September Solutions : Grades 6-9
Maine Math and Science Talent Search

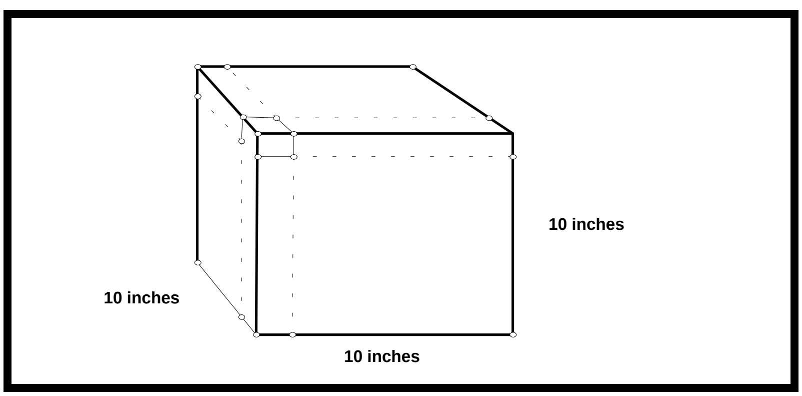
1. **Pooh and Piglet's Taxes** Winnie the Pooh and Piglet have gathered honey in the forest but must pay a tax to Heffalumps. Pooh has gathered 64 jars of honey and Piglet has 20 jars. Since they don't have any money they decide on a clever way to pay the tax. They know that Pooh's taxes consist of 5 jars of honey plus \$0.40 cents, and Piglet's taxes are 2 jars of honey, but he gets \$0.40 cents in change. So Winnie and Piglet figure out that they can pay their taxes by simply giving Heffalumps 7 jars of honey. From this information how much is each jar of honey worth ?

Solution Together Pooh and Piglet have collected 84 jars of honey so their total payment of 7 jars of honey means they pay 1/12 th the dollar value of the honey. And so Pooh's taxes of five jars of honey + \$0.40 cents should be 1/12th the value of the honey Pooh collects, and so if we call J = dollar value of a jar of honey, we have the relationship

$$\underbrace{5J + \$0.40}_{\text{taxes in dollars Pooh pays}} = \underbrace{\tfrac{1}{12}(64J)}_{\text{value of honey Pooh gathers}}$$

Solving this equation gives $J = \$1.20$

2. **Cube Problem** Consider a $10 \times 10 \times 10$ inch cube made from 1000 smaller cubes, each of size $1 \times 1 \times 1$ inch. If you look at the larger cube from all possible directions, what is the largest number of smaller cubes you can see at one time?



Solution If is clear you see the maximum number of the smaller cubes when you view the larger cube as drawn above with three sides showing. On the top of the large cube you can see 100 smaller cubes, on the left face of the larger cube you can see $9 \times 10 = 90$ *extra* smaller cubes, and on the front face you can see $9 \times 9 = 81$ smaller cubes you haven't seen on the other two faces. Hence, the total number of smaller cubes you can see at one time is 100 + 90 + 81 = 271.

3. **Clock Without Hands** A clock strikes once at 1 o'clock, 2 at 2 o'clock and so on. In addition, the clock strikes once at a quarter past the hour, once at half past the hour and once at three quarters past the hour. Suppose the hands of the clock are missing. What is the longest number of strikes of the clock for which you do *not* know the time ? And what will the time be when you finally know ?

Solution Now, if the clock strikes 3 times or 4 times at one time, then of course you know the time is 3 o'clock or 4 o'clock respectively. But if you hear the clock strikes once then you can't really tell the time. So the question is, what the longest number of consecutive times of the clock striking once before you can tell for sure the time ? Well, starting at 12:15 the clock strikes once every 15 minutes for the maximum of *seven* consecutive times at: 12:15, 12:30, 12:45, 1:00, 1:15, 1:30, and 1:45. Now when you hear the clock strike once six consecutive times you do not know the time *for sure* (you don't know if the 6th 1-strike is the last or not), but once you hear the clock strike once for the *seven* time, then you know the time on the seventh one-strike is 1:45. On the other hand if the 7th strike is not a 1-strike but a 2-strike, then this means the time is 2:00.

