of vector potential as

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.41}$$
$$F_{\mu\nu} = \frac{1}{2} e^{2\varphi} \epsilon_{\mu\nu\rho\sigma} \partial^\rho B^\sigma \tag{2.42}$$

and introduce the definitions

$$^3g_{ij} = g_{Tij} \tag{2.43}$$

$$V = g_{T\phi\phi} \tag{2.44}$$
$$\tilde{\varphi} = \varphi - \frac{1}{4}\ln V \tag{2.45}$$

Then the action can be rewritten in the form

$$S = \alpha \int d^3x\sqrt{-{}^3g}e^{-2\tilde{\varphi}}\Big[{}^3R + 4\partial_i\tilde{\varphi}\partial^i\tilde{\varphi} - \frac{1}{4}V^{-2}\partial_i V\partial^i V - 2V^{-1}\partial_i A_\phi \partial^i A_\phi - 2e^{4\tilde{\varphi}}\partial_i B_\phi \partial^i B_\phi\Big] \tag{2.46}$$

where $0 \leq \phi \leq \alpha$ and the ϕ coordinate is integrated out. This action is invariant under the transformations

$$V' = \frac{1}{\Lambda^2}V \tag{2.47}$$
$$A'_\phi = -\frac{1}{B\Lambda}(1 + BA_\phi) \tag{2.48}$$
$${}^3g'_{ij} = \Psi^2\, {}^3g_{ij} \tag{2.49}$$
$$e^{-2\tilde{\varphi}'} = \frac{1}{\Psi}e^{-2\tilde{\varphi}} \tag{2.50}$$
$$B'_\phi = -\frac{1}{E\Psi}(1 + EB_\phi) \tag{2.51}$$
$$\Lambda = (1 + BA_\phi)^2 + \frac{1}{2}B^2 V \tag{2.52}$$
$$\Psi = (1 + EB_\phi)^2 + \frac{1}{2}E^2 e^{-4\tilde{\varphi}} \tag{2.53}$$

Now we may construct new solutions by applying these transformations to axisymmetric solutions satisfying $A_i = g_{i\phi} = 0$. The analogue of the Melvin solution [42], obtained by applying these transformations to the Minkowski metric in cylindrical coordinates and after applying another gauge transformation in order to get regular vector potentials on the $\rho = 0$ axis, is [25]

$$ds^2 = \Lambda\Psi[-dt^2 + d\rho^2 + dz^2] + \frac{\rho^2 d\phi^2}{\Lambda\Psi} \tag{2.54}$$
$$e^{-2\varphi} = \frac{\Lambda}{\Psi} \tag{2.55}$$
$$A_\phi = -\frac{\hat{B}_M\rho^2}{2\Lambda} \tag{2.56}$$
$$B_\phi = -\frac{\hat{E}_M\rho^2}{2\Psi} \tag{2.57}$$
$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.58}$$
$$\tilde{F}_{\mu\nu} = \partial_\mu \tilde{B}_\nu - \partial_\nu \tilde{B}_\mu \tag{2.59}$$
$$\Lambda = 1 + \frac{1}{2}\hat{B}_M^2\rho^2 \tag{2.60}$$
$$\Psi = 1 + \frac{1}{2}\hat{E}_M^2\rho^2 \tag{2.61}$$

This solution, which is called the $U(1)^2$ Melvin solution, has a manifest duality symmetry under

$$\begin{array}{lcl} \tilde{F} & \leftrightarrow & G \\ \varphi & \leftrightarrow & -\varphi \\ \hat{B}_M & \leftrightarrow & \hat{E}_M \end{array} \tag{2.62}$$

It represents a pair of magnetic fields which are essentially uniform near the axis $\rho = 0$, with field strengths given by $\hat{E}_M$ and $\hat{B}_M$. The fields depart from uniformity away from the axis because the field energy curves the space-time. However, in practice we cannot construct such strong fields, so the physically interesting part of this solution is the region near the axis.

If we apply the transformations to the generalized C metric (2.23), we will obtain the analogue of the Ernst solution, which preserves the manifest duality symmetry of (2.23). The resulting solution is the $U(1)^2$ Ernst solution [25]

$$\begin{aligned} ds^2 &= \frac{\Lambda\Psi}{A^2(x-y)^2}\Big[F(x)\Big(G(y)dt^2 - G^{-1}(y)dy^2\Big) \\ &+ F(y)G^{-1}(x)dx^2\Big] + \frac{F(y)G(x)}{\Lambda\Psi A^2(x-y)^2}d\phi^2 \end{aligned} \tag{2.63}$$

$$e^{-2\varphi} = e^{-2\varphi_0}\frac{\Lambda}{\Psi}\left(\frac{1+\Sigma Ay}{1-\Sigma Ay}\right)\left(\frac{1-\Sigma Ax}{1+\Sigma Ax}\right) \tag{2.64}$$

$$A_\phi = -\frac{e^{\varphi_0}}{B\Lambda}\left(1+\frac{B\beta x}{1-\Sigma Ax}\right)+k \tag{2.65}$$

$$B_\phi = -\frac{e^{-\varphi_0}}{E\Psi}\left(1+\frac{E\alpha x}{1+\Sigma Ax}\right)+k' \tag{2.66}$$

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.67}$$

$$\tilde{F}_{\mu\nu} = \partial_\mu \tilde{B}_\nu - \partial_\nu \tilde{B}_\mu \tag{2.68}$$

where

$$\Lambda = \left(1+\frac{B\beta x}{1-\Sigma Ax}\right)^2 + \frac{B^2(1-x^2-r_+Ax^3)(1+r_-Ax)(1-\Sigma Ay)^2}{2A^2(x-y)^2(1-\Sigma Ax)^2} \tag{2.69}$$

$$\Psi = \left(1+\frac{E\alpha x}{1+\Sigma Ax}\right)^2 + \frac{E^2(1-x^2-r_+Ax^3)(1+r_-Ax)(1+\Sigma Ay)^2}{2A^2(x-y)^2(1+\Sigma Ax)^2} \tag{2.70}$$

$$F(\xi) = 1-\Sigma^2A^2\xi^2 \tag{2.71}$$

$$\begin{aligned} G(\xi) &= \frac{(1-\xi^2-r_+A\xi^3)(1+r_-A\xi)}{(1-\Sigma^2A^2\xi^2)} \\ &= \frac{(1-\xi^2-r_+A\xi^3)(1+r_-A\xi)}{F(\xi)} \end{aligned} \tag{2.72}$$

and

$$\alpha^2 = \frac{1}{2}(r_+-\Sigma)(r_--\Sigma)+\frac{1}{2}A^2\Sigma^3(r_--\Sigma)$$

$$= \quad Q^2 + \frac{1}{2}A^2\Sigma^3(r_- - \Sigma) \tag{2.73}$$

$$\begin{aligned} \beta^2 &= \frac{1}{2}(r_+ + \Sigma)(r_- + \Sigma) - \frac{1}{2}A^2\Sigma^3(r_- + \Sigma) \\ &= P^2 - \frac{1}{2}A^2\Sigma^3(r_- + \Sigma) \end{aligned} \tag{2.74}$$

This solution describes two oppositely-charged black holes, although the coordinate system used here only includes one of the black holes. We can, like for the C metric, make an expansion in order to describe both black holes. The black holes carry two magnetic gauge charges and the background consists of two magnetic fields, which reduce to the fields in the $U(1)^2$ Melvin solution if we go to infinity along the axis of symmetry. The constants φ_0, k and k' will be chosen so that the solution at infinity agrees with the $U(1)^2$ Melvin solution.

The solution has a manifest duality symmetry under

$$\begin{array}{rcl} \tilde{F} & \leftrightarrow & G \\ \varphi & \leftrightarrow & -\varphi \end{array} \tag{2.75}$$

and

$$\begin{array}{rcl} Q & \leftrightarrow & P \\ \Sigma & \leftrightarrow & -\Sigma \\ B & \leftrightarrow & E \end{array} \tag{2.76}$$

which implies

$$\begin{array}{rcl} k & \leftrightarrow & k' \\ \varphi_0 & \leftrightarrow & -\varphi_0 \end{array} \tag{2.77}$$

to preserve the agreement with (2.54) at infinity.

For $r_+A < 2/(3\sqrt{3})$, the function $G(\xi)$ has four real roots, which I will denote in ascending order by ξ_1, ξ_2, ξ_3 and ξ_4. It is convenient to define another function $H(\xi) = G(\xi)F(\xi)$, so that we may write [25, 43]

$$H(\xi) = -(r_+A)(r_-A)(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4) \tag{2.78}$$

I restrict the parameters so that $\xi_1 = -1/(r_-A)$ and $\xi_1 \leq \xi_2 \leq \xi_3 < \xi_4$. The surface $y = \xi_0 \equiv -1/|\Sigma|A$ is singular. This is the singular surface inside the black hole, that is, the singular surface at $r = |\Sigma|$ in the black hole solutions (2.13). As $r_- \geq |\Sigma|$, $\xi_1 \geq \xi_0$. The surface $y = \xi_1$ is the inner black hole horizon, $y = \xi_2$ is the outer black hole horizon (event horizon) and $y = \xi_3$ is the acceleration horizon. In order for the metric to have Lorentzian signature, x is restricted to the range $\xi_3 \leq x \leq \xi_4$, in which $G(x)$ is positive. At $x = \{\xi_3, \xi_4\}$, the norm of $\partial/\partial\phi$ vanishes, so these points are interpreted as the poles of the two-spheres. The axis of symmetry is $x = \xi_3, \xi_4$, with $x = \xi_3$ pointing at spatial infinity and $x = \xi_4$ pointing at the other black hole. Because of the conformal factor $(x - y)^{-2}$ in the metric, there is a divergence in the metric at $x = y$. This divergence is interpreted as the point at infinity, so y is restricted to the range $\xi_0 < y < x$. Spatial infinity is reached only along the axis, that is, when $y = x = \xi_3$, and null or time-like infinity when $y = x \neq \xi_3$. We can choose $\xi_1 < \xi_2$, in which case the black holes are non-extreme, or we can choose $\xi_1 = \xi_2$, in which case the black holes are extreme.

As in the Ernst solution [41], the background fields provide the force necessary to accelerate the black holes. To eliminate the conical singularities in this metric at $x = \xi_3$ and $x = \xi_4$ simultaneously, A must be chosen so that

$$G'(\xi_3)\Lambda(\xi_4)\Psi(\xi_4) = -G'(\xi_4)\Lambda(\xi_3)\Psi(\xi_3) \tag{2.79}$$

is satisfied. To see this, one can follow the same analyses as for the C metric given at the end of the previous section. Then, in order for the spheres to be regular, we must take the period of ϕ to be

$$\Delta\phi = \frac{4\pi L^2}{G'(\xi_3)} \tag{2.80}$$

where L is defined by

$$L^2 \equiv \Lambda(\xi_3)\Psi(\xi_3) \tag{2.81}$$

In the limit $r_+ A \ll 1$ (which means that the black holes are very small), (2.79) reduces to Newton's law, $MA \approx BP + EQ$. More generally, the condition (2.79) reduces the number of free parameters for the solution by relating the acceleration of the black holes to the other parameters. This leads us to suppose that $r_+ A \ll 1$ is in some sense a point particle limit, which appears reasonable, as this is simply a statement that the black hole is small on the scale set by the background fields. In this limit, so long as $|r_+ A y| \ll 1$ as well, one finds that $G(\xi) \approx 1 - \xi^2$ and $F(\xi) \approx 1$. The metric (2.63) in this limit becomes

$$\begin{aligned} ds^2 &\approx \frac{\Lambda\Psi}{A^2(x-y)^2}[(1-y^2)\,dt^2 - (1-y^2)^{-1}\,dy^2 \\ &+ (1-x^2)^{-1}\,dx^2] + \frac{1-x^2}{\Lambda\Psi A^2(x-y)^2}\,d\phi^2 \end{aligned} \tag{2.82}$$

with

$$\Lambda \approx 1 + \frac{1}{2}B^2\frac{1-x^2}{A^2(x-y)^2} \tag{2.83}$$

$$\Psi \approx 1 + \frac{1}{2}E^2\frac{1-x^2}{A^2(x-y)^2} \tag{2.84}$$

This is just the Melvin solution (2.54) in nonstandard coordinates. If we transform the coordinates according to

$$\rho^2 = \frac{1-x^2}{A^2(x-y)^2} \tag{2.85}$$

$$\zeta^2 = \frac{y^2-1}{A^2(x-y)^2} \tag{2.86}$$

$$\hat{t} = \zeta\sinh t \tag{2.87}$$

$$z = \zeta\cosh t \tag{2.88}$$

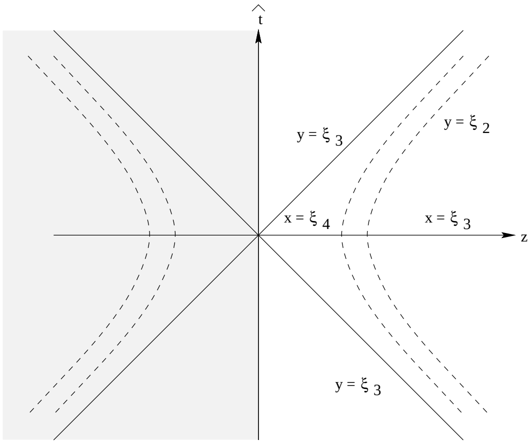

Figure 2.5: The $z, \hat{t}$ plane of the dilaton Ernst solution in the limit $r_+ A \ll 1$. The dotted lines indicate a region of size $\sim r_+$ surrounding the black hole at $y \sim \xi_2$; inside this region the geometry is approximately that of the black hole. The black hole moves on a trajectory with acceleration A, and the acceleration horizon is given by $y = \xi_3$. The coordinates used in (2.63) cover only the unshaded part of the figure.

we get the metric back in the form of (2.54). The dilaton and gauge fields in this approximation are given by

$$e^{-2\varphi} \approx e^{-2\varphi_0} \frac{\Lambda}{\Psi} \tag{2.89}$$

$$A_\phi \approx \frac{e^{\varphi_0} B \rho^2}{2\Lambda} \tag{2.90}$$

$$B_\phi \approx \frac{e^{-\varphi_0} E \rho^2}{2\Psi} \tag{2.91}$$

where k and k' have been chosen to be $\frac{e^{\varphi_0}}{BL}$ and $\frac{e^{-\varphi_0}}{EL}$ respectively, to give regularity on the axis $\rho = 0$. This is in agreement with the dilaton and gauge fields in the Melvin solution up to the arbitrary constant shift of the dilaton. The features of this approximation are sketched in figure 2.5.

The $U(1)^2$ Ernst solution also approaches (2.54) at large space-like distances, that is, when we go to infinity along the axis. Spatial infinity corresponds to $x, y \to \xi_3$ and in this limit it is convenient to use the change of coordinates given in [23]

$$x - \xi_3 = \frac{4F(\xi_3)L^2}{G'(\xi_3)A^2} \frac{\rho^2}{(\rho^2 + \zeta^2)^2} \tag{2.92}$$

$$\xi_3 - y = \frac{4F(\xi_3)L^2}{G'(\xi_3)A^2} \frac{\zeta^2}{(\rho^2 + \zeta^2)^2} \tag{2.93}$$

$$t = \frac{2\eta}{G'(\xi_3)} \tag{2.94}$$

$$\phi = \frac{2L^2\tilde{\phi}}{G'(\xi_3)} \tag{2.95}$$

The physical magnetic field parameter $\hat{B}_E$ is given by

$$\begin{aligned} \hat{B}_E &= \frac{\Delta \text{flux}}{\Delta \text{area}}|_{x=\xi_3} \\ &= \frac{1}{2}\frac{G'(\xi_3)}{L\Lambda(\xi_3)^{\frac{1}{2}}}B \end{aligned} \tag{2.96}$$

The other physical magnetic field parameter $\hat{E}_E$ is then given by

$$\hat{E}_E = \frac{1}{2}\frac{G'(\xi_3)}{L\Psi(\xi_3)^{\frac{1}{2}}}E \tag{2.97}$$

The parameters B and E are those appearing in the $U(1)^2$ Ernst solution (2.63).

