Maine Math, Science and Engineering Talent Search
Round 5 Solutions/Grades 6-12

1. **Minimum Cost Swimming Pool** We are building a 120 cubic meter swimming pool and wish to line the bottom and sides of the pool with decorative tiles of size 1 meter $\times$ 1 meter. If we are not allowed to cut the tiles, what are the dimensions of the pool that uses the least number of tiles ? We will assume that the length of the pool is greater than or equal to the width which is greater than or equal to the depth.

Solution Letting the pool have length x meters, width y meters, and depth z meters, the volume of the pool is $V = xyz$ cubic meters, and the surface area of the bottom and sides is $S_A = xy + 2z(x+y)$ square meters. Since the blocks have dimensions 1 meter by 1 meter we know the pool dimensions x, y, and z must be integers. Our goal is to find the positive integers x, y, z which minimizes $S_A = xy + 2z(x+y)$, where $z \leq y \leq x$ satisfies $xyz = 120$.

Now if $z \geq 5$, then $xyz > 120$ and so z must be 1, 2, 3, or 4. We now use the relationship $xyz = 120$ and write $xy = 120/z$ and so $S = \frac{120}{z} + 2z(x+y)$. Hence, for each $z = 1, 2, 3, 4$ we pick y, z such that $xyz = 120$, but that $y + z$ is a minimum. For example, when $z = 1$, we have $xyz = xy = 120$, and the factors x and y of 120 for which $x + y$ is a minimum can easily been checked to be $x = 12$, $y = 10$, which are shown in the first row of the following table.

z	y	x	$x+y$	xyz	$S_A = xy + 2z(x+y)$
1	10	12	12	120	164
2	6	10	16	120	124
3	5	8	13	120	118
4	5	6	11	120	118

Likewise, when $z = 2$, we have $xyz = 2xy = 120$ or $xy = 60$, hence the two factors x, y of 60 for which $x + y$ is a minimum can easily be seen to be $x = 10$, $y = 6$. When $z = 3$, 4 we get the last two rows of the above table.

The conclusion from the above table is that we need 118 square tiles to build the 120 cubic meter pool, and from these tiles we could build pools of two different shapes: 4 meter $\times$ 5 meters $\times$ 6 meters or 3 meters $\times$ 5 meters $\times$ 8 meters.

2. **Interesting Equation** Show that for any positive integer n the number $2903^n - 803^n - 464^n + 261^n$ is divisible by 1897.

Solution Writing

$2903^n - 803^n - 464^n + 261^n = 2903^n - 803^n - (464^n - 261^n)$

we use the identity $a^n - b^n = (a-b)(a^n + na^{n-1}b + ... + n\,ab^{n-1} + b^n)$

to observe

$$2903^n - 803^n \text{ is divisible by } 2903 - 803 = 2100 = 7 \cdot 300$$
$$464^n - 261^n \text{ is divisible by } 464 - 261 = 203 = 7 \cdot 29$$

Hence, our expression is divisible by 7. But, we also can write our expression as

$$2903^n - 803^n - 464^n + 261^n = 2903^n - 264^n - (803^n - 261^n)$$

hence

$$2903^n - 264^n \text{ is divisible by } 2903 - 464 = 2439 = 9 \cdot 271$$
$$803^n - 261^n \text{ is divisible by } 803 - 261 = 542 = 2 \cdot 271$$

and so our expression is also divisible by 271. But 7 and 271 are prime numbers and so the given expression is divisible by $7 \times 271 = 1897$.

3. **Unknown Number Base** The numbers in the following long division are given in some unknown base (2, 3, 4, ...). Your problem is to determine that base, and the numerator BDBC, and denominator BCB in the division.

