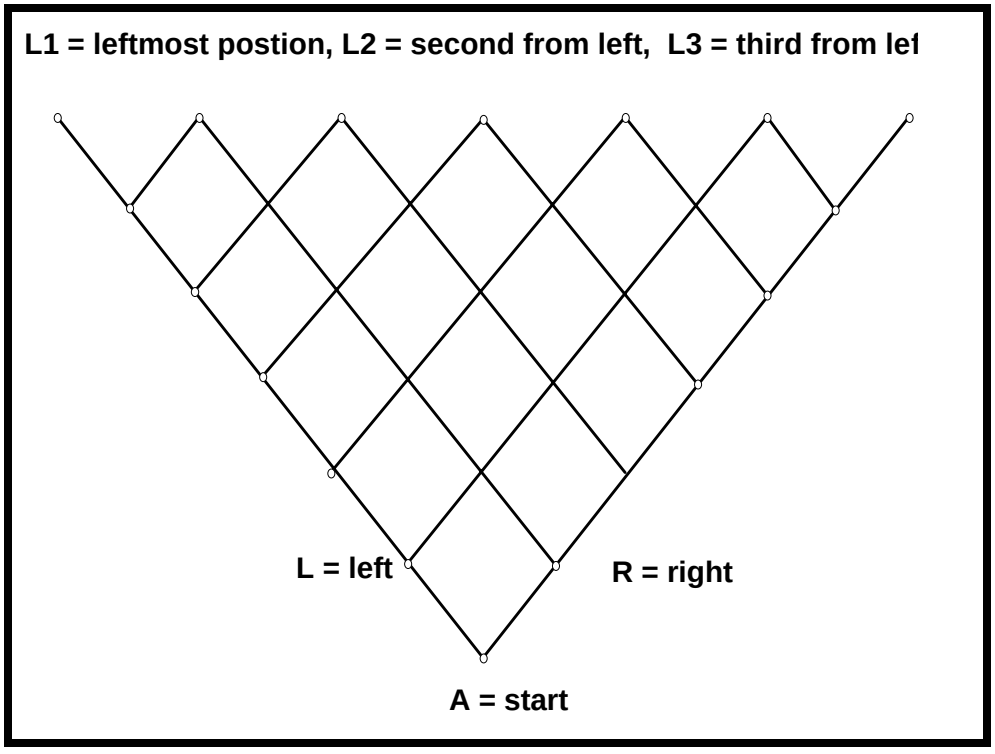
November-December Problems (Round 3)
Grades 6-12
Deadline (postmark): Jan 10, 2000

1. **The Seven Friend's Problem** Seven friends are sitting around a table. Each has a cup containing some milk; some cups are empty, some cups hold the same amount of milk, and some hold a different amount of milk. All together the amount of milk is one quart. One friend now stands up and divides all of his milk equally among the other six cups. Then the second friend gets up and divides his milk equally among the other six, and so on, each friend dividing his milk among the other six. Suppose that after this, the friends have the same amount of milk that they had at the beginning. How much milk was in each friend's cup at the beginning of this process ?

2. **Network of Roads** On the first week 2^{150} people leave the bottom row in the figure below at point A, half going to the next point in the left direction (L), and half going to the next point in the right direction (R). If the people continue this process for the next 150 weeks, half going to the left and half to the right, how many people will be at the leftmost (L_1) point, second point from the left (L_2), and third point from the left (L_3) in the 150th row ?



3. **So You Know Long Division?** In the long division below each letter represents a number. Different letters represent different numbers. Find the number that make the division correct.

$$f\;g\;a$$
$$e\,f\,\sqrt{\;a\;b\;c\;d}$$
$$h\,b$$

$$d\,i\;c$$
$$b\,e$$

$$d\,h\;d$$
$$d\,h\;d$$
----

4. **Subdividing a Cube**

$a)$ Show that a cube can be divided up into 8 smaller cubes. $\leftarrow$ 1 point
$b)$ Show that a cube can be divided up into 20 smaller cubes. $\leftarrow$ 2 points
c) Show that a cube can be divided up into 38 smaller cubes. $\leftarrow$ 3 points
$d)$ Show that a cube can be divided up into 49 smaller cubes. $\leftarrow$ 4 points

5. **Frank and Francine** Frank and Francine agree to meet at the library and wait 10 minutes for one another from the time they arrive. The both arrive randomly, independent of the other between 1:00 p.m. and 1:10 p.m. What is the probablitity they meet ?

6. **Interesting Equation** Show that

$$\tan 9^\circ - \tan 27^\circ - \tan 63^\circ + \tan 81^\circ = 4$$

Do not use a calculator to evaluate the trigonometric functions.

7. **Reciprocals and Sums** A machine originally has the number 100 in its memory. When turned on, it takes the reciprocal of every number in its memory as well as the sum of any combination of numbers and stores these reciprocals and sums back in the memory. What are the possible numbers the machine will have in its memory if it does this forever.

8. **Is it Hard or Easy ?** Solve the equation

$$log\left(3^{x^{\sqrt{x}}-(\sqrt{x})^x+2}+1\right) = 1$$

9. **Sphere Inside a Tetrahedron** For a regular tetrahedron :

(i) Find the radius of the inscribed sphere as a function of the length of the edges of the tetrahedron.

(ii) Find the radius of the circumscribed sphere as a function of the length of the edge.

(iii) Find the radius of the radius of the sphere that just touches the *edges* of the tetrahedron.

(iv) Show that the radius of the sphere that just touches the edges of the tetrahedron is the geometric mean of the radius of the inscribed sphere and the radius of the *circumscribed* sphere of the tetrahedron.

10. **Fascinating Property of the Unit Circle**

(a) The points $\boldsymbol{A_1}$, $\boldsymbol{A_2}$, and $\boldsymbol{A_3}$ divide the boundary of the unit circle (a circle with radius 1) into 3 equal parts. Show that the product of the distances of $\boldsymbol{A_1}$ to the other two points is 3.

(b) The points $\boldsymbol{A_1}$, $\boldsymbol{A_2}$, $\boldsymbol{A_3}$, and $\boldsymbol{A_4}$ divide the boundary of the unit circle into 4 equal parts. Show that that the product of the distances of $\boldsymbol{A_1}$ to each of the 3 other points is 4.

(c) The points $\boldsymbol{A_1}$, $\boldsymbol{A_2}$, $\boldsymbol{A_3}$, $\boldsymbol{A_4}$ and $\boldsymbol{A_5}$ divide the boundary of the unit circle into 5 equal parts. Show that that the product of the distances of $\boldsymbol{A_1}$ to each of the 4other points is 5.

(d) The points $\boldsymbol{A_1}$, $\boldsymbol{A_2}$, $\boldsymbol{A_3}$, $\boldsymbol{A_4}$ $\boldsymbol{A_5}$, and $\boldsymbol{A_6}$ divide the boundary of the unit circle into 6 equal parts. Show that that the product of the distances of $\boldsymbol{A_1}$ to each of the 5 other points is 6.