When we take $\rho^2 + \zeta^2$ to be large, i.e. when we look at large space-like distances, the functions Λ and Ψ in the $U(1)^2$ Ernst solution become

$$\begin{aligned} \Lambda &\rightarrow \Lambda(\xi_3)\left\{1 + \frac{4\hat{B}_E^2L^2}{2A^2G'(\xi_3)^2}(x-\xi_3)G'(\xi_3)F(\xi_3)\frac{G'(\xi_3)^2A^4}{16F(\xi_3)^2L^4}(\rho^2+\zeta^2)^2\right\} \\ &= \Lambda(\xi_3)(1 + \frac{1}{2}\hat{B}_E^2\rho^2) \end{aligned} \tag{2.98}$$

and

$$\begin{aligned} \Psi &\rightarrow \Psi(\xi_3)\left\{1 + \frac{4\hat{E}_E^2L^2}{2A^2G'(\xi_3)^2}(x-\xi_3)G'(\xi_3)F(\xi_3)\frac{G'(\xi_3)^2A^4}{16F(\xi_3)^2L^4}(\rho^2+\zeta^2)^2\right\} \\ &= \Psi(\xi_3)(1 + \frac{1}{2}\hat{E}_E^2\rho^2) \end{aligned} \tag{2.99}$$

The terms in the metric (2.63) become in this limit

$$\frac{\Lambda\Psi}{A^2(x-y)^2} \rightarrow \frac{\tilde{\Lambda}\tilde{\Psi}A^2G'(\xi_3)^2}{16F(\xi_3)^2L^2}(\rho^2+\zeta^2)^2 \tag{2.100}$$

$$\begin{aligned} F(x)G(y)dt^2 &\approx F(\xi_3)(y-\xi_3)G'(\xi_3)\frac{4\,d\eta^2}{G'(\xi_3)^2} \\ &= -\frac{16F(\xi_3)^2L^2}{A^2G'(\xi_3)^2}\frac{\zeta^2\,d\eta^2}{(\rho^2+\zeta^2)^2} \end{aligned} \tag{2.101}$$

$$-\frac{F(x)}{G(y)}\,dy^2 + \frac{F(y)}{G(x)}\,dx^2 \approx \frac{16F(\xi_3)^2L^2}{G'(\xi_3)^2A^2(\rho^2+\zeta^2)^2}(d\rho^2+d\zeta^2) \tag{2.102}$$

$$
\begin{aligned}
\frac{F(y)G(x)}{\Lambda\Psi A^2(x-y)^2}\,d\phi^2 \;&\rightarrow\; \frac{F(\xi_3)G'(x)(x-\xi_3)}{\Lambda(\xi_3)\Psi(\xi_3)\tilde{\Lambda}\tilde{\Psi}A^2}\frac{G'(\xi_3)^2A^2(\rho^2-\zeta^2)^2}{16F(\xi_3)^2L^4}\frac{4L^4}{G'(\xi_3)^2}\,d\tilde{\phi}^2 \\
&= \frac{\rho^2\,d\tilde{\phi}^2}{\tilde{\Lambda}\tilde{\Psi}}
\end{aligned}
\tag{2.103}
$$

When we put all this together we get

$$
ds^2 \approx \tilde{\Lambda}\tilde{\Psi}(-\zeta^2 d\eta^2 + d\zeta^2 + d\rho^2) + \frac{\rho^2\,d\tilde{\phi}^2}{\tilde{\Lambda}\tilde{\Psi}} \tag{2.104}
$$

with

$$
\tilde{\Lambda} \approx (1 + \frac{1}{2}\hat{B}_E^2\rho^2) \tag{2.105}
$$

$$
\tilde{\Psi} \approx (1 + \frac{1}{2}\hat{E}_E^2\rho^2) \tag{2.106}
$$

and

$$
\hat{B}_E^2 \approx \frac{B^2G'^2(\xi_3)}{4L^2\Lambda(\xi_3)} \tag{2.107}
$$

$$
\hat{E}_E^2 \approx \frac{E^2G'^2(\xi_3)}{4L^2\Psi(\xi_3)} \tag{2.108}
$$

We obtain again (2.54), when we set $\hat{t} = \zeta\sinh\eta$, $z = \zeta\cosh\eta$. For large $\rho^2+\zeta^2$, the dilaton and gauge fields tend to

$$
\begin{aligned}
e^{-2\varphi} \;&\approx\; e^{-2\varphi_0}\frac{\Lambda(\xi_3)}{\Psi(\xi_3)}\frac{\tilde{\Lambda}}{\tilde{\Psi}} && (2.109)\\
A_{\tilde{\phi}} \;&\approx\; e^{\varphi_0}\sqrt{\frac{\Psi(\xi_3)}{\Lambda(\xi_3)}}\frac{\hat{B}_E\rho^2}{2\tilde{\Lambda}} && (2.110)\\
B_{\tilde{\phi}} \;&\approx\; e^{-\varphi_0}\sqrt{\frac{\Lambda(\xi_3)}{\Psi(\xi_3)}}\frac{\hat{E}_E\rho^2}{2\tilde{\Psi}} && (2.111)
\end{aligned}
$$

When we set $e^{2\varphi_0} = \frac{\Lambda(\xi_3)}{\Psi(\xi_3)}$ we recover the gauge fields of the Melvin solution. Now we recovered the $U(1)^2$ Melvin solution at large space-like distances, we can identify the physical strength of the background fields in the Ernst solution as $\hat{E}_E$ and $\hat{B}_E$. We also can calculate the physical charges on the black holes by integrating the field tensor over two-spheres surrounding the black holes. We find

$$
\hat{P} = \frac{1}{4\pi}\int G = \frac{\Lambda(\xi_3)\Psi(\xi_3)^{\frac{3}{2}}}{G'(\xi_3)\Lambda(\xi_4)^{\frac{1}{2}}}\frac{\beta(\xi_4-\xi_3)}{(1-\Sigma A\xi_4)(1-\Sigma A\xi_3)} \tag{2.112}
$$

and

$$
\hat{Q} = \frac{1}{4\pi}\int \tilde{F} = \frac{\Psi(\xi_3)\Lambda(\xi_3)^{\frac{3}{2}}}{G'(\xi_3)\Psi(\xi_4)^{\frac{1}{2}}}\frac{\alpha(\xi_4-\xi_3)}{(1+\Sigma A\xi_4)(1+\Sigma A\xi_3)} \tag{2.113}
$$

where α and β are given by (2.73) and (2.74).

2.4 Pair creation instanton and creation rate

We now take the Euclidean section of (2.63) by setting $\tau = it$. Half the Euclidean section gives an instanton describing black hole pair production [45]. There are three possible instantons. One that describes the pair production of non-extreme black holes, one that describes the pair production of extreme black holes and one that describes a special case when $\xi_2 = \xi_3$. The last one will not be discussed, because it does not describe black hole pair production. In this case there are no nodal singularities in the metric, even when the electro-magnetic field is absent. The proper distance between the two inner black hole horizons tends to a constant when ξ_2 approaches ξ_3. The corresponding instanton is often referred to as the type I instanton [24], but its physical interpretation is not clear.

First, let us consider the non-extreme or wormhole instantons. In this case $\xi_1 < \xi_2 < \xi_3$. To obtain a positive definite metric, we must restrict y to $\xi_2 \leq y \leq \xi_3$. Now (y, τ) are also coordinates on a two-sphere, except when $x = \xi_3$. We must impose another condition on the parameters to eliminate the possible conical singularities at the black hole horizon $y = \xi_2$ and the acceleration horizon $y = \xi_3$ simultaneously, with a single choice of the period of τ. (This is equivalent to demanding that the Hawking temperature of the black hole horizon equals the Unruh temperature of the acceleration horizon.) The period of τ is taken to be

$$\Delta\tau = \frac{4\pi}{|G'(\xi_2)|} \tag{2.114}$$

and the constraint is

$$|G'(\xi_2)| = |G'(\xi_3)| \tag{2.115}$$

where $G(\xi)$ is given by (2.72). The constraint gives

$$\begin{aligned} &(1 - \frac{\xi_2^2}{\xi_0^2})(\xi_2 - \xi_1)(\xi_3 - \xi_2)(\xi_4 - \xi_2)\frac{\xi_0^2}{(\xi_0^2 - \xi_2^2)^2} = \\ &\quad (1 - \frac{\xi_3^2}{\xi_0^2})(\xi_3 - \xi_1)(\xi_3 - \xi_2)(\xi_4 - \xi_3)\frac{\xi_0^2}{(\xi_0^2 - \xi_3^2)^2} \end{aligned} \tag{2.116}$$

which yields to

$$\left(\frac{\xi_2^2 - \xi_0^2}{\xi_3^2 - \xi_0^2}\right)\left(\frac{\xi_3 - \xi_1}{\xi_2 - \xi_1}\right) = \frac{\xi_4 - \xi_2}{\xi_4 - \xi_3} \tag{2.117}$$

This condition provides a further restriction on the black hole parameters. It may be thought of as determining the mass of the black hole in terms of its charges. Or more precisely, we can solve it for $r_- A$ in terms of $r_+ A$ and ΣA.

The whole Euclidean section is a bounce, that is an instanton-anti-instanton pair joined along a space-like slice. Its topology is $S^2 \times S^2 - \{pt\}$, where the removed point is $x = y = \xi_3$.

For the extremal instantons is $\xi_1 = \xi_2$. To ensure regularity at the acceleration horizon we must now take for the period of τ:

$$\Delta\tau = \frac{4\pi}{|G'(\xi_3)|} \tag{2.118}$$

The event horizon is at infinite distance in all space-like directions, so we do not have to worry about a conical singularity there. The range of y in the Euclidean section is now $\xi_2 < y \leq \xi_3$. The (y, τ) are now polar coordinates on an R^2, except when $x = \xi_3$. The extremal bounce has topology $S^2 \times R^2 - \{pt\}$ and the instanton can be interpreted as creating a pair of extremal black holes, with infinite throats.

Having described the pair creation instanton, we now can calculate the pair creation rate. The amplitude for pair creation in a background field is given by the path integral

$$\Psi = \int \mathcal{D}[g]\mathcal{D}[A]\mathcal{D}[B]\, e^{-I} \tag{2.119}$$

where the action I in the path integral is the action (2.1). The integral is over all metrics and gauge fields which interpolate between the background fields at infinity and a space-like slice which contains the pair of black holes. If there is an appropriate instanton, we assume that Ψ will be approximately $\Psi \approx e^{-I}$, where I is now the action of the instanton. The pair creation rate is given by the modulus of this amplitude, so it will be approximately $\Gamma \approx e^{-I_b}$, where I_b is the action of the bounce.

We will evaluate the action by a Hamiltonian decomposition [44]. The solutions we are interested in are stationary. The time-translation Killing vector has fixed points at the black hole event horizon. So there will be a contribution from a neighbourhood of each horizon. The Euclidean action then takes the form (see appendix C)

$$I = \beta H - \frac{1}{4}(\Delta A + A_{bh}) \tag{2.120}$$

with $\beta = \Delta\tau$ the period in imaginary time, H the Hamiltonian, ΔA the difference in area of the acceleration horizon in the physical solution (i.e. the Ernst solution) and in the background (i.e. the Melvin solution) and A_{bh} the black hole horizon. In the extreme case the term proportional to A_{bh} vanishes, as the black hole event horizon is not a part of the Euclidean section. Thus the Euclidean action in the extreme case is given by

$$I = \beta H - \frac{1}{4}\Delta A \tag{2.121}$$

As can be seen in appendix C, the Hamiltonian (which is defined with respect to the background space-time) can be expressed as

$$H = \int_\Sigma N\mathcal{H} - \frac{1}{8\pi}\int_{S^\infty} N({}^2K - {}^2K_0) \tag{2.122}$$

with N the lapse, $\mathcal{H}$ the Hamiltonian constrained, ${}^2K$ the trace of the two-dimensional extrinsic curvature of the boundary near infinity and ${}^2K_0$ the analogous quantity for the background space-time. The only nonzero contribution comes from the gravitational surface term, because the constraint vanishes on solutions.

We need to introduce a boundary near infinity, so we can calculate the extrinsic curvature in the instanton and the background solution. Then we can calculate the surface term in the Hamiltonian. To ensure that the boundary used in both calculations is the same, we need to match the intrinsic features (the induced metric, the gauge fields and the value of the dilaton on the boundary) of the boundary. We do this by calculating the features at the boundary in the $U(1)^2$ Ernst metric and in the $U(1)^2$ Melvin solution. Then we will match the features so that they era the same in these two metrics at the boundary.

This boundary is at infinity, $x = y = \xi_3$. To calculate the induced metric in the $U(1)^2$ Ernst solution, we use the coordinate transformations

$$\begin{aligned} \phi &= \frac{2L^2}{G'(\xi_3)}\phi' && (2.123)\\ t &= \frac{2}{G'(\xi_3)}t' && (2.124)\end{aligned}$$

and we take the boundary to lie at

$$\begin{aligned} x &= \xi_3 + \epsilon_E \chi && (2.125)\\ y &= \xi_3 + \epsilon_E(\chi - 1) && (2.126)\end{aligned}$$

with $0 \leq \chi \leq 1$. The term with $d\phi^2$ in the $U(1)^2$ Ernst metric then becomes

$$\begin{aligned} \frac{F(y)G(x)}{\Lambda\Psi A^2(x-y)^2}\,d\phi^2 &\approx [F(\xi_3) + \epsilon_E(\chi-1)F'(\xi_3) + \frac{1}{2}\epsilon_E^2(\chi-1)^2F''(\xi_3)]\\ &\quad [\epsilon_E\chi G'(\xi_3)^2 + \frac{1}{2}\epsilon_E^2\chi^2G''(\xi_3)]\frac{4L^4}{G'(\xi_3)L^2\lambda\psi A^2\epsilon_E^2}\,d\phi'^2\\ &= \frac{\epsilon_E\chi H'(\xi_3)4L^4}{L^2\lambda\psi A^2\epsilon_E^2 G'(\xi_3)^2}\left[1 + \frac{1}{2}\epsilon_E\chi\frac{H''(\xi_3)}{H'(\xi_3)} - \frac{\epsilon_E F'(\xi_3)}{F(\xi_3)}\right]d\phi'^2\\ &= \frac{4F(\xi_3)L^2\chi}{\lambda\psi A^2\epsilon_E G'(\xi_3)}\left[1 + \frac{1}{2}\epsilon_E\chi\frac{H''(\xi_3)}{H'(\xi_3)} - \frac{\epsilon_E F'(\xi_3)}{F(\xi_3)}\right]d\phi'^2 && (2.127)\end{aligned}$$

The term with dy^2 becomes

$$\begin{aligned} -\frac{\Lambda\Psi F(x)}{A^2(x-y)^2G(y)}\,dy^2 &\approx -\frac{L^2\lambda\psi}{A^2}\left[\frac{F(\xi_3)+\epsilon_E\chi F'(\xi_3)}{\epsilon_E(\chi-1)G'(\xi_3)}\right]d\chi^2\\ &= -\frac{L^2\lambda\psi}{A^2\epsilon_E G'(\xi_3)}\left[\frac{1}{\chi-1} + \frac{\chi\epsilon_E F'(\xi_3)}{(\chi-1)F(\xi_3)}\right]d\chi^2 && (2.128)\end{aligned}$$

The term with dx^2 becomes

$$\begin{aligned} \frac{\Lambda\Psi F(y)}{A^2(x-y)^2G(x)}\,dx^2 &\approx \frac{L^2\lambda\psi}{A^2}\left[\frac{F(\xi_3)+\epsilon_E(\chi-1)F'(\xi_3)}{\epsilon_E\chi G'(\xi_3)}\right]d\chi^2\\ &= \frac{L^2\lambda\psi F(\xi_3)}{A^2\epsilon_E G'(\xi_3)}\left[\frac{1}{\chi} + \frac{\epsilon_E(\chi-1)F'(\xi_3)}{\chi F(\xi_3)}\right]d\chi^2 && (2.129)\end{aligned}$$

The induced metric on the boundary for the $U(1)^2$ Ernst metric is then given by,

$$
\begin{aligned}
{}^{(2)}ds^2 = \frac{L^2F(\xi_3)}{A^2\epsilon_E G'(\xi_3)}\Bigg\{&-\frac{\lambda\psi\, d\chi^2}{\chi(\chi-1)}\left[1+\epsilon_E(2\chi-1)\frac{F'(\xi_3)}{F(\xi_3)}\right] \\
&+\frac{4\chi}{\lambda\psi}\left[1+\epsilon_E\chi\frac{H''(\xi_3)}{2H'(\xi_3)}-\epsilon_E\frac{F'(\xi_3)}{F(\xi_3)}\right]d\phi'^2\Bigg\}
\end{aligned}
\tag{2.130}
$$

with

$$
\lambda = 1+\frac{2L^2\hat{B}_E^2F(\xi_3)\chi}{A^2\epsilon_E G'(\xi_3)}\left(1+\frac{1}{2}\epsilon_E\chi\frac{H''(\xi_3)}{H'(\xi_3)}+\frac{2\Sigma A\epsilon_E}{1-\Sigma A\xi_3}\right) \tag{2.131}
$$

and

$$
\psi = 1+\frac{2L^2\hat{E}_E^2F(\xi_3)\chi}{A^2\epsilon_E G'(\xi_3)}\left(1+\frac{1}{2}\epsilon_E\chi\frac{H''(\xi_3)}{H'(\xi_3)}-\frac{2\Sigma A\epsilon_E}{1+\Sigma A\xi_3}\right) \tag{2.132}
$$

The gauge potentials on the boundary for the $U(1)^2$ Ernst solution are

$$
A_{\phi'} = \frac{Le^{\varphi_0}}{\Lambda(\xi_3)\hat{B}_E}\left[1-\frac{A^2\epsilon_E G'(\xi_3)}{2L^2F(\xi_3)\hat{B}_E^2\chi}\right] \tag{2.133}
$$

and

$$
B_{\phi'} = \frac{Le^{-\varphi_0}}{\Psi(\xi_3)\hat{E}_E}\left[1-\frac{A^2\epsilon_E G'(\xi_3)}{2L^2F(\xi_3)\hat{E}_E^2\chi}\right] \tag{2.134}
$$

The dilaton at the boundary is

$$
e^{-2\varphi} = e^{-2\varphi_0}\frac{\Lambda(\xi_3)\lambda}{\Psi(\xi_3)\psi}\left(1-\frac{2\Sigma A\epsilon_E}{1-\Sigma^2A^2\xi_3^2}\right) \tag{2.135}
$$

Everything is evaluated only up to second nontrivial order in ϵ_E, because at the end we will take the limit $\epsilon_E \to 0$ and then the higher-order terms will not contribute to the Hamiltonian.