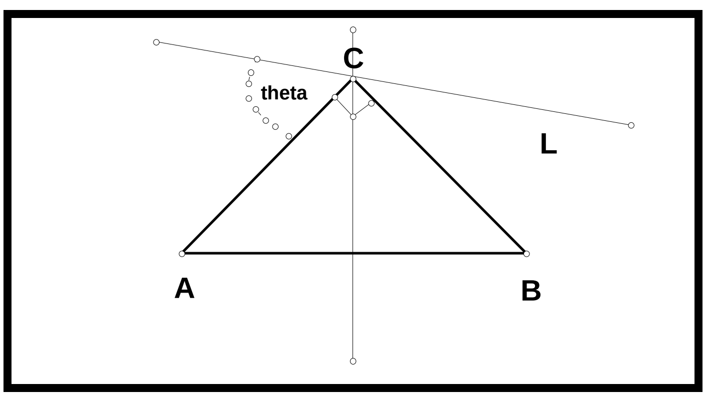
Solution: The solution written at the right is found from the following reason:

```
              B D B C                         2 2 2 3
    B C B √ A B C D C B A          2 3 2 √ 1 2 3 2 3 2 1
            A E A D                      1 0 1 2
            -----------                  ----------
             B B E C                           2 0 0 3
             B E C C                           2 0 3 3
            -------------                  -----------
                A B E B                      1 2 0 2
```

```
A E A D            1 0 1 2
------------       ----------
  A C C A            1 3 3 1
  A C E A            1 3 0 1
  -----------        -----------
      C E                3 0
```

Solution Calling the unknown base n, we know the digits are 0, 1, 2, ... $n-1$. We first observe from the division that E $=0$, and also that B $+$ C $=$ $n+0=n$. Also observe $1+$ A $+ 0 =$ A $+ 1$ $=$ B and so A $+$ B $=$ C, C $>$ A, B. Furthermore, we see C $+$ D $= n$ $+$ Q $=$ C $+$ 2B, and so D $=$ 2B. We also see that the product BC ends in A, and that $\mathrm{B}^2 =$ D. From $\mathrm{B}^2 =$ D and D $=$ 2B we get B $= 2$. Therefore D $= 2$, A $= 1$, C $= 3$, and finally the base $n = 5$. We have the division in base 5 above

4. **Isosceles Triangle Problem** Consider the right isosceles triangle ABC drawn below. We rotate the triangle around a line L which passes through the vertex C, making an angle θ with the side AC, where $0 \leq \theta \leq 90°$. (i.e. the line does not cross the interior of the triangle. Find volume of the rotated object as a function of θ.



Solution

Calling a the common length of legs of the triangle, and A_1 and B_1 the projection of the points A and B, respectively, on the line L, as shown in the above diagram, it is clear that the volume of revolution of the triangle around the line is directly proportional to the length of the line segment A_1B_2. Hence, we simply determine the angle that makes the length of this segment a maximum and minimum. We have

$$\begin{aligned} A_1B_1 &= A_1C + CB_1 = a \cos\theta + a\cos(90° - \theta) \\ &= a\,(cos\,\theta + cos\,(90 - \theta)) \\ &= 2a\,cos\,45°\,cos\,(\theta - 45) \end{aligned}$$

which attains its maximum and values at

$$\begin{aligned} &max\, A_1B_1 = 2a\,cos\,45° = a\sqrt{2} \quad \text{when } \theta = 45° \\ &min\, A_1B_1 = 2a\,cos\,45°\,cos\,45° = a \quad \text{when } \theta = 0° \text{ or } 90° \end{aligned}$$

Hence

$$V_{max} = \tfrac{2a^2\pi}{6} \cdot a\sqrt{2} = \tfrac{a^3\sqrt{2}}{3}\pi \qquad\qquad V_{min} = \tfrac{2a^2\pi}{6} \cdot a = \tfrac{a^3}{3}\pi$$

5. **Algebra Anyone ?** Solve the equation $x\,(x+a)(x+2a)(x+3a) = b^4$.
Solution We write

$$\begin{aligned} x(x+3a) = x^2 + 3ax,\ (x+a)(x+2a) = x^2 + 3ax + 2a^2 \\ z = x^2 + 3ax \end{aligned}$$

we get

$z(z+2a^2) = b^4$or $z^2 + 2a^2x - b^4 = 0$

which has solutions $z = -a^2 \pm \sqrt{a^4 + b^4}$. Hence, we get the four roots

$$\begin{aligned} x^2 + 3ax + (a^2 - \sqrt{a^4+b^4}) = 0 \Rightarrow x_{1,2} = \tfrac{-3a \pm \sqrt{5a^2 + 4\sqrt{a^4+b^4}}}{2} \\ x^2 + 3ax + (a^2 + \sqrt{a^4+b^4}) = 0 \Rightarrow x_{3,4} = \tfrac{-3a \pm \sqrt{5a^2 - 4\sqrt{a^4+b^4}}}{2} \end{aligned}$$