4. **Currency in a Faraway Country** A country has a currency system consisting of a crown, which is paper money, and smaller coins of 1/2 crown, 1/3 crown, 1/4 crown, 1/5 crown. What is the most money in coins a person can have without any collection of the coins being equal to a crown.

Solution It is clear we can't have 2 of the 1/2 crown coins, 3 of the 1/3 crown coins, 4 of the 1/4 crown coins, or 5 of the 1/5 crown coins. The most money in coins you can have without any collection equal to one crown is obtained by selecting

2 $(1/3)$ crown piece, 3 (1/4) crown piece, 4 (1/5) crown pieces

for a total of

$$2\left(\tfrac{1}{3}\right) + 3\left(\tfrac{1}{4}\right) + 4\left(\tfrac{1}{5}\right) = \tfrac{133}{60} = 2\,\tfrac{13}{60} \text{ crowns}$$

You could also select

1 (1/2) crown piece, 2 (1/3) crown piece, 1 (1/4) crown piece, 4 (1/5) crown piece

which would not result in any combination of coins giving 1 crown, which would also give

$$1\left(\tfrac{1}{2}\right) + 2\left(\tfrac{1}{3}\right) + 1\left(\tfrac{1}{4}\right) + 4\left(\tfrac{1}{5}\right) = \tfrac{133}{60} = 2\,\tfrac{13}{60} \text{ crowns}$$

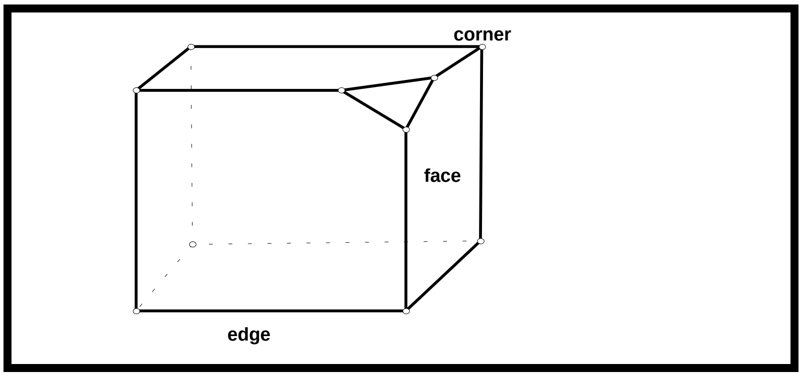
5. **The Million Problem** What two whole numbers, neither containing zeros, when multiplied together equal exactly 1,000,000 ?

Solution This may seem difficult, but when we break numbers into their component parts everything is easy. If we write

$$1{,}000{,}000 = 10^6 = (2 \times 5)^6 = 2^6 \times 5^6 = 64 \times 15{,}625$$

we see that the two numbers are 64 and 15,625. Note that we can do the same thing with $100 = 10^2 = (2 \times 5)^2 = 2^2 \times 5^2 = 4 \times 25$, which means the two numbers that give 100 are 4 and 25. You can find for yourself the two numbers whose product is 1,000,000,000.

6. **Interesting Geometric Shapes** Start with a cube of dimensions $3 \times 3 \times 3$ inches and snip off the 8 corner points. We have shown one of the eight cutoff corners below. How many faces (sides), corners (corner points), and edges (straight edges) will the new object have ?



Solution If we cut off the eight corner points, we still have the six original faces (although not squares now) but we will have added 8 more faces (the little triangular faces at the corners), and so now we have a total of $6 + 8 = 14$ faces. Now for the corner pints, each of the original corner points is erased but replaced by *three* corner points. Thus, the new object has a total of $8 \times 3 = 24$ corner points. Finally, the original cube had 12 edges. By cutting off the little corners like we did we keep the 12 original edges (although they are shorter) and add 3 more edges at each of the 8 triangular faces. Hence, the total number of edges in the new object is $12 + 8\,(3) = 36$. In summary, we have

	faces	**corner points**	**edges**
original cube	6	8	12
new object	14	24	36

7. **Radical Expression** Believe it or not $\sqrt{(n+3)(n+2)(n+1)\,n\ +\ 1}$ is an integer for any $n = 1,\ 2,\ 3,\ ....$. In fact for a certain integer n it is equal to a two-digit number, where n is the number in the tens place, and $n + 1$ is the number in the 1s position. What is the two-digit number?