For the $U(1)^2$ Melvin solution we take the boundary at

$$
x = -1+\epsilon_M\chi[1+\epsilon_E f(\chi)] \tag{2.136}
$$

$$
y = -1+\epsilon_M(\chi-1)[1+\epsilon_E g(\chi)] \tag{2.137}
$$

in the accelerated coordinate system (2.82). Of course there are more choices possible. The induced metric on the boundary in this case is

$$
\begin{aligned}
{}^{(2)}ds^2 &= \frac{-\Lambda\Psi}{2\chi(\chi-1)\bar{A}^2\epsilon_M}\{1-\epsilon_E(\chi-1)f(\chi)+\epsilon_E\chi g(\chi) \\
&- 2\epsilon_E\chi(\chi-1)[f'(\chi)-g'(\chi)]-2\epsilon_E[\chi f(\chi)-(\chi-1)g(\chi)]\}d\chi^2 \\
&+ \frac{2\chi}{\Lambda\Psi\bar{A}^2\epsilon_M}\{1-\frac{1}{2}\epsilon_M\chi+\epsilon_E f(\chi)-2\epsilon_E[\chi f(\chi)-(\chi-1)g(\chi)]\}d\phi^2
\end{aligned}
\tag{2.138}
$$

with

$$\Lambda = 1 + \frac{\hat{B}_M^2 \chi}{\bar{A}^2 \epsilon_M}\{1 - \frac{1}{2}\epsilon_M \chi + \epsilon_E f(\chi) - 2\epsilon_E[\chi f(\chi) - (\chi - 1)g(\chi)]\} \tag{2.139}$$

$$\Psi = 1 + \frac{\hat{E}_M^2 \chi}{\bar{A}^2 \epsilon_M}\{1 - \frac{1}{2}\epsilon_M \chi + \epsilon_E f(\chi) - 2\epsilon_E[\chi f(\chi) - (\chi - 1)g(\chi)]\} \tag{2.140}$$

The gauge potentials on the boundary in the $U(1)^2$ Melvin solution are

$$A_\phi = \frac{1}{\hat{B}_M}\left[1 - \frac{\bar{A}^2 \epsilon_M}{\hat{B}_M^2 \chi}\right] \tag{2.141}$$

and

$$B_\phi = \frac{1}{\hat{E}_M}\left[1 - \frac{\bar{A}^2 \epsilon_M}{\hat{E}_M^2 \chi}\right] \tag{2.142}$$

and the dilaton at the boundary in $U(1)^2$ Melvin solution is

$$e^{-2\varphi} = \frac{\Lambda}{\Psi} \tag{2.143}$$

(where Λ and Ψ are given by (2.139) and (2.140).) Note that $\bar{A}$ represents a choice of coordinates in the Melvin metric. We can fix this coordinate freedom by taking

$$\bar{A}^2 = -\frac{G'(\xi_3)}{2L^2 F(\xi_3)}\frac{H'(\xi_3)}{H''(\xi_3)}A^2 \tag{2.144}$$

Furthermore, when we write

$$\begin{aligned} e^{\varphi_0} &= \left(\frac{\Lambda(\xi_3)}{\Psi(\xi_3)}\right)^{\frac{1}{2}}(1 - \gamma\epsilon_E) && (2.145)\\ \hat{B}_M &= \hat{B}_E(1 + \alpha\epsilon_E) && (2.146)\\ \hat{E}_M &= \hat{E}_E(1 + \beta\epsilon_E) && (2.147) \end{aligned}$$

we find that the intrinsic metric, gauge potentials and dilaton on the boundary can all be matched by taking

$$\begin{aligned} \epsilon_M &= -\frac{H''(\xi_3)}{H'(\xi_3)}\epsilon_E && (2.148)\\ f(\chi) &= \frac{F'(\xi_3)}{F(\xi_3)}(4\chi - 3) && (2.149)\\ g(\chi) &= \frac{F'(\xi_3)}{F(\xi_3)}(4\chi - 1) && (2.150) \end{aligned}$$

and

$$\gamma = \alpha = -\beta = \frac{\Sigma A}{1 - \Sigma^2 A^2 \xi_3^2} \tag{2.151}$$

The lapse functions are also matched by these conditions. The lapse function at the boundary for the $U(1)^2$ Ernst metric is given by

$$N = \left[\frac{4L^2 F(\xi_3)(1-\chi)\lambda\psi}{A^2\epsilon_E G'(\xi_3)}\right]^{\frac{1}{2}} \left[1 + \frac{1}{4}\epsilon_E(\chi-1)\frac{H''(\xi_3)}{H'(\xi_3)} + \frac{1}{2}\epsilon_E\frac{F'(\xi_3)}{F(\xi_3)}\right] \tag{2.152}$$

For the $U(1)^2$ Melvin solution, the lapse function is given by

$$\begin{aligned} N = \left[\frac{2(1-\chi)\Lambda\Psi}{\bar{A}^2\epsilon_M}\right]^{\frac{1}{2}} & \left\{1 - \frac{1}{4}\epsilon_M(\chi-1) + \frac{1}{2}\epsilon_E g(\chi)\right. \\ & \left. -\epsilon_E[\chi f(\chi) - (\chi-1)g(\chi)]\right\} \end{aligned} \tag{2.153}$$

It can be checked that the matching conditions (2.144-2.151) make (2.152) and (2.153) equal as well.

We now can also calculate the Hamiltonian. The extrinsic curvature of the boundary embedded in the $U(1)^2$ Ernst solution is

$${}^2K = \frac{A\epsilon_E^{\frac{1}{2}} G'(\xi_3)^{\frac{1}{2}}}{LF(\xi_3)^{\frac{1}{2}}\lambda\psi}\left[1 + \frac{1}{4}\epsilon_E\frac{H''(\xi_3)}{H'(\xi_3)}(4\chi-3) - \frac{1}{2}\epsilon_E\frac{F'(\xi_3)}{F(\xi_3)}(4\chi-3)\right] \tag{2.154}$$

The extrinsic curvature of the boundary embedded in the $U(1)^2$ Melvin solution is

$${}^2K_0 = \frac{\bar{A}\epsilon_M^{\frac{1}{2}}\sqrt{2}}{\Lambda\Psi}\left[1 - \frac{1}{4}\epsilon_M(4\chi-3) - \frac{1}{2}\epsilon_E\frac{F'(\xi_3)}{F(\xi_3)}(24\chi-13)\right] \tag{2.155}$$

When we use the matching conditions (2.144-2.151), we get

$${}^2K - {}^2K_0 = \frac{5A\epsilon_E^{\frac{3}{2}} G'(\xi_3)^{\frac{1}{2}} F'(\xi_3)}{LF(\xi_3)^{\frac{1}{2}}\lambda\psi F(\xi_3)}(2\chi-1) \tag{2.156}$$

Thus, taking the limit $\epsilon_E \to 0$, the Hamiltonian is

$$\begin{aligned} H &= -\frac{1}{4}\int_0^1 d\chi\, N\sqrt{h}({}^2K - {}^2K_0) \\ &= -\frac{5L^2F'(\xi_3)}{A^2G'(\xi_3)}\int_0^1 d\chi\,(2\chi-1) \\ &= 0 \end{aligned} \tag{2.157}$$

And the action is given by

$$I = -\frac{1}{4}(\Delta A + A_{bh}) \tag{2.158}$$

in the non-extreme case and by

$$I = -\frac{1}{4}\Delta A \tag{2.159}$$

in the extreme case, where ΔA is the difference in area of the acceleration horizon in the Ernst and Melvin metrics and A_{bh} is the area of the black hole horizon.

The area of the black hole horizon is (with $\Delta\phi = \frac{4\pi}{G'(\xi_3)}$ and $\xi_3 \leq x \leq \xi_4$)

$$\begin{aligned} A_{bh} &= \int_{y=\xi_2} \sqrt{g_{xx}g_{\phi\phi}}\, dx d\phi \\ &= \frac{4\pi F(\xi_2)L^2}{A^2 G'(\xi_3)} \frac{(\xi_4-\xi_3)}{(\xi_3-\xi_2)(\xi_4-\xi_2)} \end{aligned} \tag{2.160}$$

To calculate the difference in area of the acceleration horizon (ΔA), we need to introduce a boundary, as we did in calculating the Hamiltonian. We take this boundary to be a large circle at $x = \xi_3 + \epsilon_E$. The area of the acceleration horizon in the $U(1)^2$ Ernst solution up to this boundary is

$$\begin{aligned} A_E &= \int_{y=\xi_3} \sqrt{g_{xx}g_{\phi\phi}}\, dx d\phi \\ &= \frac{F(\xi_3)\Delta\phi}{A^2} \int_{x=\xi_3+\epsilon_E}^{x=\xi_4} \frac{dx}{(x-\xi_3)^2} \\ &= -\frac{4\pi F(\xi_3)L^2}{A^2 G'(\xi_3)(\xi_4-\xi_3)} + \pi\rho_E^2 \end{aligned} \tag{2.161}$$

with

$$\rho_E^2 = \frac{4F(\xi_3)L^2}{G'(\xi_3)A^2\epsilon_E} \tag{2.162}$$

The area of the acceleration horizon in the $U(1)^2$ Melvin solution up to a boundary at $\rho = \rho_M$ is

$$\begin{aligned} A_M &= 2\pi \int_{\rho=0}^{\rho=\rho_M} \rho\, d\rho \\ &= \pi\rho_M^2 \end{aligned} \tag{2.163}$$

Note that there is no ambiguity in the choice of boundary in the Melvin solution here, $\rho = \rho_M$ is the only choice which obeys the Killing symmetry.

Now we need to match the proper length of he boundary, the integrals of both gauge potentials around the boundary and the value of the dilaton at the boundary. The proper length of the boundary in the $U(1)^2$ Ernst solution is

$$\begin{aligned} l_E = \frac{4\pi}{\hat{E}_E\hat{B}_E\rho_E} \Bigg[1 - \frac{F(\xi_3)L^2}{G'(\xi_3)A^2}\frac{H''(\xi_3)}{H'(\xi_3)}\frac{1}{\rho_E^2} + \frac{2F(\xi_3)L^2}{G'(\xi_3)A^2}\frac{F'(\xi_3)}{F(\xi_3)}\frac{1}{\rho_E^2} \\ - \frac{1}{\hat{E}_E^2\rho_E^2} - \frac{1}{\hat{B}_E^2\rho_E^2} \Bigg] \end{aligned} \tag{2.164}$$

In the $U(1)^2$ Melvin solution, the proper length of the boundary is

$$l_M = \frac{4\pi}{\hat{E}_M\hat{B}_M\rho_M} \left[1 - \frac{1}{\hat{E}_M^2\rho_M^2} - \frac{1}{\hat{B}_M^2\rho_M^2} \right] \tag{2.165}$$

The integrals of the gauge potentials around the boundary in the $U(1)^2$ Ernst solution are

$$\oint A_\phi\, d\phi = \frac{2\pi}{\hat{B}_E}\frac{e^{\varphi_0}\Psi(\xi_3)^{\frac{1}{2}}}{\Lambda(\xi_3)^{\frac{1}{2}}}\left(1-\frac{2}{\hat{B}_E^2\rho_E^2}\right) \tag{2.166}$$

and

$$\oint B_\phi\, d\phi = \frac{2\pi}{\hat{E}_E}\frac{e^{-\varphi_0}\Lambda(\xi_3)^{\frac{1}{2}}}{\Psi(\xi_3)^{\frac{1}{2}}}\left(1-\frac{2}{\hat{E}_E^2\rho_E^2}\right) \tag{2.167}$$

For the $U(1)^2$ Melvin solution they are given by

$$\oint A_\phi\, d\phi = \frac{2\pi}{\hat{B}_M}\left(1-\frac{2}{\hat{B}_M^2\rho_M^2}\right) \tag{2.168}$$

and

$$\oint B_\phi\, d\phi = \frac{2\pi}{\hat{E}_M}\left(1-\frac{2}{\hat{E}_M^2\rho_M^2}\right) \tag{2.169}$$

Finally, the dilaton at the boundary in the $U(1)^2$ Ernst solution is

$$e^{-2\varphi} = e^{-2\varphi_0}\frac{\Lambda(\xi_3)\hat{B}_E^2}{\Psi(\xi_3)\hat{E}_E^2}\left[1+\frac{2\Sigma A}{(1-\Sigma^2A^2\xi_3^2)}\frac{4F(\xi_3)L^2}{G'(\xi_3)A^2\rho_E^2}+\frac{2}{\hat{B}_E^2\rho_E^2}-\frac{2}{\hat{E}_E^2\rho_E^2}\right] \tag{2.170}$$

In the $U(1)^2$ Melvin solution the dilaton at the boundary is

$$e^{-2\varphi} = \frac{\hat{B}_M^2}{\hat{E}_E^2}\left[1+\frac{2}{\hat{B}_M^2\rho_M^2}-\frac{2}{\hat{E}_M^2\rho_M^2}\right] \tag{2.171}$$

When we write e^{φ_0}, $\hat{B}_M$ and $\hat{E}_M$ as in (2.145), (2.146) and (2.147) and

$$\rho_M = \rho_E\left\{1+\frac{1}{\rho_E^2}\frac{F(\xi_3)L^2}{G'(\xi_3)A^2}\left[\frac{H''(\xi_3)}{H'(\xi_3)}-\frac{2F'(\xi_3)}{F(\xi_3)}\right]\right\} \tag{2.172}$$

the proper length of the boundary, the integrals of the gauge fields and the dilaton can all be matched.

Now the difference in area can be evaluated, letting $\epsilon_E \to 0$,

$$\begin{aligned}\Delta A &= A_E - A_M\\ &= -\frac{4\pi L^2F(\xi_3)}{G'(\xi_3)A^2}\left[\frac{1}{(\xi_4-\xi_3)}+\frac{H''(\xi_3)}{2H'(\xi_3)}-\frac{F'(\xi_3)}{F(\xi_3)}\right]\\ &= -\frac{4\pi L^2F(\xi_3)}{G'(\xi_3)A^2}\left[\frac{(\xi_2-\xi_1)}{(\xi_3-\xi_2)(\xi_3-\xi_1)}+\frac{2}{(\xi_3-\xi_1)}-\frac{F'(\xi_3)}{F(\xi_3)}\right]\end{aligned} \tag{2.173}$$

Thus, in the extreme case, when $\xi_2=\xi_1$, we have

$$I = -\frac{1}{4}\Delta A = \frac{2\pi L^2F(\xi_3)}{G'(\xi_3)A^2}\left[\frac{1}{(\xi_3-\xi_1)}-\frac{F'(\xi_3)}{2F(\xi_3)}\right] \tag{2.174}$$

While for the non-extreme case we have

$$\begin{aligned}
I &= -\frac{1}{4}(\Delta A + A_{bh}) \\
&= \frac{\pi L^2 F(\xi_3)}{G'(\xi_3)A^2}\left[\frac{2}{(\xi_3-\xi_1)} - \frac{F'(\xi_3)}{F(\xi_3)} + \frac{(\xi_2-\xi_1)}{(\xi_3-\xi_2)(\xi_3-\xi_1)} - \frac{F(\xi_2)(\xi_4-\xi_3)}{F(\xi_3)(\xi_3-\xi_2)(\xi_4-\xi_2)}\right] \\
&= \frac{2\pi L^2 F(\xi_3)}{G'(\xi_3)A^2}\left[\frac{1}{(\xi_3-\xi_1)} - \frac{F'(\xi_3)}{2F(\xi_3)}\right] \qquad (2.175)
\end{aligned}$$

where the instanton condition (2.117) is used to cancel the last two terms. We see that the action is the same in both cases, so the action of the bounce is given by

$$I_b = \frac{2\pi L^2 F(\xi_3)}{G'(\xi_3)A^2}\left[\frac{1}{(\xi_3-\xi_1)} - \frac{F'(\xi_3)}{2F(\xi_3)}\right] \qquad (2.176)$$

In the point-particle limit, $r_+ A \ll 1$, both the extreme and non-extreme black holes satisfy $r_+ \approx r_-$ [21]. In this limit the action reduces to

$$\begin{aligned}
I_b &\approx \frac{\pi r_-}{A} \\
&\approx \frac{\pi M^2}{BP + EQ} \qquad (2.177)
\end{aligned}$$

where Newton's law is used in the second step. The pair creation rate is e^{-I_b}, so we recover the Schwinger result (generalized to the case of two gauge fields) in this limit. So small black holes are pair produced with the same rate as we would expect for some hypothetical particles with the same mass and charges. In particular, the pair creation rate will be very small for realistic fields, as we must have $M > M_{pl}$ for the semi-classical approximation to be valid.

2.5 Comparison with other theories

The pair production rate for charged black holes can be calculated in other theories as well. First we consider the Einstein-Maxwell action [43]

$$I = \frac{1}{16\pi}\int_M (-R + F^2) \qquad (2.178)$$

where R is the scalar curvature, $F_{\mu\nu}$ is the Maxwell field. The background field is now given by the Melvin metric

$$\begin{aligned}
ds^2 &= \Lambda^2[-dt^2 + dz^2 + d\rho^2] + \left(\frac{\rho}{\Lambda}\right)^2 d\phi^2 \qquad (2.179) \\
A_\phi &= \frac{\hat{B}_M \rho^2}{2\Lambda} \qquad (2.180) \\
\Lambda &= 1 + \frac{1}{4}\hat{B}_M^2 \rho^2 \qquad (2.181) \\
F^2 &= \frac{2\hat{B}_M^2}{\Lambda^4} \qquad (2.182)
\end{aligned}$$

(note that we took the background to be magnetic here.) The parameter $\hat{B}_M$ gives the value of the magnetic field on the axis. The Ernst solution is given by

$$\begin{aligned} ds^2 &= \frac{\Lambda^2}{(x-y)^2A^2}[G(y)\,dt^2 - G(y)^{-1}\,dy^2 + G(x)^{-1}\,dx^2] \\ &+ \frac{G(x)}{(x-y)^2A^2\Lambda^2}\,d\phi^2 \qquad (2.183) \\ A_\phi &= -\frac{2}{B\Lambda}[1+Bqx]+k \qquad (2.184) \\ \Lambda &\equiv \Lambda(x,y) = [1+\frac{1}{2}Bqx]^2 + \frac{B^2}{4A^2(x-y)^2}G(x) \qquad (2.185) \\ G(\xi) &= (1+r_-A\xi)(1-\xi^2-r_+A\xi^3) \qquad (2.186) \end{aligned}$$

and $q^2 = r_+r_-$. This solution again represents two oppositely charged black holes uniformly accelerating in a background magnetic field.

The function $G(\xi)$ has four roots, $\xi_1 = \frac{-1}{r_-A} \le \xi_2 \le \xi_3 < \xi_4$. The roots have the same interpretation as the roots in the $U(1)^2$ Ernst solution. The function G may now be written as

$$G(\xi) = 1(r_+A)(r_-A)(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4) \qquad (2.187)$$

The coordinate x is restricted to the range $\xi_3 \le x \le \xi_4$ in order for the metric to have Lorentzian signature. The range of y is $-\infty < y < x$.

To ensure that the metric is free of conical singularities at both poles ($x = \xi_3, \xi_4$), we must impose the condition

$$G'(\xi_3)\Lambda(\xi_4)^2 = -G'(\xi_4)\Lambda(\xi_3)^2 \qquad (2.188)$$

When this condition is satisfied, the spheres are regular as long as ϕ has the period

$$\Delta\phi = \frac{4\pi L^2}{G'(\xi_3)} \qquad (2.189)$$

with $L \equiv \Lambda(\xi_3)$.