6. **Spheres and Tetrahedrons** Starting with a sphere of volume V, inscribe a regular tetrahedron inside the sphere (the corners of the tetrahedron just touch the sphere). We then inscribe a sphere inside this tetrahedron, and then another

tetrahedron inside this sphere, and then another tetrahedron inside this sphere, and so on, getting smaller and smaller spheres and tetrahedrons. What is the sum of the volumes of all the spheres ? What is the sum of the volumes of all the tetrahedrons ?

Solution We let r be the radius of the original sphere and $r_1 > r_2 > r_3 >$ ... be the radii of the consecutive spheres. We saw in Problem 9 of the November/December (Round 4) problems that $r_1 = \frac{r}{3}$, $r_2 = \frac{r_1}{3} = (\frac{1}{3})^2 r$, $r_3 = \frac{r_2}{3} = (\frac{1}{3})^3 r$, $r_n = (\frac{1}{3})^n r$, and since the volumes of these spheres are directly proportional to the cube of their radii, we have $V_1 = \frac{1}{27} V$, $V_2 = (\frac{1}{27})^2 V$, ... $V_n = (\frac{1}{27})^n V$. But these volumes form a geometric sequence with first term V and multiplier 1/27, and so the sum is $S = \frac{V}{1-(1/27)} = \frac{27}{26} V$.

For the second part, to find the sum of the volumes of the tetrahedrons, we recall from Problem 9 of the November/December problems that the radius r of a sphere inscribed in a tetrahedron of side a is $r = \frac{a}{4}\sqrt{6}$, or $a = r\sqrt{\frac{8}{3}}$. Hence, the volume of the first tetrahedron is $V_{tetrahedron} = \frac{8r^3}{9\sqrt{3}}$. We also know that the volume of a regular tetrahedron inscribed in a sphere is directly proportional to the volume of a the sphere, hence the volumes of the tetrahedrons form a geometric sequence with first term $\frac{8r^3}{9\sqrt{3}}$ and multiplier $\frac{1}{27}$, and so the sum of the volumes is

$$\text{Sum of Tetrahedron Volumes} = \frac{V}{1-(1/27)} = \frac{8r^3}{9\sqrt{3}} \cdot \frac{27}{26} = \frac{12r^3}{13\sqrt{3}}$$

We can also use the formula for the volume of a sphere $V_{sphere} = \frac{4}{3}\pi r^3$, or $r^3 = \frac{3V}{4\pi}$, to write the sum of the volumes of the tetrahedrons in terms of the volume V of the original sphere, getting

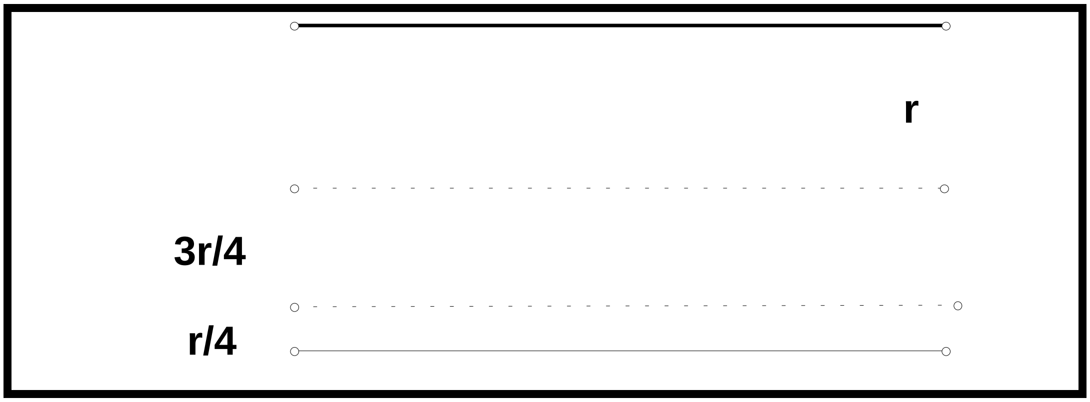
$$\text{Sum of Tetrahedron Volumes} = \frac{12 \cdot 3V}{13 \cdot 4 \cdot \pi\sqrt{3}} = \frac{3V\sqrt{3}}{13\pi}$$