Solution We find the value of n by solving

$$\sqrt{(n+3)(n+2)(n+1)\,n+1} \;=\; 10n \;+\; (n+1) \;=\; 11n+1$$

Now if $11n + 1$ is a two digit number, then we know that n is one of the integers $n = 1,\ 2,$ ..., 8 (when $n = 9$ we have that $11\ (9) + 1 = 100$ is not a two-digit number. Hence, we simply see which n satisfies the equation. Doing this, we get

$$
\begin{aligned}
n = 1 &\Rightarrow \sqrt{4 \cdot 3 \cdot 2 \cdot 1 + 1} = 5 \neq 11(1) + 1 \\
n = 2 &\Rightarrow \sqrt{5 \cdot 4 \cdot 3 \cdot 2 + 1} = 11 \neq 11(2) + 1 \\
n = 3 &\Rightarrow \sqrt{6 \cdot 5 \cdot 4 \cdot 3 + 1} = 19 \neq 11(3) + 1 \\
n = 4 &\Rightarrow \sqrt{7 \cdot 6 \cdot 5 \cdot 4 + 1} = 29 \neq 11(4) + 1 \\
n = 5 &\Rightarrow \sqrt{8 \cdot 7 \cdot 6 \cdot 5 + 1} = 41 \neq 11(5) + 1 \\
n = 6 &\Rightarrow \sqrt{9 \cdot 8 \cdot 7 \cdot 6 + 1} = 55 \neq 11(6) + 1 \\
n = 7 &\Rightarrow \sqrt{10 \cdot 9 \cdot 8 \cdot 7 + 1} = 71 \neq 11(7) + 1 \\
\boldsymbol{n = 8} &\boldsymbol{\Rightarrow \sqrt{11 \cdot 10 \cdot 9 \cdot 8 + 1} = 89 = 11(8) + 1}
\end{aligned}
$$

Hence, we see that $n = 8$, and so the two-digit number is 89.

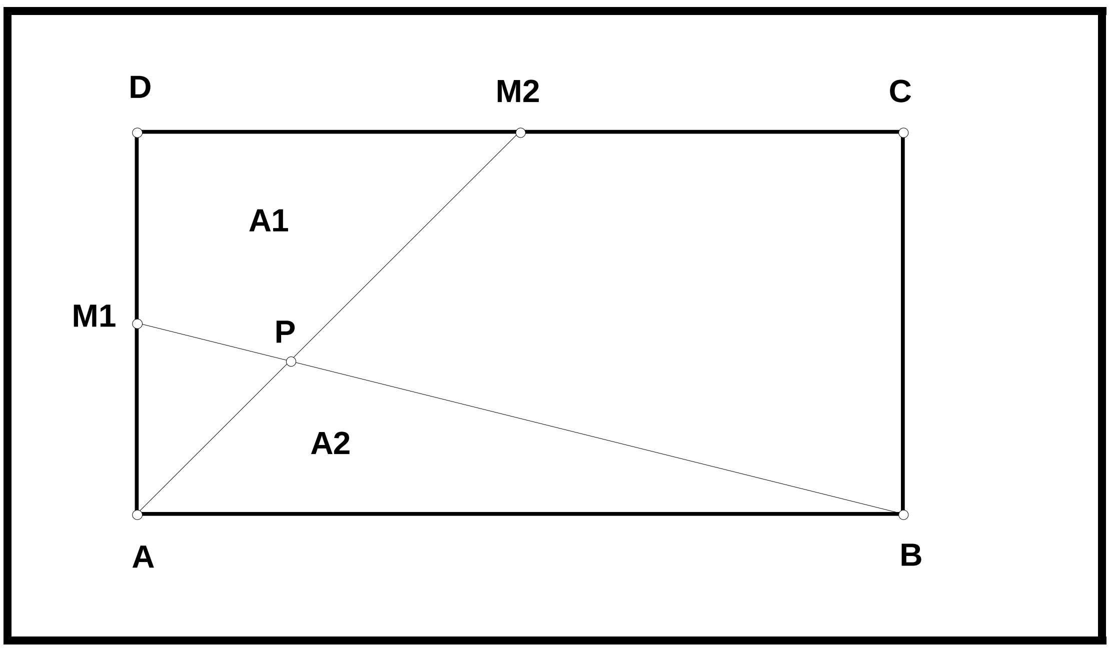
8. **Mary's Pets** Mary has some pet chickens and rabbits. To determine the number of her pets she counted a total of 50 heads and 120 legs. How many chickens and rabbits does Mary have ?