We choose the constant k to be $\frac{2}{B\sqrt{L}}$ so that there will be regularity on the axis $x = \xi_4$. Define the physical magnetic field parameter

$$\hat{B}_E = \frac{BG'(\xi_3)}{2L^{\frac{3}{2}}} \qquad (2.190)$$

which is the value of the magnetic field on the axis at infinity. The physical charge of the black hole is defined by

$$\hat{q} = \frac{1}{4\pi}\int F = q\frac{L^{\frac{3}{2}}(\xi_4-\xi_3)}{G'(\xi_3)(1+\frac{1}{2}qB\xi_4)} \qquad (2.191)$$

When we also define $m = \frac{1}{2}(r_+ + r_-)$, we can see that the solution (2.183) depends on four parameters. These are the physical magnetic field $\hat{B}_E$, the physical magnetic charge $\hat{q}$ and A and m, which may be interpreted as measures of the acceleration and the mass of the black hole respectively.

The Ernst metric approaches the Melvin metric when we set the black hole parameters m and q (or r_+ and r_-) to zero. Also at large spatial distances (that is $x, y \to \xi_3$) the Ernst solution reduces to the Melvin solution.

We now take the Euclidean section of the Ernst solution, by setting $\tau = it$. In the non-extremal case, $\xi_1 < \xi_2$, the range of y is taken to be $\xi_2 \leq y \leq \xi_3$ to obtain a positive definite metric. To avoid conical singularities at the acceleration and black hole horizons, we take the period of τ to be

$$\beta = \Delta\tau = \frac{4\pi}{G'(\xi_3)} \tag{2.192}$$

and require

$$G'(\xi_2) = -G'(\xi_3) \tag{2.193}$$

This yields to

$$\left(\frac{\xi_2 - \xi_1}{\xi_3 - \xi_1}\right)(\xi_4 - \xi_2) = \xi_4 - \xi_3 \tag{2.194}$$

which can be simplified to

$$\xi_2 - \xi_1 = \xi_4 - \xi_3 \tag{2.195}$$

If the black holes are extreme ($\xi_1 = \xi_2$), the black hole horizon lies at infinite spatial distance from the acceleration horizon. The range of y is $\xi_2 < y \leq \xi_3$ and the period of τ is given by (2.192).

We now have to calculate the action. This can again be done by a Hamiltonian decomposition. It can be shown (in a similar way as for the $U(1)^2$ theory) that the Hamiltonian vanishes. The Euclidean action than becomes

$$I = -\frac{1}{4}(\Delta A + A_{bh}) \tag{2.196}$$

in the non extreme case, and

$$I = -\frac{1}{4}\Delta A \tag{2.197}$$

in the extreme case. The area of the black hole event horizon is

$$A_{bh} = \int_{y=\xi_4} \sqrt{g_{xx}g_{\phi\phi}}\, dx\, d\phi \tag{2.198}$$

The difference area can be calculated to be

$$\Delta A = -\frac{4\pi L^2}{G'(\xi_3)A^2}\left[\frac{1}{(\xi_3 - \xi_2)} + \frac{1}{(\xi_3 - \xi_1)}\right] \tag{2.199}$$

In the extreme case, where $\xi_2 = \xi_1$, the action then becomes:

$$\begin{aligned} I &= -\frac{1}{4}\Delta A \\ &= \frac{2\pi L^2}{G'(\xi_3)A^2(\xi_3-\xi_1)} \end{aligned} \tag{2.200}$$

In the non-extreme case the action is

$$\begin{aligned} I &= -\frac{1}{4}(\Delta A + A_{bh}) \\ &= \frac{\pi L^2}{G'(\xi_3)A^2}\left[\frac{2}{(\xi_3-\xi_1)} + \frac{\xi_2-\xi_1}{(\xi_3-\xi_2)(\xi_3-\xi_1)} - \frac{\xi_4-\xi_3}{(\xi_4-\xi_3)(\xi_3-\xi_2)}\right] \\ &= \frac{2\pi}{G'(\xi_3)A^2(\xi_3-\xi_1)} \end{aligned} \tag{2.201}$$

which is the same as in the extreme case. When we set $\Sigma = 0$ (which implies $F(\xi) = 1$) in the $U(1)^2$ theory, we see that (2.201) and (2.176) agree.

One can also use a generalization of the action (2.178) which includes a dilaton [43]

$$I = \frac{1}{16\pi}\int_M [-R + 2(\nabla\varphi)^2 + e^{-2a\varphi}F^2] \tag{2.202}$$

where φ is the dilaton and a is a parameter governing the strength of the dilaton coupling. The generalization of the Melvin solution to this case is given by

$$\begin{aligned} ds^2 &= \Lambda^{\frac{2}{1+a^2}}[-dt^2 + dz^2 + d\rho^2] \\ &\quad + \Lambda^{-\frac{2}{1+a^2}}\rho^2\, d\phi^2 \end{aligned} \tag{2.203}$$

$$e^{-2a\varphi} = \Lambda^{\frac{2a^2}{1+a^2}} \tag{2.204}$$

$$A_\phi = \frac{\hat{B}_M\rho^2}{2\Lambda} \tag{2.205}$$

$$\Lambda = 1 + \frac{1}{4}(1+a^2)\hat{B}_M^2\rho^2 \tag{2.206}$$

The generalization of the Ernst metric is given by

$$\begin{aligned} ds^2 &= \frac{\Lambda^{\frac{2}{1+a^2}}}{(x-y)^2A^2}\{F(x)[G(y)\,dt^2 - G^{-1}(y)\,dy^2] + F(y)G^{-1}(x)\,dx^2\} \\ &\quad + \frac{1}{(x-y)^2A^2\Lambda^{\frac{2a^2}{1+a^2}}}F(y)G(x)\,d\phi^2 \end{aligned} \tag{2.207}$$

$$e^{-2a\varphi} = e^{-2a\varphi_0}\Lambda^{\frac{2a^2}{1+a^2}}\frac{F(y)}{F(x)} \tag{2.208}$$

$$A_\phi = -\frac{2e^{a\varphi_0}}{(1+a^2)B\Lambda}\left[1 + \frac{1}{2}(1+a^2)Bqx\right] + k \tag{2.209}$$

The functions $\Lambda \equiv \Lambda(x,y)$, $F(\xi)$ and $G(\xi)$ are now given by

$$\begin{aligned}
\Lambda &= \left[1+\frac{1}{2}(1+a^2)Bqx\right]^2 + \frac{(1+a^2)B^2}{4A^2(x-y)^2}G(x)F(x) && (2.210)\\
F(\xi) &= (1+r_-A\xi)^{\frac{2a^2}{1+a^2}} && (2.211)\\
G(\xi) &= (1-\xi^2-r_+A\xi^3)(1+r_-A\xi)^{\frac{1-a^2}{1+a^2}} && (2.212)
\end{aligned}$$

and $q^2 = \frac{r_+r_-}{1+a^2}$.

It is useful to define the function:

$$H(\xi) \equiv G(\xi)F(\xi) = -(r_+A)(r_-A)(\xi-\xi_1)(\xi-\xi 2)(\xi-\xi_3)(\xi-\xi_4) \tag{2.213}$$

where $\xi_1 = -\frac{1}{r_-A} \leq \xi_2 \leq \xi_3 < \xi_4$ are the roots of the function $G(\xi)$. These roots have again the same interpretation as in the $U(1)^2$ Ernst solution. Now define $L = \Lambda^{\frac{1}{1+a^2}}(\xi_3)$, and set $k = \frac{2e^{a\varphi_0}}{BL^{\frac{1+a^2}{2}}(1+a^2)}$. We now find that the physical magnetic field and the charge are

$$\hat{B}_E = \frac{BG'(\xi_3)}{2L^{\frac{3+a^2}{2}}} \tag{2.214}$$

and

$$\hat{q} = q\frac{e^{a\varphi_0}L^{\frac{3-a^2}{2}}(\xi_4-\xi_3)}{G'(\xi_3)(1+\frac{1+a^2}{2}qB\xi_4)} \tag{2.215}$$

In order for the metric to have the right signature, x is restricted to the range $\xi_3 \leq x \leq \xi_4$. The condition which ensures that the conical singularities at both poles are eliminated, now becomes

$$G'(\xi_3)\Lambda(\xi_4)^{\frac{2}{1+a^2}} = -G'(\xi_4)\Lambda(\xi_3)^{\frac{2}{1+a^2}} \tag{2.216}$$

and the period of ϕ is taken to be $\Delta\phi = \frac{4\pi L^2}{G'(\xi_3)}$.

When the black hole parameters are put to zero, the dilaton Ernst metric approaches the dilaton Melvin metric. This is also the case at large spatial distances.

We now take the Euclidean section by putting $\tau = it$. In the non-extremal case, the range of y is restricted to $\xi_2 \leq \xi_3$. To eliminate the conical singularities we have to choose the period of τ to be $\Delta\tau = \frac{4\pi}{G'(\xi_3)}$ and impose the condition

$$G'(\xi_2) = -G'(\xi_3) \tag{2.217}$$

This condition yields to

$$\left(\frac{\xi_2-\xi_1}{\xi_3-\xi_1}\right)^{\frac{1-a^2}{1+a^2}}(\xi_4-\xi_2) = (\xi_4-\xi_3) \tag{2.218}$$

In the extreme case the black hole horizon is at infinite distances from the acceleration horizon. The range of y is $\xi_2 < y \leq \xi_3$ and the period of τ is again $\frac{4\pi}{G'(\xi_3)}$.

We now have to calculate the action, which can again be done by a Hamiltonian decomposition. Here we find also that the Hamiltonian vanishes. The Euclidean action is thus again given by (2.196) in the non-extreme case and by (2.197) in the extreme case.

The area of the black hole horizon is now given by

$$A_{bh} = \frac{4\pi F(\xi_2)L^2}{A^2 G'(\xi_3)} \frac{(\xi_4 - \xi_3)}{(\xi_3 - \xi_2)(\xi_4 - \xi_2)} \tag{2.219}$$

The difference in area of the acceleration horizon in the physical solution and in the background can be calculated to be:

$$\Delta A = -\frac{4\pi L^2 F(\xi_3)}{A^2 G'(\xi_3)} \left[\frac{(\xi_2 - \xi_1)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} + \frac{F'(\xi_3)}{a^2 F(\xi_3)} \right] \tag{2.220}$$

In the extreme case ($\xi_2 = \xi_1$) the action becomes

$$I = \frac{\pi L^2 F'(\xi_3)}{a^2 A^2 G'(\xi_3)} \tag{2.221}$$

and in the non-extreme case

$$\begin{aligned} I &= \frac{\pi L^2}{A^2 G'(\xi_3)} \left[\frac{F'(\xi_3)}{a^2} + \frac{F(\xi_3)(\xi_2 - \xi_1)}{(\xi_3 - \xi_2)(\xi_3 - \xi_1)} - \frac{F(\xi_2)(\xi_4 - \xi_3)}{(\xi_4 - \xi_2)(\xi_3 - \xi_2)} \right] \\ &= \frac{\pi L^2 F'(\xi_3)}{a^2 A^2 g'(\xi_3)} \end{aligned} \tag{2.222}$$

Thus the actions are the same in both cases. If we now take either $Q = 0$ or $P = 0$ in the $U(1)^2$ theory and take $a = 1$, we see that the actions (2.222) and (2.176) again agree.

Chapter 3

Creation of black hole pairs in a cosmological background

3.1 Another background

We have studied the pair creation of oppositely-charged black holes in an electro-magnetic field. The production here is possible because the negative potential energy of the created pair of black holes in the electro-magnetic background field balances their rest mass energy. Black holes can, however, also be pair created by a cosmological background. A positive cosmological constant supplies the necessary negative potential energy.

We will examine the pair creation of electrically or magnetically charged black holes in a cosmological background. We will find that some of the most interesting results of the electro-magnetic case can be reproduced in the cosmological case. In particular, the pair creation rate is still determined by the entropy of the solutions. Indeed, there is an interesting similarity between the instantons we will find here and the instantons we found in the electro-magnetic case. These similarities encourage the idea that the results of these instanton calculations represent real quantum gravity effects, and will not be qualitatively modified by the inclusion of quantum corrections. In other words, since the main features are similar in these two disparate models, they might represent features that should also be present in the full quantum theory.

In the early universe, the cosmological constant plays an important role, as it is generally assumed that the universe began with a period of exponential expansion, called inflation. In generical models of chaotic inflation, the effective cosmological constant typically starts out large and then decreases slowly until inflation ends when the effective cosmological constant vanishes. It is possible that tiny black holes are formed in pairs during this inflation area due to the cosmological constant.

3.2 Cold, lukewarm and Nariai black holes

In this section, we will discuss the cold, lukewarm, ultracold and (charged) Nariai black holes. The names stand for a certain configuration and temperature of the black hole horizons.

The action for the Einstein-Maxwell theory with a cosmological constant included, is given by

$$I = -\frac{1}{16\pi}\int d^4x\sqrt{-g}(R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}) + \frac{1}{8\pi}\int_{\Sigma} d^3\sqrt{h}K \tag{3.1}$$

with R the Ricci scalar, $g_{\mu\nu}$ the metric, Λ the cosmological constant and $F_{\mu\nu}$ the electromagnetic field tensor. The boundary is Σ, which has metric h_{ij}, and K is the trace of the extrinsic curvature K_{ij} of Σ.

The cosmological Reissner-Nordström solutions can be written as [47]

$$ds^2 = -V(r)\,dt^2 + \frac{1}{V(r)}\,dr^2 + r^2\,d\Omega^2 \tag{3.2}$$

This solution is also known as the Reissner-Nordström de Sitter solution. Take the vector potential to have the non-vanishing components

$$A_t = \frac{Q}{r} \tag{3.3}$$

$$A_\phi = -H\cos\theta \tag{3.4}$$

where Q is the electric charge and H is the magnetic charge. The corresponding field strength is given by

$$F_{01} = \frac{Q}{r^2} \tag{3.5}$$

$$F_{23} = \frac{H}{r^2} \tag{3.6}$$

Furthermore, define

$$Z^2 \equiv Q^2 + H^2 \tag{3.7}$$

Now we can write the function $V(r)$ as

$$V(r) = 1 - \frac{2M}{r} + \frac{Z^2}{r^2} - \frac{1}{3}\Lambda r^2 \tag{3.8}$$

with M the mass. There is a true singularity at the origin. For large r, the metric is asymptotic (anti-) de Sitter with a cosmological constant Λ.

When $\Lambda = 0$, the metric (3.2) reduces to the Reissner-Nordström metric (1.14). As we saw, this metric has two horizons, at radii $r_\pm = M \pm (M^2 - Z^2)^{\frac{1}{2}}$. For nonzero Λ, the horizons can be found from the quartic algebraic equation ($V(r) = 0$)

$$-\frac{1}{3}\Lambda r^4 + r^2 - 2Mr + Z^2 = 0 \tag{3.9}$$

When Λ is sufficiently small compared to the local scales M^{-2}, the configuration of the horizons will display the same qualitative behaviour as in flat space. There are two horizons at the roots $r = r_\pm$ for $Z^2 < (Z^2)_{\rm crit}$, where $(Z^2)_{\rm crit}$ is some critical value. These two horizons coalesce if $Z^2 = (Z^2)_{\rm crit}$. If $Z^2 > (Z^2)_{\rm crit}$, then the double root bifurcates to a conjugate pair of complex roots. In this case we have a naked singularity. The actual value of $(Z^2)_{\rm crit}$ can be determined by solving the cubic equation Z^2:

$$M^2 - Z^2 - \frac{1}{3}\Lambda(27M^4 - 36M^2Z^2 + 8Z^4) - \frac{16}{9}\Lambda^2 Z^6 = 0 \tag{3.10}$$

Now we will look at the temperature of the horizons. The Hawking temperature is given by (1.3)

$$T = \frac{\kappa}{2\pi} \tag{3.11}$$

To obtain this temperature at the horizon we thus have to calculate the surface gravity, κ, at the horizon. When we examine the behavior of the metric (3.2) near a root ρ of $V(r)$, the surface gravity at the horizon is given by [8]

$$\kappa = \frac{1}{2}|V'(\rho)| \tag{3.12}$$

and the Hawking temperature can be written as

$$T = \frac{1}{4\pi}|V'(\rho)| \tag{3.13}$$

where the accent means derivation with respect to r.

Now, we will discuss some different configurations of the horizons. These different configurations are referred to as the cold, lukewarm, ultracold and (charged) Nariai black holes. We will also see that these black holes have different temperatures at there horizons.

From (3.13) we see that horizons with vanishing Hawking temperature give rise to simultaneous roots ρ of V and V'. This happens when $V(r)$ has a double root. Then the function V can be written as

$$V_{\rm cold}(r) = \left(1 - \frac{\rho}{r}\right)^2 \left[1 - \frac{1}{3}\Lambda(r^2 + 2\rho r + 3\rho^2)\right] \tag{3.14}$$

The corresponding critical relationships between mass, charge, horizon radius and cosmological charge are given by

$$M = \rho\left(1 - \frac{2}{3}\Lambda\rho^2\right) \tag{3.15}$$

$$Z^2 = \rho^2\left(1 - \Lambda\rho^2\right) \tag{3.16}$$

Eliminating ρ from these critical relationships leads to the relationship (3.10) between the mass and the critical charge. This solution is known as the cold solution.

If $\Lambda \leq 0$, then all positive values of ρ, M and Z^2 are admitted. When one of them is given, the other two are fixed by (3.15) and (3.16).