7. **The Case of the Missing Hand** We all know how to tell time by looking at the minute and hour hands of a clock, but did you know you can still tell the time if the minute hand is missing? That's right, suppose the minute hand of your clock falls off so the clock only has the hour hand. How would you tell the exact time ? We don't expect you to determine whether the time is AM or PM.

Solution Every time the hour hand goes from one hour to the next, the minute hand goes all the way around the clock. In other words, the minute hand goes around 12 times faster than the hour hand. Hence, the exact time past a given hour will be 12 times the fraction the hour hand is past the given hour. For example, if the hour hand reads 3.50 (half way between 3 and 4 o'clock), then the time is 30 $= 12 \times 0.50$ minutes past the hour of 3. Likewise, if the hour hand is 4.37, then the exact time is 4.44 $= 12 \times .37$ minutes past the hour of 4 (or equivalently, 4 minutes and 26.4 seconds past the hour of 4). In other words, you simply multiply the fractional part that the hour hand reads (like 0.37 above) times 12 and that is the number of minutes past the given hour. In general, if we denote h as the position of the hour hand, then the rule for determining the time is

if $0 \leq h < 1$, the time is $12\,h$ minutes past noon (or midnight)
if $1 \leq h < 2$, the time is $12\,(h-1)$ minutes past 1
if $2 \leq h < 3$, the time is $12\,(h-2)$ minutes past 2
if $3 \leq h < 4$, the time is $12\,(h-3)$ minutes past 3
if $4 \leq h < 5$, the time is $12\,(h-4)$ minutes past 4
if $5 \leq h < 6$, the time is $12\,(h-5)$ minutes past 5
if $6 \leq h < 7$, the time is $12\,(h-6)$ minutes past 6
if $7 \leq h < 8$, the time is $12\,(h-7)$ minutes past 7
if $8 \leq h < 9$, the time is $12\,(h-8)$ minutes past 8
if $9 \leq h < 10$, the time is $12\,(h-9)$ minutes past 9
if $10 \leq h < 11$, the time is $12\,(h-10)$ minutes past 10
if $11 \leq h < 12$, the time is $12\,(h-11)$ minutes past 11

8. **Archimedes Anyone?** A solid cylinder shown below floats in water, its length parallel to the surface, and $r/4$ inches under water, where r is the radius of the cylinder. Determine the density (weight per unit volume) of the cylinder. Note: Archimedes principle states that the force pushing up on an object is equal to the weight of the water the object displaces.



Solution From Archimedes Principle we know that the ratio of the density of the cylinder $(D_{cylinder})$ to the density of the water (D_{water}) is the same as the ratio of the volume of the immersed cylinder water $(V_{immersed})$ to the volume of the complete cylinder $(V_{entire\,cylinder})$. In other words