Solution If we call C = number of chickens and R = number of rabbits, then

$$
\begin{aligned}
C + R &= 50 \\
2C + 4R &= 120
\end{aligned}
$$

which has the solution $C = 40,\ R = 10.$ Hence, Mary has 40 chickens and 10 rabbits.

9. **Rectangle Problem** Consider the rectangle drawn below where M_1 and M_2 are midpoints of neighboring sides of the triangle. We now draw lines between these points and the corner points A and B, thus forming two regions with areas A_1 and A_2. Find the ratio of the areas A_1/A_2 .



Solution The areas of the two triangles AM_2D and ABM_1 are the same since

$$\text{area } AM_2D = \tfrac{1}{2}\,|AD|\,|DM_2| = \tfrac{1}{4}\,\text{(area of the rectangle)}$$
$$\text{area } ABM_1 = \tfrac{1}{2}\,|AB|\,|AD| = \tfrac{1}{4}\,\text{(area of the rectangle)}$$

We can now find the

$$A_1 = \text{area of } AM_2D - \text{area of } APM_1 = \tfrac{1}{4}\text{(area of rectangle)} - \text{area of } APM_1$$
$$A_2 = \text{area of } ABM_1 - \text{area of } APM_1 = \tfrac{1}{4}\text{(area of rectangle)} - \text{area of } APM_1$$

Hence, we have $A_1 = A_2$ and so the ratio is 1.

10. **Interesting Equation** Solve the equation

$$\tfrac{x+6}{1995} + \tfrac{x+5}{1996} + \tfrac{x+4}{1997} = \tfrac{x+3}{1998} + \tfrac{x+2}{1999} + \tfrac{x+1}{2000}$$

Solution Equations like this often take a little ingenuity. Sometimes things work and sometimes they don't. In this case if we add 1 to each fraction, we get

$$\left(\tfrac{x+6}{1995}+1\right) + \left(\tfrac{x+5}{1996}+1\right) + \left(\tfrac{x+4}{1997}+1\right) = \left(\tfrac{x+3}{1998}+1\right) + \left(\tfrac{x+2}{1999}+1\right) + \left(\tfrac{x+1}{2000}+1\right)$$

or

$$\left(\tfrac{x+6}{1995}+\tfrac{1995}{1995}\right) + \left(\tfrac{x+5}{1996}+\tfrac{1996}{1996}\right) + \left(\tfrac{x+4}{1997}+\tfrac{1997}{1997}\right)$$
$$= \left(\tfrac{x+3}{1998}+\tfrac{1998}{1998}\right) + \left(\tfrac{x+2}{1999}+\tfrac{1999}{1999}\right) + \left(\tfrac{x+1}{2000}+\tfrac{2000}{2000}\right)$$

or

$$\tfrac{x+2001}{1995} + \tfrac{x+2001}{1996} + \tfrac{x+2001}{1997} = \tfrac{x+2001}{1998} + \tfrac{x+2001}{1999} + \tfrac{x+2001}{2000}$$

or

$$\tfrac{x+2001}{1995} + \tfrac{x+2001}{1996} + \tfrac{x+2001}{1997} - \tfrac{x+2001}{1998} - \tfrac{x+2001}{1999} - \tfrac{x+2001}{2000} = 0$$

or

$$(x+2001)\left(\tfrac{1}{1996} + \tfrac{1}{1997} + \tfrac{1}{1998} - \tfrac{1}{1999} - \tfrac{1}{2000}\right) = 0$$

But the second factor is not zero, and so we have $x + 2001 = 0,$ or $x = -2001.$