If $\Lambda > 0$, then there is a maximum horizon radius $\rho = \rho_{\text{max}} = \Lambda^{-\frac{1}{2}}$ allowed. At this maximum radius the charge vanishes. The metric at this maximum radius is known as the Nariai metric and is given by

$$V_{\text{Nariai}}(r) = -\frac{r^2}{3\rho^2}\left(1 - \frac{\rho}{r}\right)^2\left(1 + \frac{2\rho}{r}\right) \tag{3.17}$$

and

$$\begin{aligned} M &= \frac{1}{3}\rho_{\text{max}} && (3.18)\\ Z^2 &= 0 && (3.19)\\ \Lambda &= \rho_{\text{max}}^{-2} && (3.20) \end{aligned}$$

For $0 < \rho < \rho_{\text{max}}$, there is an extra positive root b to $V(r)$. This root is given by

$$b = \sqrt{3\frac{1}{\Lambda} - 2\rho^2} - \rho \tag{3.21}$$

When $0 < \rho^2 < \frac{1}{2\Lambda}$, this root b is outside the cold horizon at ρ. If $\frac{1}{2\Lambda} < \rho^2 < \rho_{\text{max}}^2$, then the horizon at b is inside the cold horizon at ρ (this solution is also called the charged Nariai solution.) We can invert (3.21) and write M, Z^2 and Λ in terms of ρ and b only

$$\begin{aligned} M &= \frac{\rho(b+\rho)^2}{b^2 + 2\rho b + 3\rho^2} && (3.22)\\ Z^2 &= \frac{b\rho^2(b+2\rho)}{b^2 + 2\rho b + 3\rho^2} && (3.23)\\ \Lambda &= \frac{3}{b^2 + 2\rho b + 3\rho^2} && (3.24) \end{aligned}$$

The function (3.14) can then be written as

$$V_{\text{cold}}(r) = -\frac{r^2}{b^2 + 2\rho b + 3\rho^2}\left(1 - \frac{\rho}{r}\right)^2\left(1 - \frac{b}{r}\right)\left(1 + \frac{2\rho + b}{r}\right) \tag{3.25}$$

The temperature at the horizon at $r = b$ is

$$T_b = \frac{b}{2\pi(b^2 + 2\rho b + 3\rho^2)}\left(1 - \frac{\rho}{b}\right)^2\left(1 + \frac{\rho}{b}\right) \tag{3.26}$$

If we take $\rho = 0$, the temperature becomes

$$\begin{aligned} T_b &= \frac{1}{2\pi b}\\ &= \frac{1}{2\pi}\sqrt{\frac{1}{3}\Lambda} && (3.27) \end{aligned}$$

This is the result for the background of thermal radiation in a pure de Sitter cosmology.

When $\rho^2 = \frac{1}{2\Lambda}$, there is a special case. The horizons at b and ρ coincide and $V(r)$ has a triple root. This is the ultracold solution, which has an ultracold horizon at $r = \rho$. The Hawking temperature at this horizon vanishes. This solution can be described by

$$V_{\text{ultracold}}(r) = -\frac{r^2}{6\rho^2}\left(1-\frac{\rho}{r}\right)^3\left(1+\frac{3\rho}{r}\right) \tag{3.28}$$

and

$$\begin{aligned} M &= \frac{2}{3}\rho & (3.29)\\ Z^2 &= \frac{1}{2}\rho^2 & (3.30)\\ \Lambda &= \frac{1}{2}\rho^{-2} & (3.31)\end{aligned}$$

The ratio charge to mass is $\frac{Z}{M} = \frac{3}{2\sqrt{2}}$ and is at its largest here for any given positive value of Λ.

A de Sitter space has an isotropic background of thermal radiation. It seems rather natural that a black hole in such a space has, in its final state, an outer horizon at the same temperature as the surrounding bath. Such a black hole is known as a lukewarm one.

We take the outer black hole horizon at radius a and a de Sitter edge at radius b. The Hawking temperature is the same at a and b, which gives us the algebraic problem

$$\begin{aligned} V(a) &= V(b) = 0 & (3.32)\\ V'(a) &= \pm V'(b) & (3.33)\end{aligned}$$

From these conditions the form of the function V follows

$$\begin{aligned} V_{\text{lukewarm}}(r) &= -\frac{r^2}{(a+b)^2}\left(1-\frac{a}{r}\right)\left(1-\frac{b}{r}\right)\left(1+\frac{a+b}{r}-\frac{ab}{r^2}\right)\\ &= \left(1-\frac{ab}{(a+b)r}\right)^2-\frac{r^2}{(a+b)^2} & (3.34)\end{aligned}$$

When we compare this to the $V(r)$ of the Reissner-Nordström de Sitter metric given by (3.2), we see that the mass, charge and cosmological constant are given by

$$\begin{aligned} M &= \frac{ab}{a+b} & (3.35)\\ Z^2 &= \frac{a^2b^2}{(a+b)^2} & (3.36)\\ \Lambda &= \frac{3}{(a+b)^2} & (3.37)\end{aligned}$$

We see that $M^2 = Z^2$, which is the same relationship for the mass and charge of an extreme Reissner-Nordström black hole. The Hawking temperature at $r = a$ and $r = b$ is

$$T_a = T_b = \frac{|b-a|}{2\pi(a+b)^2} \tag{3.38}$$

The black hole has an inner horizon at

$$\rho_- = \frac{1}{2}\sqrt{a^2 + 6ab + b^2} - \frac{1}{2}(a+b) \tag{3.39}$$

When we take $a \to 0$, the expressions reduce to the results for empty de Sitter space. The expressions correspond to an extreme Reissner-Nordström black hole in flat space when $b \to \infty$. If a and b coincide, we get a cold black hole with $M = \frac{1}{2}a$, $Z^2 = \frac{1}{4}a^2$ and $\Lambda = \frac{3}{4}a^{-2}$ and an extra horizon at $\rho_- = (\sqrt{2}-1)a$.

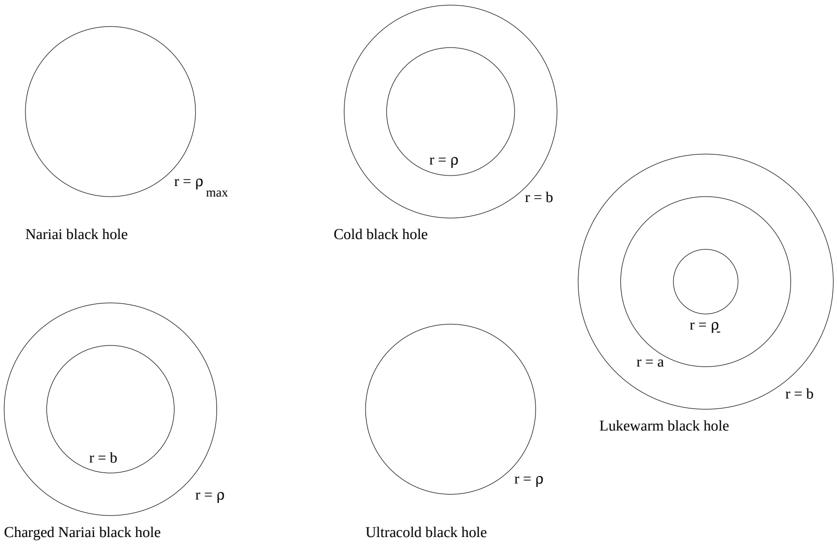

Figure 3.1: The different configurations of the horizons. In the Nariai black hole $r = \rho_{max}$ is a double root. In the cold and charged Nariai solutions $r = \rho$ is a double root. In the ultracold solution $r = \rho$ is a triple root. In the lukewarm solution $r = b$ is a de Sitter edge, $r = a$ the outer black hole horizon and $r = \rho_-$ the inner black hole horizon.

In figure 3.1 we see the different configurations of the horizons visualized.

3.3 Cosmological C metric

There is a generalization of the C metric to include a cosmological constant [49]. In the usual C metric one can not eliminate the possible conical singularities at both poles simultaneously. In the last chapter, there was an electro-magnetic field added in order to be able to eliminate both conical singularities simultaneously. If we add a cosmological constant to the C metric, one might think that this constant can provide the necessary force to accelerate the black holes away from each other. We thus might obtain a completely regular metric.

The cosmological C metric can be written as

$$ds^2 = \frac{1}{A^2(x-y)^2}\left[H(y)\,dt^2 - H^{-1}(y)\,dy^2 + G^{-1}(x)\,dx^2 + G(x)\,d\phi^2\right] \quad (3.40)$$

$$G(x) = 1 - x^2(1+r_+Ax)(1+r_-Ax) - \frac{\Lambda}{3A^2} \quad (3.41)$$

$$H(y) = 1 - y^2(1+r_+Ay)(1+r_-Ay) \quad (3.42)$$

The cosmological constant Λ will be assumed to be positive. The gauge field is given by

$$F = -q\,dx \wedge d\phi \quad (3.43)$$

in the magnetic case and

$$F = -q\,dt \wedge dy \quad (3.44)$$

in the electric case, and $q = \sqrt{r_+r_-}$. This metric describes two oppositely charged black holes, accelerating away from each other. However, the coordinates used here describe only one of the black holes. An analytical extension can be made to include both black holes. The roots of $G(x)$ will be denoted by x_1, x_2, x_3, x_4 in ascending order. The roots of $H(y)$ will be denoted by y_1, y_2, y_3, y_4, also in ascending order. The parameters Λ, r_+, r_- and A will be restricted so that all these roots are real. As with the usual C metric, $y = y_1$ is interpreted as the inner black hole horizon, $y = y_2$ as the outer black hole horizon and $y = y_3$ as the acceleration horizon.

We will restrict the coordinate x to $x_3 \leq x \leq x_4$ in order to obtain a metric of the appropriate signature. The coordinate y is restricted to $-\infty < y \leq x$. In order to have a regular solution, we must avoid having conical singularities at $x = x_3$ or $x = x_4$, so we must impose the condition,

$$G'(x_3) = -G'(x_4) \quad (3.45)$$

and choose the period of ϕ to be $\Delta\phi = \frac{4\pi}{|G'(x_3)|}$. This constraint corresponds to

$$(x_3 - x_4)(x_3 - x_2)(x_3 - x_1) = (x_3 - x_4)(x_4 - x_2)(x_4 - x_1) \quad (3.46)$$

which is satisfied when $x_3 = x_2$ or when $x_3 = x_4$. The condition assumes that x_3 is a finite distance from x_4. If one takes $x_3 = x_2$, then $x = x_3$ would lie at an infinite proper distance from any other point, so that there would be no conical singularity there. That is, the (x, ϕ) sections would no longer be compact. This is analogous to what happens with the Euclidean section of an extreme black hole [43]. However, for a positive cosmological constant, $H(y)$ will have just two real roots if $x_3 = x_2$, so this contradicts our restriction of the parameters. Therefore, the only way to satisfy the constraint is by taking $x_3 = x_4$.

This seems to imply that the (x, ϕ) section shrinks to a point, but this is just due to a poor choice of coordinate system. In the limit that $x_3 = x_4$ the proper distance between

x_3 and x_4 remains finite, as can be seen by the following coordinate transformation. The limit $x_3 \to x_4$ corresponds to the limit $A \to \sqrt{\frac{1}{3}\Lambda}$ from above. Let us write

$$1 - \frac{\Lambda}{3A^2} = \epsilon^2 \tag{3.47}$$

so that the appropriate limit is $\epsilon \to 0$. With this parameterization, $G(x)$ is written as

$$G(x) = \epsilon^2 - x^2(1 + r_+ Ax)(1 + r_- Ax) \tag{3.48}$$

so $x_3 \approx -\epsilon$ and $x_4 \approx \epsilon$. Now we take

$$x = \epsilon \cos\theta \tag{3.49}$$

$$\phi = \frac{\chi}{\epsilon} \tag{3.50}$$

In the limit $\epsilon \to 0$, the metric becomes

$$ds^2 = \frac{3}{\Lambda y^2}[H(y)\, dt^2 - H^{-1}(y)\, dy^2 + d\theta^2 + \sin^2\theta\, d\chi^2] \tag{3.51}$$

And if we make a further coordinate transformation

$$r = -\frac{\sqrt{3\Lambda^{-1}}}{y} \tag{3.52}$$

$$T = \sqrt{\frac{3}{\Lambda t}} \tag{3.53}$$

the metric can be written as

$$ds^2 = -V(r)\, dT^2 + V(r)^{-1}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\chi^2) \tag{3.54}$$

where

$$V(r) = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right) - \frac{\Lambda R^2}{3} \tag{3.55}$$

The period of χ is $\Delta\chi = 2\pi$. The range of r is from 0 to ∞, θ runs from 0 to π. The gauge field is now

$$F = q \sin\theta\, d\theta \wedge d\chi \tag{3.56}$$

in the magnetic case and

$$F = -\frac{q}{r^2}\, dT \wedge dr \tag{3.57}$$

in the electric case. Thus we see that this can be identified as a Reissner-Nordström de Sitter solution with charge $q = \sqrt{r_+ r_-}$ and 'mass' $M = \frac{1}{2}(r_+ + r_-)$. Note that if we set $\Lambda = 0$, the limit $x_3 = x_4$ is just the limit $A \to 0$ and what we have just done specializes to the familiar statement that the C metric reduces to the Reissner-Nordström metric in this limit.

3.4 Instantons in Reissner-Nordström de Sitter space

We have seen in section 3.3 that the C metric reduces to the Reissner-Nordström de Sitter metric (3.2):

$$ds^2 = -V(r)\,dt^2 + \frac{1}{V(r)}\,dr^2 + r^2\,d\Omega^2 \tag{3.58}$$

with the function $V(r)$ given by

$$V(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{1}{3}\Lambda r^2 \tag{3.59}$$

The gauge field in the electric and magnetic case are given by

$$F = -\frac{Q}{r^2}\,dt \wedge dr \tag{3.60}$$

$$F = Q \sin\theta\, d\theta \wedge d\phi \tag{3.61}$$

respectively. The independent parameters M, Q and Λ are all positive, and we do not look at dyonic states. The coordinates used here, describe one black hole. But as before, we can make an analytical extension, so there are actually two black holes.

The function $V(r)$ has four roots r_1, r_2, r_3 and r_4 in ascending order. The first root r_1 is negative, so this one has no physical significance. The root at $r = r_2$ is the inner black hole horizon. The one at $r = r_3$ is the outer black hole horizon and the one at $r = r_4$ is the cosmological (acceleration) horizon.

The Lorentzian section is $0 \leq r < \infty$. When we now replace t by $i\tau$, we get the Euclidean analytic continuation and we can calculate the instantons. In order to have a positive-definite metric we must take $r_3 \leq r \leq r_4$. There are possible conical singularities at $r = r_3$ and $r = r_4$.

When $r_3 = r_2$, the range for r is $r_3 < r \leq r_4$, because the double root in $V(r)$ implies that the proper distance from any point to $r = r_3$ along space-like directions is infinite. To get a regular instanton we must take the periodicity of τ

$$\Delta\tau = \frac{2\pi}{\kappa_4} \tag{3.62}$$

where κ_4 is the surface gravity of the horizon at $r = r_4$. This instanton is the cold instanton, were we use the terminology we introduced in section 3.2.

When $r_3 \neq r_2$, then it follows that $\kappa_3 = \kappa_4$ and the periodicity is again given by (3.62). There are two ways to satisfy this condition. One is to set $r_3 = r_4$. This solution will be referred to as the charged Nariai instanton. The other way is to set $Q = M$, which implies $\kappa_3 = \kappa_4$. This solution will be referred to as the lukewarm instanton. There is also a special case where $r_2 = r_3 = r_4$. This is the ultracold instanton. We already saw the form of the function $V(r)$ in the various cases in section 3.2 [47], but we will review them here, for the instantons.

If there is a double root ρ of $V(r)$, we can write $V(r)$ as in (3.14) and

$$M = \rho\left(1 - \frac{2}{3}\Lambda\rho^2\right) \tag{3.63}$$

$$Q^2 = \rho^2\left(1 - \Lambda\rho^2\right) \tag{3.64}$$

We take $\Lambda > 0$ and thus $0 < \rho < \Lambda^{-\frac{1}{2}}$. The other positive root, which gives an horizon at $r = b$, is again given by (3.21). If $0 < \rho^2 < \frac{1}{2}\Lambda^{-1}$, then $b > \rho$, and $r_2 = r_3 = \rho$. This is the cold instanton. If $\frac{1}{2}\Lambda^{-1} < \rho^2 < \Lambda^{-1}$, then $b < \rho$, and $r_3 = r_4 = \rho$. This instanton is the charged Nariai instanton. When $\rho^2 = \frac{1}{2}\Lambda^{-1}$, we have a horizon at $b = \rho = r_2 = r_3 = r_4$. This is the ultracold instanton. We can again rewrite the function $V(r)$ as in (3.25).

When there is no double-root in $V(r)$, then it must have two roots r_3 and r_4 such that $\kappa_3 = \frac{1}{2}|V'(r_3)| = \kappa_4 = \frac{1}{2}|V'(r_4)|$. This fixes the form of $V(r)$ as (see (3.34))

$$V_{\text{lukewarm}}(r) = \left(1 - \frac{r_3 r_4}{(r_3 + r_4) r}\right)^2 - \frac{r^2}{(r_3 + r_4)^2} \tag{3.65}$$

From this we see that $Q = M$. This is the lukewarm instanton.

We will now briefly discuss the various instantons. First let us examine the lukewarm instanton. The topology of the Euclidean section is $S^2 \times S^2$. The temperature of the two horizons is given by

$$T = \frac{1}{2\pi}\sqrt{\frac{\Lambda}{3}\left(1 - 4M\sqrt{\frac{\Lambda}{3}}\right)} \tag{3.66}$$

The Lorentzian section describes two black holes in de Sitter space. Thus the instanton describes the pair creation of non-extreme black holes in thermal equilibrium with the cosmological acceleration radiation.

For the cold instanton, the horizon at $r = r_3$ is at infinite distance. The Euclidean section has topology $S^2 \times R^2$. There is a boundary B^∞ at internal infinity $r = r_3 = \rho$. The relation between mass and charge is given parametrically by (3.63) and (3.64) and is displayed in figure 3.2. The temperature of the horizon at $r = r_4$ is given by (3.26). The Lorentzian section describes two extreme black holes in de Sitter space. The cold instanton therefore describes the pair creation of extreme black holes in thermal equilibrium with the acceleration radiation.

When $r_3 \neq r_2$ and $r_3 = r_4$, we have the charged Nariai instanton. The topology of the Euclidean section in this case is $S^2 \times S^2$. The relation between mass and charge is still given by (3.63) and (3.64) and is also displayed in figure 3.2. Although this solution is just a special case of Reissner-Nordström de Sitter, the singularity retreats to infinite proper distance when $r_3 = r_4$. Thus there is no longer a global event horizon and the Lorentzian section is just the product of a two-dimensional de Sitter space and a two-sphere of fixed radius, $dS^2 \times S^2$. Therefore this instanton does not represent the pair production of black holes.