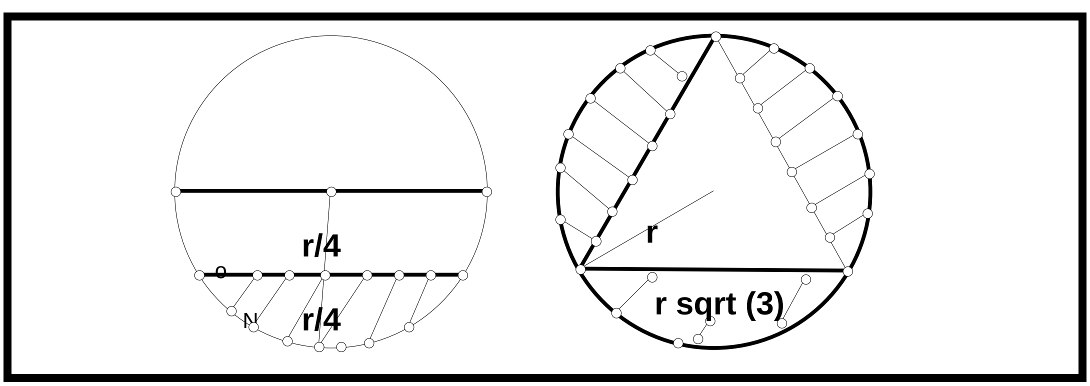
$$\frac{D_{cylinder}}{D_{water}} = \frac{V_{immersed}}{V_{entire\,cylinder}}$$

Hence, if we draw a cylinder of radius r and height h immersed in water as shown below, where the shaded area represents the portion of the cylinder immersed in water, we have

$$V_{immersed} = \text{area of base } \times \text{height } = \tfrac{1}{3}\,(\pi r^2 - \tfrac{3\sqrt{3}\,r^2}{4})\,h,$$
$$V_{entire\,cylinder} = \pi r^2 h$$

$$\frac{V_{immersed}}{V_{entire\,cylinder}} = \frac{\frac{1}{3}(\pi r^2 - \frac{3\sqrt{3}\,r^2}{4})\,h}{\pi r^2 h} = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} \approx 0.1955$$

Hence, $\frac{D_{cylinder}}{D_{water}} \approx 0.1955 \Rightarrow D_{cylinder} = 0.1955 \times D_{water}$

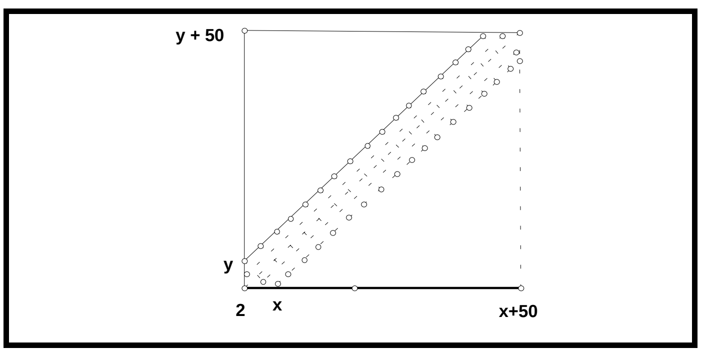


9. Frank and Francine have a study date and meet at the library between 2 pm and 2:50 pm. They each wait 10 minutes before leaving. If they both arrive at random and independent of the other, what is the probability they will meet ?

Solution We assume Francine arrives x minutes past 2 and Frank arrives y minutes past 2, where x and y are between 0 and 50, inclusive. Now Frank and Francine will meet if $x - 10 \le y \le x + 10$ which is represented by the shaded region in the diagram below. Hence, the probability P that Frank and Francine will meet is the ratio of the area of the shaded region, which is $50^2 - 2\,(\frac{1}{2}^2 \cdot 40^2)$, to the area of the total region, which is 50^2. Hence, the chance that Frank and Francine will meet is

$$P = \frac{shaded\,area}{total\,area} = \frac{50^2 - \frac{1}{2}\,40^2}{50^2} = 0.36$$

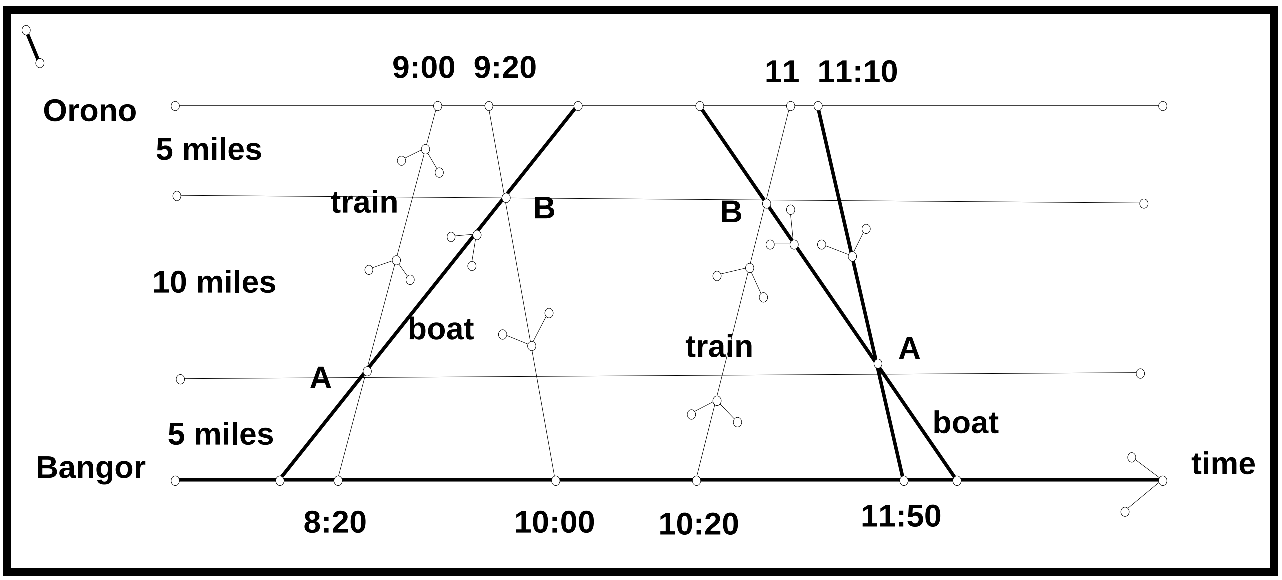
In other words a 36% chance.



10. **Orono and Bangor** Orono and Bangor are 20 miles apart on the same side of a moving river. A train runs next to the river and travels 30 miles per hour. The train makes two round trips from Bangor to Orono and back. On the first trip it leaves Bangor at 8:20 a.m. and arrives back in Bangor at 10 a.m. On the second trip it leaves Bangor at 10:20 a.m. and arrives back at 11:50 am. There is also a boat that makes a round trip from Bangor to Orono. The boat leaves before the train but is caught by the train 1/4th the way from Bangor to Orono. Later on the way to Orono the boat is met by the returning train 3/4th the way from Bangor and Orono. On the return trip the boat meets the train coming from Bangor on the second trip

3/4ths the way from Bangor to Orono, and later is passed by the returning train 1/4th the way from Bangor to Orono. What is the direction of flow of the river and what is its rate of flow ?

Solution We draw the picture below showing the position of the train and boat on the vertical axis and time on the horizontal axis.



We first observe that it takes the train 40 minutes to make a one-way trip between Bangor and Orono, and thus the train leaving Bangor at 8:20 a.m. and 10:20 a.m. arrives in Orono at 9:00 a.m. and 11:00 a.m. Furthermore, the train must leave Orono at 9:20 a.m. and 11:10 a.m. on the two trips to get back in Bangor at the given times of 10 am and 11:50 am. Hence, the train has a 10 min layover in Orono on the first trip, and a 20 minute layover on the second trip. To determine the direction the river flows, we know that the distance from point A to point B is 10 miles and that the train travels a total of 10 + 5 + 5 $=$ 20 miles between the time it passes the boat for the first time and meets it coming back; hence the time it takes the train to go from A to B is 40 + 20 $=$ 60 minutes. By a similar argument, the time it takes the boat to go from B to A is 40 + 10 $=$ 50 minutes. Hence, the boat moves at 10 m/h when traveling from Bangor to Orono, and 12 m/h when going from Orono to Bangor. Therefore, the river flows from Orono to Bangor.

Now calling $R_b =$ rate of the boat in still water, $R_r =$ rate the river flows, we have R_b- $R_r =$ 10 m/h and $R_b +$ $R_r =$ 12 m/h, and so solving these equation, we get $R_b = 11\,\mathrm{m/h}$, $R_r = 1\,\mathrm{m/h}$..