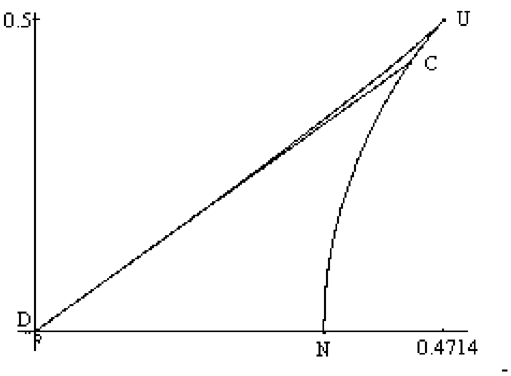

Figure 3.2: The values of Q and M for which instantons can be obtained in the cosmological case. The plot is of the dimensionless quantities $Q\sqrt{\Lambda}$ versus $M\sqrt{\Lambda}$. The curve DU represents the cold solutions, DC represents the lukewarm solutions and NU represents the charged Nariai solutions. The point at D is de Sitter space, while U is the ultracold case, and N is the Nariai solution. Taken from [46].

In the ultracold case is $r_2 = r_3 = r_4$. This can be seen as the limit $b = \rho$ in the Reissner-Nordström de Sitter metric (3.2), with $V(r)$ given in (3.14). The mass and charge are again given by (3.63) and (3.64) with $\rho = \frac{1}{\sqrt{2\Lambda}}$, so they can be written as

$$M = \frac{2}{3\sqrt{2\Lambda}} \tag{3.67}$$

and

$$Q^2 = \frac{1}{4\Lambda} \tag{3.68}$$

Now, let us take $\rho = \frac{1}{\sqrt{2\Lambda}} - \epsilon$ and $b = \frac{1}{\sqrt{2\Lambda}} + \epsilon$ and consider the limit $\epsilon \rightarrow 0$. Then we can construct two different metrics.

First we define new coordinates R and ψ by

$$r = \frac{1}{\sqrt{2\Lambda}} + \epsilon \cos\left(\sqrt{\frac{4}{3}\epsilon(2\Lambda)^{\frac{3}{2}}R}\right) \tag{3.69}$$

$$\psi = \frac{4}{3}(2\Lambda)^{\frac{3}{2}}\epsilon^2\tau \tag{3.70}$$

in the metric (3.2), with $t \rightarrow i\tau$. Then we can rewrite (3.25) as

$$V_{\mathrm{cold}}(r) = \frac{2}{3}(2\Lambda)^{\frac{3}{2}}\epsilon^3 \sin^2\left[\sqrt{\frac{4}{3}\epsilon(2\Lambda)^{\frac{3}{2}}R}\right]\left[\cos\left(\sqrt{\frac{4}{3}\epsilon(2\Lambda)^{\frac{3}{2}}R}\right) + 1\right] \tag{3.71}$$

In the limit $\epsilon \rightarrow 0$ (the ultracold limit), the metric becomes

$$ds^2 = R^2\, d\psi^2 + dR^2 + \frac{1}{2\Lambda}(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{3.72}$$

This instanton has topology $S^2 \times R^2$. The internal infinity B^∞ is now at $R = \infty$. The gauge fields are given by

$$F = \frac{1}{2\sqrt{\Lambda}} \sin\theta \, d\theta \wedge d\phi \tag{3.73}$$

in the magnetic case, and

$$F = -i\sqrt{\Lambda} R \, dR \wedge d\psi \tag{3.74}$$

in the electric case.

To obtain the other metric, define the coordinates x and γ by

$$r = \frac{1}{\sqrt{2\Lambda}} + \sqrt{\frac{2}{3}(2\Lambda)^{\frac{3}{2}}}\epsilon^{\frac{3}{2}} x \tag{3.75}$$

$$\gamma = \sqrt{\frac{2}{3}(2\Lambda)^{\frac{3}{2}}}\epsilon^{\frac{3}{2}} \tau \tag{3.76}$$

Then the function (3.25) becomes

$$V_{\mathrm{cold}}(r) = \frac{2}{3}\epsilon^3 (2\Lambda)^{\frac{3}{2}} \left(1 + \sqrt{\frac{2}{3}}\epsilon(2\Lambda)^{\frac{3}{2}} x\right)^2 \left(1 - \sqrt{\frac{2}{3}}\epsilon(2\Lambda)^{\frac{3}{2}} x\right) \tag{3.77}$$

The metric, in the limit $\epsilon \to 0$ (the ultracold limit), becomes

$$ds^2 = d\gamma^2 + dx^2 + \frac{1}{2\Lambda}(d\theta^2 + \sin^2\theta \, d\phi^2) \tag{3.78}$$

The topology of this instanton is also $S^2 \times R^2$. The internal infinity B^∞ has two components, $x = \pm\infty$. The gauge field in the magnetic case is given by

$$F = \frac{1}{2\sqrt{\Lambda}} \sin\theta \, d\theta \wedge d\phi \tag{3.79}$$

and in the electric case by

$$F = i\sqrt{\Lambda} \, d\gamma \wedge dx \tag{3.80}$$

Thus for sufficiently small charge, there are two solutions, the lukewarm and cold solutions. The lukewarm solution corresponds to the creation of pairs of non-extreme black holes. The cold solution corresponds to the pair creation of extreme black holes. When the mass is given, the charge of the cold solution is larger then the charge of the lukewarm solution. If the mass reaches $M = \frac{1}{3\sqrt{\Lambda}}$, then there is a third solution, the charged Nariai solution. This solution has charge lower than the charge of the other two solutions. When the mass reaches $M = \frac{3}{4\sqrt{3\Lambda}}$, the lukewarm and charged Nariai solutions coincide. There is no lukewarm solution which has a larger mass. When the mass reaches $M = \frac{2}{3\sqrt{2\Lambda}}$, the cold and charged Nariai solutions coincide in the ultracold solution. There is no regular solution with a larger mass. The ratio $\frac{Q}{M} = \frac{3}{2\sqrt{2}}$ is here at its largest.

There is an analogy with the Ernst solutions. The lukewarm and charged Nariai solutions can be thought of as non-extreme instanton. The cold solution can be thought of as the extreme instanton. The plots of the mass versus the charge for the Reissner-Nordström de Sitter and Ernst instantons are given in the figures 3.2 and 3.3 respectively. The features of these two plots are similar.

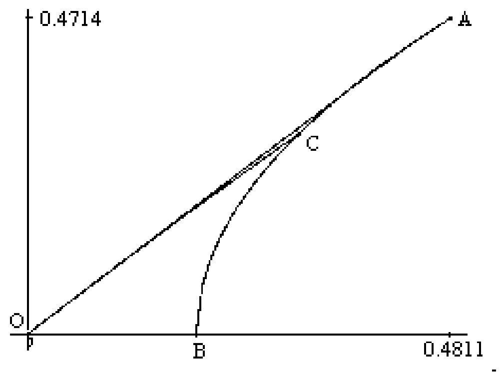

Figure 3.3: The values of Q and M for which instantons can be obtained in the electromagnetic case. The plot is of the dimensionless quantities qA versus mA. The curve OA represents the extreme solutions, OC represents the non-extreme (lukewarm) solutions and BA represent the charged Nariai instantons. The point at O is the Melvin solution. Taken from [46].

3.5 Pair creation rate

The pair creation of non-extreme charged black holes in a cosmological background is described by propagation from nothing to a surface Σ with topology $S^2 \times S^1$, since such a surface may be thought of as a Wheeler wormhole attached to an S^3, and S^3 is the topology of the spatial sections of de Sitter space. The pair production of extreme charged black holes is similarly described by propagation to a surface with topology $S^2 \times R^1$ [46, 48].

The amplitude for the creation of a pair of charged black holes is again

$$\Psi = \int \mathcal{D}[g]\mathcal{D}[A]\, e^{-I} \tag{3.81}$$

The path integral is over all metrics and gauge fields which agree with the given boundary data on Σ. Like the amplitude given by (2.119), this amplitude is dominated by a contribution from a Euclidean solution (the instanton) which has boundary Σ and satisfies the boundary conditions there. For pair creation of black holes, the instanton is in fact half of $S^2 \times S^2$. When there is an appropriate instanton, the amplitude is approximately $\Psi \approx e^{-I}$. In the case we studied in the previous chapter, the creation rate is given by $\Psi^2 = \Gamma = e^{-I_b}$, where I_b is the action of the bounce. This action is easier to calculate, but in passing from Ψ to Γ there is information lost about the boundary data on the surface on which the bounce is sliced in half. When there is a boundary at infinity, this is not very important. However, in the cosmological case this information is crucial.

The actions for the magnetic and electric charged black holes differ [48]. Also the nature of the boundary conditions, which we impose on the magnetic and electric solutions, differ. If we consider a single black hole, we can choose a particular charge sector in the magnetic case, but we have to introduce a chemical potential for the charge in the electric case.

So, we can impose the magnetic charge as a boundary condition at infinity, but we can only impose the chemical potential, and not the electric charge, as a boundary condition in the electric case. Thus the partition function in the magnetic case is a function of the temperature and charge, $Z(\beta, Q)$, while in the electric case the partition function is a function of the chemical potential w, rather then the charge, $Z(\beta, w)$.

We can calculate $Z(\beta, w)$ in the electric case directly by using (3.81) with an action which is adopted to holding the electric charge fixed. To make the action give the classical equations of motion under a variation which holds the electric charge on the boundary fixed, we need to include an additional surface term in the action. This will make the action of dual electric and magnetic solutions identical.

In the case of pair created black holes, we also have to fix the charge on the boundary. However, in the cosmological case, the $S^2 \times S^2$ instanton has no boundary, so it seems not possible to have a chemical potential in this case. But we actually consider the amplitude to propagate from nothing to a three surface Σ with topology $S^2 \times S^1$, and we can impose the potential w on the boundary Σ.

The variation of the action (3.1) is

$$\begin{aligned} \delta I &= \text{(terms giving the equations of motion)} \\ &+ \text{(gravitational boundary terms)} \\ &+ \frac{1}{4\pi} \int_\Sigma d^3x \sqrt{h} F^{\mu\nu} n_\mu \delta A_\nu \end{aligned} \tag{3.82}$$

where n_μ is the normal to Σ and h_{ij} is the induced metric on Σ. We see that the variation of (3.1) will only give the equations of motion if the variation is at fixed gauge potential on the boundary, A_i. It is not necessary to fix the component $A_\mu n^\mu$ normal to the boundary.

For the magnetic solution, fixing the gauge potential fixes the charge on each of the black holes, as the magnetic charge is just given by the integral of F_{ij} over a two-sphere lying in the boundary. In the electric case, fixing the gauge potential A_i can be seen as fixing the chemical potential w. Holding the charge fixed in the electric case is equivalent to fixing $n_\mu F^{\mu i}$ on the boundary, as the electric charge is given by the integral of the dual of F over a two-sphere lying in the boundary. Therefore, the appropriate action is

$$I_{\rm el} = I - \frac{1}{4\pi} \int_\Sigma d^3x \sqrt{h} F^{\mu\nu} n_\mu A_\nu \tag{3.83}$$

as its variation is

$$\begin{aligned} \delta I &= \text{(terms giving the equations of motion)} \\ &+ \text{(gravitational boundary terms)} \\ &- \frac{1}{4\pi} \int_\Sigma d^3x \, \delta(\sqrt{h} F^{\mu\nu} n_\mu) A_\nu \end{aligned} \tag{3.84}$$

So it gives the equations of motion when $\sqrt{h} n_\mu F^{\mu i}$, and thus the electric charge, is held fixed.

Thus, when we want to examine the pair creation, the appropriate action will be (3.1) in the magnetic case and (3.83) in the electric case. We identify Σ with a surface of zero extrinsic curvature in the Euclidean section, so the gravitational boundary term will make no contribution to the action. Thus the action (3.1) for the instanton will just be half of the whole Euclidean section.

Now, we will consider the instantons derived in section 3.4. We will calculate the actions for these instantons in the magnetic and electric case. We use the action (3.1) for the magnetic instanton and the action (3.83) for the electric instanton.

For the lukewarm solution, $F^2 = \frac{2Q^2}{r^4}$ in the magnetic case. The action (3.1) then becomes

$$\begin{aligned} I_{\rm lukewarm} &= -\frac{\Lambda V^{(4)}}{8\pi} + \frac{1}{16\pi}\int d^4x\,\sqrt{g}F^2 \\ &= -\beta\Lambda\frac{r_4^3 - r_3^3}{12} + \frac{1}{4}Q^2\beta\left(\frac{1}{r_3} - \frac{1}{r_4}\right) \\ &= -\frac{3\pi}{2\Lambda} + \pi M\sqrt{\frac{3}{\Lambda}} \end{aligned} \tag{3.85}$$

with $V^{(4)}$ the four-volume of the instanton and β the period of τ. In the last step we made use of the relations (3.34) to (3.37) (with $Z^2 = Q^2$) and $\beta = \frac{2\pi}{\kappa_4}$ (and $\kappa_4 = \frac{1}{2}V'(r_4)$.) In the electric case $F^2 = -\frac{2Q^2}{r^4}$ and the action (3.1) is

$$\begin{aligned} I^{\rm e}_{\rm lukewarm} &= -\beta\Lambda\frac{r_4^3 - r_3^3}{12} - \frac{1}{4}Q^2\beta\left(\frac{1}{r_3} - \frac{1}{r_4}\right) \\ &= -\frac{3\pi}{2\Lambda} \end{aligned} \tag{3.86}$$

Now we must calculate the additional boundary term in (3.83). To do this we have to take a gauge for the Maxwell field. A suitable choice here is

$$A = -i\frac{Q\tau}{r^2}dr \tag{3.87}$$

We take a coordinate system such that the boundary is the surface $\tau = 0, \frac{1}{2}\beta$ in the Euclidean section. We take the integral in the r direction on the boundary to run from the black hole horizon to the acceleration horizon along $\tau = 0$ and back along $\tau = \frac{1}{2}\beta$. The additional term is

$$\begin{aligned} \frac{1}{4\pi}\int_\Sigma d^3x\,\sqrt{h}F^{\mu\nu}n_\mu A_\nu &= -\frac{Q^2\beta}{2}\left(\frac{1}{r_3} - \frac{1}{r_4}\right) \\ &= -\pi M\sqrt{\frac{3}{\Lambda}} \end{aligned} \tag{3.88}$$

The action in the electric case thus is

$$\begin{aligned} I^{\rm el}_{\rm lukewarm} &= I^{\rm e}_{\rm lukewarm} - \frac{1}{4\pi}\int_\Sigma d^3x\,\sqrt{h}F^{\mu\nu}n_\mu A_\nu \\ &= -\frac{3\pi}{2\Lambda} + \pi M\sqrt{\frac{3}{\Lambda}} \end{aligned} \tag{3.89}$$

We thus see that the actions in the electric and magnetic case are the same. Because $M < \frac{1}{4}\sqrt{\frac{3}{\Lambda}}$, the action lies in the range $-\frac{3\pi}{2\Lambda} \leq I_{\text{lukewarm}} \leq -\frac{3\pi}{4\Lambda}$.

For the cold solution, we have $F^2 = frac2Q^2r^4$ in the magnetic case, and the action (3.1) is

$$\begin{aligned} I_{\text{cold}} &= \beta\Lambda\frac{b^3-\rho^3}{12} + \frac{1}{4}Q^2\beta\left(\frac{1}{\rho}-\frac{1}{b}\right) \\ &= -\frac{1}{2}\pi b^2 \end{aligned} \tag{3.90}$$

Here we used $\beta = \frac{2\pi}{\kappa_b}$ (with $\kappa_b = \frac{1}{2}V'(b)$) and the relations (3.22) to (3.25). In the electric case, $F^2 = -\frac{2Q^2}{r^4}$, and the action (3.1) becomes

$$\begin{aligned} I^{\text{e}}_{\text{cold}} &= \beta\Lambda\frac{b^3-\rho^3}{12} - \frac{1}{4}Q^2\beta\left(\frac{1}{\rho}-\frac{1}{b}\right) \\ &= -\frac{1}{2}\pi b^2 - \frac{1}{2}Q^2\beta\left(\frac{1}{\rho}-\frac{1}{b}\right) \end{aligned} \tag{3.91}$$

To calculate the additional boundary term in (3.83), we take the gauge potential

$$A = -iQ\left(\frac{1}{r}-\frac{1}{b}\right)d\tau \tag{3.92}$$

We will take the boundary B^∞ at $r = \rho + \epsilon$, and take the limit $\epsilon \to 0$. The integral over Σ consists of two parts. One part of the integral is from $r = \rho + \epsilon$ to the acceleration horizon at $r = b$ along $\tau = 0$ and back along $\tau = \frac{1}{2}\beta$. The other part is an integral over the internal infinity $r = \rho + \epsilon$, which is in the direction of decreasing τ. The additional boundary term in (3.83) then is

$$\frac{1}{4\pi}\int_\Sigma d^3x\,\sqrt{h}F^{\mu\nu}n_\mu A_\nu = -\frac{1}{2}Q^2\beta\left(\frac{1}{\rho}-\frac{1}{b}\right) \tag{3.93}$$

Thus the action in the electric case becomes

$$I^{\text{el}}_{\text{cold}} = -\frac{1}{2}\pi b^2 \tag{3.94}$$

Again, the action in the magnetic case is the same as the action in the electric case. The action lies in the range $-\frac{3\pi}{2\Lambda} \leq I_{\text{cold}} \leq -\frac{\pi}{4\Lambda}$.

For the charged Nariai solution, we first have to make a coordinate transformation [48]. Set $r_3 = \rho - \epsilon$ and $r_4 = \rho + \epsilon$, then the function $V(r)$ becomes

$$V(r) = -\frac{\Lambda}{3r^2}(r-\rho-\epsilon)(r-\rho+\epsilon)(r-r_1)(r-r_2) \tag{3.95}$$

Now we make the coordinate transformation

$$r = \rho + \epsilon\cos\chi \tag{3.96}$$
$$\psi = A\epsilon\tau \tag{3.97}$$

where

$$A = \frac{\Lambda}{3\rho^2}(\rho - r_1)(\rho - r_2) \tag{3.98}$$

The function $V(r)$ then is

$$V(r) \approx A\epsilon^2 \sin^2 \chi \tag{3.99}$$

Thus, in the limit $\epsilon \to 0$, the metric can be rewritten as

$$ds^2 = \frac{1}{A}(d\chi^2 + \sin^2 \chi \, d\psi^2) + \frac{1}{B}(d\theta^2 + \sin^2 \theta \, d\phi^2) \tag{3.100}$$

where both χ and θ run from 0 to π, ψ and ϕ both have period 2π and $B = \frac{1}{\rho^2} > A$ (this corresponds to real charge). The cosmological constant is given by $\Lambda = \frac{1}{2}(A + B)$. The gauge field in the magnetic case is

$$F = Q \sin \theta \, d\theta \wedge d\phi \tag{3.101}$$

and in the electric case

$$F = -iQ\frac{B}{A} \sin \chi \, d\chi \wedge d\psi \tag{3.102}$$

and $Q^2 = \frac{B-A}{2B^2}$.

In the magnetic case, where $F^2 = \frac{2Q^2}{B^2}$, the action (3.1) is

$$\begin{aligned} I_{\text{Nariai}} &= -\frac{\Lambda\pi}{AB} + \frac{Q^2 B\pi}{A} \\ &= -\frac{\pi}{B} \end{aligned} \tag{3.103}$$

In the electric case, we have $F^2 = -\frac{2Q^2}{B^2}$, and the action (3.1) is

$$\begin{aligned} I^{\text{e}}_{\text{Nariai}} &= -\frac{\Lambda\pi}{AB} - \frac{Q^2 B\pi}{A} \\ &= -\frac{\pi}{A} \end{aligned} \tag{3.104}$$

A suitable gauge potential is

$$A = i\frac{QB}{A}\psi \sin \chi \, d\chi \tag{3.105}$$

We take the boundary to be the surface $\psi = 0$, $\psi = \pi$, and integrate from the black hole horizon ($\chi = \pi$) to the acceleration horizon ($\chi = 0$) along $\psi = 0$ and back along $\psi = \pi$. The additional boundary term in (3.83) then becomes

$$\begin{aligned} \frac{1}{4\pi}\int_\Sigma d^3x \sqrt{h} F^{\mu\nu} n_\mu A_\nu &= -\frac{1}{4\pi}\frac{Q^2 B}{A}\int \psi \sin \chi \sin \theta \, d\chi \, d\theta \, d\phi \\ &= -2\pi Q^2 \frac{B}{A} \end{aligned} \tag{3.106}$$

Thus the action in the electric case is

$$\begin{aligned} I^{\rm el}_{\rm Nariai} &= -\frac{\pi}{A} + \frac{2\pi Q^2 B}{A} \\ &= -\frac{\pi}{b} \end{aligned} \tag{3.107}$$

Thus the action for the electric case equals the action for the magnetic case and lies in the range $-\frac{\pi}{\Lambda} \leq I_{\rm Nariai} \leq -\frac{\pi}{2\Lambda}$.

For the ultracold case, we have constructed two metrics. In both cases we have $F^2 = 2\Lambda$ in the magnetic solution and $F^2 = -2\Lambda$ in the electric solution. The volume contribution to the action (3.1) in the magnetic case vanishes.

First, we consider the metric (3.72). The action (3.1) then is

$$\begin{aligned} I^{1}_{\rm ultracold} &= -\frac{1}{8\pi}\int_{B^\infty} \sqrt{h}K \\ &= -\frac{\pi}{4\Lambda} \end{aligned} \tag{3.108}$$

For the electric case, the action (3.1) is

$$\begin{aligned} I^{\rm 1e}_{\rm ultracold} &= -\frac{\Lambda V^{(4)}}{4\pi} - \frac{1}{8\pi}\int_{B^\infty} \sqrt{h}K \\ &= -\frac{1}{4}\pi R_0^2 - \frac{\pi}{4\Lambda} \end{aligned} \tag{3.109}$$

(the boundary B^∞ in this case is the surface $R = R_0$). A suitable gauge potential is

$$A = -\frac{i}{2}\sqrt{\Lambda}R^2\, d\psi \tag{3.110}$$

Take Σ to be the surfaces $\psi = 0$, $\psi = \pi$, together with the semi-circle at $R = R_0$ lying between them. Now take the integral around the boundary to be from $R = R_0$ to $R = 0$ along $\psi = 0$ and back along $\psi = \pi$, and around $R = R_0$ in the direction of decreasing ψ. Then, the additional boundary term in (3.83) is

$$\frac{1}{4\pi}\int_\Sigma d^3x\, \sqrt{h}F^{\mu\nu}n_\mu A_\nu = -\frac{1}{4}\pi R_0^2 \tag{3.111}$$

and the action (3.83) becomes

$$I^{\rm 1el}_{\rm ultracold} = -\frac{\pi}{4\Lambda} \tag{3.112}$$

Again the actions in the electric and magnetic cases are the same. The action also agrees with the limit of the action of the cold solution as it approaches the ultracold solution.

Now, we consider the metric (3.78). The action in the magnetic case, $I^2_{\rm ultracold}$, vanishes, as the extrinsic curvature surface term at $x = \pm x_0$ vanishes as well. If we consider the action for the region between the two surfaces $\gamma = \pm\gamma_0$, we get in the electric case

$$\begin{aligned} I^{\rm 2e}_{\rm ultracold} &= -\frac{\Lambda V^{(4)}}{4\pi} \\ &= -2x_0\gamma_0 \end{aligned} \tag{3.113}$$

We can take the gauge potential to be

$$A = -ix\sqrt{\Lambda}\,d\gamma \tag{3.114}$$

and the additional boundary term becomes

$$\frac{1}{4\pi}\int_{\Sigma} d^3x\,\sqrt{h}F^{\mu\nu}n_\mu A_\nu = -2x_0\gamma_0 \tag{3.115}$$

Thus the action is $I^{2\mathrm{el}}_{\mathrm{ultracold}} = 0$ also.

Now we can calculate the approximation of the amplitude of the pair production process. When we take the square of this amplitude we get the production rate. The de Sitter space has action $I_{\text{de Sitter}} = -\frac{3\pi}{2\Lambda}$, which is the lower bound of the action for the cold and lukewarm instantons. So, the rate at which the black hole pairs are created relative to the rate at which the de Sitter space itself is created, is less than one. Thus the creation of black hole pairs is suppressed.

The instantons are all cosmological solutions, so there is no asymptotic region in the Euclidean section. This can be interpreted as meaning that these solutions are closed systems and thus necessarily have fixed energy. The solutions should therefore be interpreted as a contribution to a microcanonical ensemble, because this is the thermodynamical ensemble at fixed energy. Then the partition function $Z^2 = \Psi^2$ is the density of states and the entropy will be given by $S = \ln Z$. The contribution to the entropy from the instantons is then just

$$S = -2I \tag{3.116}$$

The entropy turns out to be just a quarter of the total area of the horizons which appear in the instanton (which we denote by A.)

In the lukewarm instanton, there are two horizons, one at $r = r_3$ and one at $r = r_4$, so

$$\frac{1}{4}A = \pi r_3^2 + \pi r_4^2 \tag{3.117}$$

Because

$$r_{3,4} = \frac{1}{2}\left(\sqrt{\frac{3}{\Lambda}} \pm \sqrt{\frac{3}{\Lambda} - 4M\sqrt{\frac{3}{\Lambda}}}\right) \tag{3.118}$$

the area becomes

$$\frac{1}{4}A = \frac{3\pi}{\Lambda} - 2\pi M\sqrt{\frac{3}{\Lambda}} \tag{3.119}$$

The entropy then becomes

$$\begin{aligned} S_{\text{lukewarm}} &= -2I_{\text{lukewarm}} \\ &= \frac{1}{4}A \end{aligned} \tag{3.120}$$

In the cold case, there is only the acceleration horizon at $r = b$. Its area is given by

$$A = 4\pi b^2 \tag{3.121}$$

so the entropy now is

$$
\begin{aligned}
S_{\text{cold}} &= -2I_{\text{cold}} \\
&= \pi b^2 \\
&= \frac{1}{4}A
\end{aligned}
\tag{3.122}
$$

In the charged Nariai case, there are two horizons. Both horizons have area $A = \frac{4\pi}{B}$. The entropy is thus given by

$$
\begin{aligned}
S_{\text{Nariai}} &= -2I_{\text{Nariai}} \\
&= \frac{2\pi}{B} \\
&= \frac{1}{4}A
\end{aligned}
\tag{3.123}
$$

In the case of the first type of the ultracold solution, the surface R is interpreted as a Rindler horizon. The area of this horizon is

$$
A = \frac{2\pi}{\Lambda} \tag{3.124}
$$

and thus the entropy becomes

$$
\begin{aligned}
S^1_{\text{ultracold}} &= -2I^1_{\text{ultracold}} \\
&= \frac{\pi}{2\Lambda} \\
&= \frac{1}{4}A
\end{aligned}
\tag{3.125}
$$

For the other type there are no horizons and the entropy vanishes

$$
S^2_{\text{ultracold}} = -2I^2_{\text{ultracold}} = 0 \tag{3.126}
$$

Thus the usual relation between the entropy and the area of the horizons extends to all these solutions. Just as for the Ernst instantons, the pair creation of extreme black holes is suppressed relative to the pair creation of non-extreme black holes by a factor $e^{S_{bh}}$, where S_{bh} is the entropy associated with the black hole horizon.

Conclusion

We have studied the pair creation process of small charged black holes in order to give a minor attribution to the discussions about the information paradox and quantum gravity. First we examined the creation in an electro-magnetic background. We used the action of a truncation of low energy string theory, which contains two $U(1)$ gauge fields and a dilaton field. As a solution to this action we found the $U(1)^2$ Ernst metric, which is asymptotic to the $U(1)^2$ Melvin metric. We saw that this solution describes two oppositely charged black holes, accelerating away from each other. The nodal singularities can be removed now, due to the electro-magnetic background.

In order to find the instanton describing the pair production, we made an analytical continuation to Euclidean space. We found that the pair creation of extreme black holes seems to be suppressed relative to the pair creation of non-extreme black holes. But we calculated the pair creation rate for both the extreme and non-extreme black holes and we found that the action is the same and finite in both cases. In the point-particle limit, the pair production rate reduces to the result of the Schwinger process (generalized to two gauge fields.)

We have also compared the result of the $U(1)^2$ theory with the Einstein-Maxwell theory and a generalization of the latter with a dilaton field. When the appropriate parameters are put to zero, the result in all three theories agree.

Secondly, we studied the pair creation of charged black holes in a cosmological background. The action we used was the one for the Einstein-Maxwell theory with a cosmological constant included. The solution to this action was the Reissner-Nordström de Sitter metric. There were different classes of black holes, namely the cold, ultracold, (charged) Nariai and lukewarm ones. For these different black holes, we calculated the instantons and found similarities with the instantons in the electro-magnetic instantons. Here we also saw that the pair creation of extreme black holes is suppressed relative to the pair creation of non-extreme black holes. We calculated the pair creation rate and found that it is finite in the cosmological background also.

It might be interesting to include a dilaton in the cosmological case as well. Also, the effects of quantum corrections on the pair production rate can be studied to get a better picture of what is happening during the pair creation.

Appendix A

Penrose diagram

The Minkowski metric (1.8) in spherical coordinates is given by

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{A.1}$$

Now we define light-cone coordinates according to

$$\begin{aligned} u &= t - r \\ v &= t + r \end{aligned} \tag{A.2}$$

The metric then takes the form

$$ds^2 = -dvdu + \frac{1}{4}(v-u)^2 d\Omega^2 \tag{A.3}$$

In figure A.1 are the relation between (r,t) and (u,v) and various asymptotic regions indicated. These asymptotic regions are

$$\begin{aligned}
i^+ &= t \to +\infty \text{ (at fixed } r) &&= \text{future time-like infinity} \\
i^- &= t \to -\infty \text{ (at fixed } r) &&= \text{past time-like infinity} \\
i^0 &= r \to \infty \text{ (at fixed } t) &&= \text{space-like infinity} \\
I^+ &= v \to \infty \text{ (at fixed } u) &&= \text{future null infinity} \\
I^- &= u \to -\infty \text{ (at fixed } v) &&= \text{past null infinity}
\end{aligned} \tag{A.4}$$

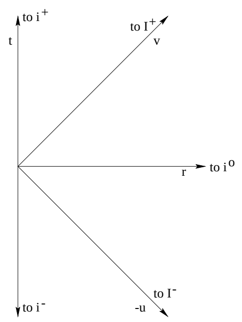

Figure A.1: Relation between (r,t) and (u,v) coordinates.

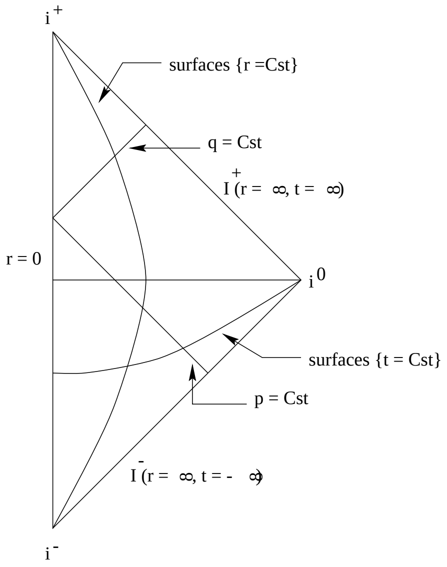

Figure A.2: A Penrose diagram.

But it is a bit awkward to study I^+ and I^- in (u, v) coordinates because they are at infinite values of u and v. We therefore introduce the coordinates (p, q)

$$\begin{aligned} v = t + r = \tan p \\ u = t - r = \tan q \end{aligned} \tag{A.5}$$

The metric now takes the form

$$ds^2 = \sec^2 p \sec^2 q(-dpdq + \frac{1}{4}\sin^2(p-q)d\Omega^2) \tag{A.6}$$

which is conformal to the metric

$$ds'^2 = -4dpdq + \sin^2(p-q)d\Omega^2 \tag{A.7}$$

We now define the coordinates

$$\begin{aligned} t' = p + q \\ r' = p - q \end{aligned} \tag{A.8}$$

and the metric becomes

$$ds'^2 = -(dt')^2 + (dr')^2 + \sin^2 r' d\Omega' \tag{A.9}$$

We can now represent the conformal structure of infinity by drawing a diagram of the (t', r') plane, see figure A.2. Such a diagram shall be called a Penrose diagram.

Each point in this diagram represents a twosphere. Radial null geodesics are represented by straight lines at $\pm 45^o$.

See also [6].

Appendix B

Instantons

In this appendix I will roughly follow [50]. A nice book on this subject is for instance [51].

An instanton[1] is a localized solution of Euclidean field equations with finite Euclidean action. The Euclidean version of a theory, or a set of field equations, involves replacing the Minkowskian metric $\eta^{\mu\nu}$ by the Euclidean metric $\delta^{\mu\nu}$ (where the indices now run from 1 to 4.) The components of the space-time coordinate vector x^μ are still real. However the Euclidean theory is left invariant by $SO(4)$ transformations rather then Lorentz transformations. The Euclidean system can be thought of as an analytical continuation of the Minkowskian system. This is done by taking the Euclidean time $x^4 = \tau = it$ (where t is the Minkowskian time.)

Certain features of Minkowskian quantum field theories can be conveniently studied by beginning with classical solutions of the Euclidean version of its field equations. One of these features is quantum tunneling between degenerate classical ground states. This tunneling process can be described with the instanton solution.

Take for instance a particle of unit mass moving in a potential in one dimension

$$H = \frac{1}{2}p^2 + V(x) \tag{B.1}$$

The Euclidean expression for the transition function can be written as a path integral

$$\langle x_f | e^{-\frac{1}{\hbar}HT} | x_i \rangle = N \int \mathcal{D}x \, e^{-\frac{1}{\hbar}S} \tag{B.2}$$

where H is the Hamiltonian, T is a time interval, N is a normalization factor, S is the Euclidean action and $\langle x_i|$ and $|x_f\rangle$ are the initial and final states respectively. We expand the Hamiltonian in a complete set of energy eigenstates

$$H|n\rangle = E_n|n\rangle \tag{B.3}$$

Because the set of energy eigenstates is complete, we can write

$$\sum_n |n\rangle\langle n| = 1 \tag{B.4}$$

[1]The name 'instanton' was invented by 't Hooft [52] because these solutions are localized in x_4 (the imaginary time coordinate) as well.

We then get for the lefthandside of (B.2)

$$\langle x_f|e^{-\frac{1}{\hbar}HT}|x_i\rangle = \sum_n e^{\frac{1}{\hbar}E_nT}\langle x_f|n\rangle\langle n|x_i\rangle \tag{B.5}$$

On the righthandside, the Euclidean action is

$$S = \int_{-\frac{1}{2}T}^{\frac{1}{2}T} dt \left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x)\right] \tag{B.6}$$

and $\mathcal{D}x$ denotes integration over all functions $x(t)$ which obey the boundary conditions $x(-\frac{1}{2}T) = x_i$ and $x(\frac{1}{2}T) = x_f$. If $\bar{x}$ is any function obeying the boundary conditions, then a general function obeying the boundary conditions can be written as

$$x(t) = \bar{x}(t) + \sum_n c_n x_n(t) \tag{B.7}$$

where the x_n's are a complete set of real orthonormal functions vanishing at the boundaries.

The righthandside of equation (B.2) is of interest because it can readily be evaluated in the semiclassical (small $\hbar$) limit. In this case the functional integral is dominated by the stationary points of S. For simplicity, let us assume that there is only one such stationary point, which we denote by $\bar{x}$. In this point the variational derivative of the action vanishes:

$$\frac{\delta S}{\delta \bar{x}} = -\frac{d^2\bar{x}}{dt^2} + V'(\bar{x}) = 0 \tag{B.8}$$

Further, let us choose the x_n's to be eigenfunctions of the second variational derivative of S at $\bar{x}$. The second variational derivative of S is

$$\frac{\delta^2 S}{\delta \bar{x}^2} = -\frac{d^2}{dt^2} + V''(\bar{x}) \tag{B.9}$$

The eigenfunctions are thus given by the equation:

$$-\frac{d^2 x_n}{dt^2} + V''(\bar{x})x_n = \lambda_n x_n \tag{B.10}$$

where λ_n is the eigenvalue for x_n. Then in the limit of small $\hbar$, the action can be expanded around $\bar{x}$ and the transition function becomes

$$\langle x_f|e^{-\frac{1}{\hbar}HT}|x_i\rangle = \exp\frac{1}{\hbar}\{S(\bar{x}) + \frac{1}{2}\int dt_1 dt_2 \frac{\delta^2 S(x)}{\delta x^2}|_{x=\bar{x}}x(t_1)x(t_2) + \ldots\} \tag{B.11}$$

The term proportional to $\frac{\delta S(x)}{\delta x}$ is suppressed, because $\bar{x}$ is a stationary point, so $\delta S(\bar{x}) = 0$. When we now replace the continuous time variable by a finite number of discrete points, we can write the term $\int dt_1 dt_2 \frac{\delta^2 S}{\delta x^2}x(t_1)x(t_2)$ as (x, Ax). Here A is a matrix proportional to $\frac{\delta^2}{\delta x^2}$. We now can write the transition function as follows

$$\begin{aligned}\langle x_f|e^{-\frac{1}{\hbar}HT}|x_i\rangle &= Ne^{-\frac{1}{\hbar}S(\bar{x})}\int dx_1\ldots dx_n e^{-\frac{1}{2\hbar}(x,Ax)+\ldots} \\ &= Ne^{-\frac{1}{\hbar}S(\bar{x})}\prod_n \lambda_n^{-\frac{1}{2}}[1+\mathcal{O}(\hbar)] \\ &= Ne^{-\frac{1}{\hbar}S(\bar{x})}[\det(-\partial_t^2 + V''(\bar{x}))]^{-\frac{1}{2}}[1+\mathcal{O}(\hbar)]\end{aligned} \tag{B.12}$$

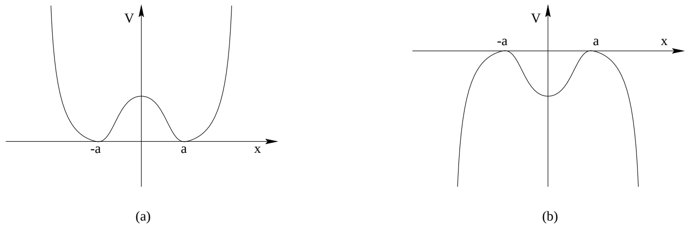

Figure B.1: Double well potential.

(here we have assumed that the eigenvalues are positive.) If there are several stationary points, in general one has to sum over all of them.

Equation (B.8) is the equation of motion for a particle of unit mass moving in a potential minus V. Thus

$$E = \frac{1}{2}\left(\frac{d\bar{x}}{dt}\right)^2 - V(\bar{x}) \tag{B.13}$$

is a constant of motion. This can be used to determine the qualitative features of the solutions of equation (B.8) by inspection.

Let us now look at a particle which is moving in a double well potential (see figure B.1.) We take this potential even, i.e. $V(x) = V(-x)$, and call its extrema $\pm a$. We now want to compute the transition functions

$$\langle a|e^{-\frac{1}{\hbar}HT}|a\rangle = \langle -a|e^{-\frac{1}{\hbar}HT}|-a\rangle \tag{B.14}$$

and

$$\langle a|e^{-\frac{1}{\hbar}HT}|-a\rangle = \langle -a|e^{-\frac{1}{\hbar}HT}|a\rangle \tag{B.15}$$

The matrix elements are equal because of the reflection symmetry $x \leftrightarrow -x$. We can do this by approximating the functional integral by its semiclassical limit (B.12). Just as before, the first step is to find solutions of the classical Euclidean equation of motion, (B.8), consistent with the boundary conditions.

Of course, two such solutions are those in which the particle stays fixed on top of one

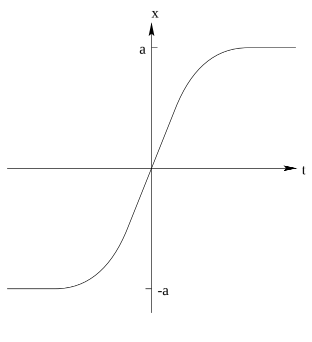

Figure B.2: The instanton.

of the two hills in figure B.1b. However, there is another interesting solution, where the particle begins at the top of one hill at time $-\frac{1}{2}T$ and moves to the top of the other hill and arrives there at time $\frac{1}{2}T$.

When we take T to infinity, we are dealing with a solution of the equation of motion with vanishing E. Whence

$$\frac{dx}{dt} = \sqrt{2V(x)} \tag{B.16}$$

Equivalently

$$t = t_1 + \int_0^x dx'[2V(x')]^{-\frac{1}{2}} \tag{B.17}$$

where t_1 is an integration constant, the time at which x vanishes. This solution is sketched in figure B.2 and is called an instanton at time t_1. The anti-instanton is simply the instanton with the particle moving from a to $-a$ and replacing t by $-t$.

Appendix C

Derivation of the Hamiltonian

In this appendix we will derive the Hamiltonian for a gravitational action. We will follow the discussions in [7, 44].

C.1 No inner boundaries

We take a Lorentzian action for a metric g and generic matter fields φ

$$I(g,\varphi) = \int_M \left[\frac{R}{16\pi} + L_m(g,\varphi)\right] + \frac{1}{8\pi}\oint_{\partial M} K \tag{C.1}$$

where R is the scalar curvature of g, L_m is the matter Lagrangian and K is the trace of the extrinsic curvature[1] of the boundary. Assume that L_m includes at most first order derivatives. This action is well defined for spatially compact geometries, but diverges for non-compact ones. To define the action for non-compact geometries, one must choose a reference background g_0, φ_0. This background is required to be a static solution to the field equations. The physical action then is the difference

$$I_p(g,\varphi) \equiv I(g,\varphi) - I(g_0,\varphi_0) \tag{C.2}$$

The action I_p is finite for a class of fields g, φ which asymptotically approach g_0, φ_0 in the following sense. Fix a boundary near infinity Σ^∞, and require that g, φ induce the same fields on this boundary as g_0, φ_0.

For asymptotically flat space-times, the appropriate background is flat space with zero matter fields. The physical action (C.2) then reduces to

$$I_p(g,\varphi) = \int_M \left[\frac{R}{16\pi} + L_m(g,\varphi)\right] + \frac{1}{8\pi}\oint_{\partial M} (K - K_0) \tag{C.3}$$

where K_0 is the trace of the extrinsic curvature of the boundary embedded in flat space-time.

[1] Let ξ^a be the unit tangent field of congruence of time-like geodesics orthogonal to Σ. Then the extrinsic curvature K_{ab} of Σ is defined as $K_{ab} = \nabla_a \xi_b$.

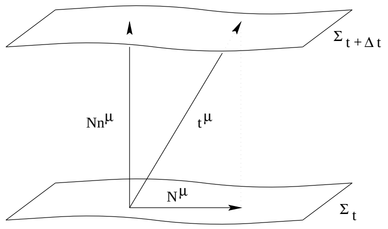

Figure C.1: A space-time diagram illustrating the definition of the lapse function N and the shift vector N^μ.

The physical action is given by (C.2), so the physical Hamiltonian is the difference between the Hamiltonian computed from (C.1) and the one computed for the background. First we will write the action (C.1) in a Hamiltonian form.

To begin we choose a time function t and a time-like vector field ('time flow') t^μ on M satisfying $t^\mu \nabla t = 1$. Then we introduce a family of space-like surfaces Σ_t labeled by t. Given a metric $g_{\mu\nu}$, it is convenient to decompose t^μ into its normal and tangential parts with respect to the surfaces Σ_t. To do so, we introduce the lapse function and the shift vector. The lapse function, N, is defined by

$$\begin{aligned} N &= -g_{\mu\nu} t^\mu n^\nu \\ &= (n^\mu \nabla_\mu t)^{-1} \end{aligned} \tag{C.4}$$

and the shift vector, N^μ, is defined by

$$N^\mu = h^\mu{}_\nu t^\nu \tag{C.5}$$

with n^μ the unit normal to Σ_t and $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ the induced spatial metric on Σ_t. Now we can write the time-like vector field as

$$t^\mu = N n^\mu + N^\mu \tag{C.6}$$

(see figure C.1). For now we assume that there are no inner boundaries, so the surfaces Σ_t do not intersect and are complete. It is convenient to choose the surfaces Σ_t so that they meet the boundary near infinity Σ^∞ orthogonally. This is not essential, but it simplifies the analysis. The boundary ∂M thus consists of an initial and final surface with unit normal n^μ and a surface near infinity Σ^∞ on which n^μ is tangent.

The curvature tensor gives us the following relation [7]

$$\begin{aligned} R_{\mu\nu\rho\sigma} h^{\mu\rho} h^{\nu\sigma} &= R_{\mu\nu\rho\sigma}(g^{\mu\rho} + n^\mu n^\rho)(g^{\nu\sigma} + n^\nu n^\sigma) \\ &= R + 2R_{\mu\rho} n^\nu n^\rho \\ &= 2G_{\mu\rho} n^\mu n^\rho \end{aligned} \tag{C.7}$$

where the last step follows from the Einstein equation $G_{\mu\rho} = R_{\mu\rho} - \frac{1}{2}R$. The scalar curvature, R, then can be expressed as

$$R = 2(G_{\mu\nu} n^\mu n^\nu - R_{\mu\nu} n^\mu n^\nu) \tag{C.8}$$

where $G_{\mu\nu}$ is the Einstein tensor and $R_{\mu\nu}$ is the Ricci tensor. The first term can be written as

$$2G_{\mu\nu}n^\mu n^\nu = {}^3R - K_{\mu\nu}K^{\mu\nu} + K^2 \tag{C.9}$$

The second term can be evaluated by commuting covariant derivatives on n^μ with the result

$$\begin{aligned} R_{\mu\nu}n^\mu n^\nu &= R^\rho_{\mu\rho\nu}n^\mu n^\nu \\ &= -n^\mu(\nabla_\mu\nabla_\rho - \nabla_\rho\nabla_\mu)n^\rho \\ &= (\nabla_\mu n^\mu)(\nabla_\rho n^\rho) - (\nabla_\rho n^\mu)(\nabla_\mu n^\rho) - \nabla(n^\mu\nabla_\rho n^\rho) + \nabla_\rho(n^\mu\nabla_\mu n^\rho) \\ &= K^2 - K_{\mu\nu}K^{\mu\nu} - \nabla_\mu(n^\mu\nabla_\nu n^\nu) + \nabla_\nu(n^\mu\nabla_\mu n^\nu) \end{aligned} \tag{C.10}$$

Substitute these expressions in (C.8) and the scalar curvature becomes

$$R = {}^3R + K_{\mu\nu}K^{\mu\nu} - K^2 + 2\nabla_\mu(n^\mu\nabla_\nu n^\nu) - 2\nabla_\nu(n^\mu\nabla_\mu n^\nu) \tag{C.11}$$

When we substitute this in the action (C.1), the two total derivative terms in (C.11) give rise to boundary conditions. The first term is proportional to n^μ and thus only contributes to the initial and final boundary and completely cancels the $\oint K$ term on these surfaces (this can be seen by applying Gauss's theorem.) Thus the action then becomes

$$I = \int_M \{\frac{1}{16\pi}({}^3R + K_{\mu\nu}K^{\mu\nu} - K^2) + L_m - \frac{1}{8\pi}\nabla_\nu(n^\mu\nabla_\mu n^\nu)\} \tag{C.12}$$

The second term is orthogonal to n^μ and therefore only contributes to the surface integral near infinity. If r^μ is the unit normal to Σ^∞, then the integral over this surface becomes

$$\frac{1}{8\pi}\int_{\Sigma^\infty} \nabla_\mu r^\mu + r_\nu n^\mu\nabla_\mu n^\nu = \frac{1}{8\pi}\int_{\Sigma^\infty}(g^{\mu\nu} - n^\mu n^\nu)\nabla_\mu r_\nu \tag{C.13}$$

The surface Σ^∞ is foliated by a family of two surfaces S_t^∞ coming from its intersection with Σ_t. The integrand in (C.13) is the trace of the two dimensional extrinsic curvature ${}^2K$ of S_t^∞ in Σ_t. Thus the action (C.1) takes the form

$$I = \int N\,dt \left[\frac{1}{16\pi}\int_{\Sigma^\infty}\sqrt{{}^3g}({}^3R + K_{\mu\nu}K^{\mu\nu} - K^2 + 16\pi L_m) + \frac{1}{8\pi}\int_{S_t^\infty}{}^2K\right] \tag{C.14}$$

with ${}^3g$ the induced metric on Σ_t.

Now introduce the canonical momenta $p^{\mu\nu}$ and p conjugated to respectively ${}^3g_{\mu\nu}$ and φ. Let us consider the case when the matter does not contain gauge fields. The extrinsic curvature $K_{\mu\nu}$ is related to the time derivative of the three metric ${}^3\dot{g}_{\mu\nu}$ by

$$K_{\mu\nu} = \frac{1}{2N}\left[{}^3\dot{g}_{\mu\nu} - 2D_{(\mu}N_{\nu)}\right] \tag{C.15}$$

where D_μ is the covariant derivative[2] associated with ${}^3g_{\mu\nu}$. When we write the action in a form that does not contain derivatives of the shift vector, we now obtain another surface term $-2\int_{S_t^\infty} N^\mu p_{\mu\nu} r^\nu$. Then the action takes the form

$$I = \int dt \left[\int_{\Sigma_t} (p^{\mu\nu}\, {}^3\dot{g}_{\mu\nu} + p\dot{\varphi} - N\mathcal{H} - N^\mu \mathcal{H}_\mu) + \frac{1}{8\pi}\int_{S_t^\infty} (N\, {}^2K - N^\mu p_{\mu\nu} r^\nu) \right] \tag{C.16}$$

with $\mathcal{H}$ the Hamiltonian constraint and $\mathcal{H}_\mu$ the momentum constraint. These constraints contain contributions from the matter as well as the gravitational field. From (C.16) we see that the Hamiltonian is given by

$$H = \int_{\Sigma_t} (N\mathcal{H} + N^\mu \mathcal{H}_\mu) - \frac{1}{8\pi}\int_{S_t^\infty} (N\, {}^2K - N^\mu p_{\mu\nu} r^\nu) \tag{C.17}$$

Now we must derive the Hamiltonian for the reference background in order to calculate the physical Hamiltonian. When we repeat the above analysis, we find that the momenta $p_0^{\mu\nu}$ and p_0 and the constraints vanish, because this background is a stationary solution to the field equations. If we label the static slices so that $N_0 + N$ on Σ^∞, the reference Hamiltonian is simply

$$H_0 = -\frac{1}{8\pi}\int_{S_t^\infty} N\, {}^2K_0 \tag{C.18}$$

Then the physical Hamiltonian is

$$\begin{aligned} H_p &\equiv H - H_0 \\ &= \int_{\Sigma_t} (N\mathcal{H} + N^\mu \mathcal{H}_\mu) - \frac{1}{8\pi}\int_{S_t^\infty} [N({}^2K - {}^2K_0) - N^\mu p_{\mu\nu} r^\nu] \end{aligned} \tag{C.19}$$

C.2 Inner boundaries and the Euclidean action

In the last section we considered the case where the only boundary of the surfaces Σ_t was at infinity. But often the surfaces have an inner boundary as well. We shall consider the situation where the surfaces Σ_t have an internal infinity. This case arises both for an extreme horizon, where the intersection between the past and future horizons has receded to an internal infinity, or for space-times having more than one asymptotic region (such as the maximally extended Schwarzschild solution.) We need to introduce another asymptotic boundary surface $\Sigma^{-\infty}$. There will be a surface term $\frac{1}{8\pi}\oint N\, {}^2K$ on this boundary.

Now we consider the Euclidean action

$$\tilde{I} = -\frac{1}{16\pi}\int_M (R + 16\pi L_m) - \frac{1}{8\pi}\oint_{\partial M} K \tag{C.20}$$

In a static or stationary solution the time derivatives ${}^3\dot{g}_{\mu\nu}$ and $\dot{\varphi}$ are zero. Thus the action for a region between surfaces Σ_t an imaginary time distance β apart is

$$\tilde{I} = \beta H \tag{C.21}$$

[2]The covariant derivative associated with $g_{\mu\nu}$ means that $\nabla_\mu g_{\nu\rho} = 0$.

If the stationary time surfaces Σ_t do not intersect, then the imaginary time coordinate can be periodically identified with any period β. However, when the stationary time surfaces intersect at a horizon S_h, the periodicity β is fixed by regularity of the Euclidean solution at S_h. The action of the region swept out by the surfaces Σ_t between their inner and outer boundaries is again $\tilde{I} = \beta H$. But this is not the action of the full four dimensional solution, it is the action of the solution without the two surface S_h. The contribution to the action from a little tubular neighbourhood surrounding the two surface S_h is just $-\frac{1}{4}A$, with A the area of S_h. Thus we obtain for the action

$$\tilde{I} = \beta H - \frac{1}{4}A \tag{C.22}$$

Until now we have not yet taken into account the reference background. Consider a background that does not contain a two surface S_h on which the stationary time surfaces intersect. The background must be identified with the same period in imaginary time at infinity as the solution under consideration in order for the induced metrics on Σ^∞ to agree. One thus obtains $\tilde{I}_0 = \beta H_0$ for the background which leads to the familiar result

$$\tilde{I}_p = \beta H_p - \frac{1}{4}A_{bh} \tag{C.23}$$

for non-extreme black holes and

$$\tilde{I}_p = \beta H_p \tag{C.24}$$

for extreme black holes.

We now consider the Euclidean version of solutions with acceleration horizons. The action (C.22) holds in this case also. Again the periodicity in imaginary time in the background (β_0) must agree with that in the original space-time (β), because the periodicity is determined by regularity of the Euclidean space-time on the axis (which now extends out at infinity.) Repeating the argument above (C.22) we find that the background satisfies

$$\tilde{I}_0 = \beta H_0 - \frac{1}{4}A_0 \tag{C.25}$$

The physical Euclidean action is related to the physical Hamiltonian by

$$\tilde{I}_p = \beta H_p - \frac{1}{4}(\Delta A + A_{bh}) \tag{C.26}$$

in the non-extreme case and

$$\tilde{I}_p = \beta H_p - \frac{1}{4}\Delta A \tag{C.27}$$

in the extreme case, with ΔA the difference between the area of S_h in the original space-time and its area in the reference background.

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