Two problems on complex cosines

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In this note we will solve two interconnected problems from the MathLinks discussion

`http://www.mathlinks.ro/Forum/viewtopic.php?t=67939`

We start with a theorem:

Theorem 1. Let φ be a complex number, and let $x_1 = 2\cos\varphi$. Let $k \geq 1$ be an integer, and let x_2, x_3, ..., x_k be $k-1$ complex numbers. Then, the chain of equations

$$x_1 = \frac{1}{x_1} + x_2 = \frac{1}{x_2} + x_3 = ... = \frac{1}{x_{k-1}} + x_k \tag{1}$$

(if $k = 1$, then this chain of equations has to be regarded as the zero assertion, i. e. as the assertion which is always true) holds if and only if the equation $x_m = \frac{\sin((m+1)\varphi)}{\sin(m\varphi)}$ holds for every $m \in \{1, 2, ..., k\}$. Hereby, in the case when $\sin(m\varphi) = 0$, the equation $x_m = \frac{\sin((m+1)\varphi)}{\sin(m\varphi)}$ is to be understood as follows:

- If φ is an integer multiple of π, then $\sin(m\varphi) = \sin((m+1)\varphi) = 0$, and the number $\frac{\sin((m+1)\varphi)}{\sin(m\varphi)}$ has to be understood as $\lim\limits_{\psi\to\varphi} \frac{\sin((m+1)\psi)}{\sin(m\psi)}$.
- If φ is not an integer multiple of π and we have $\sin(m\varphi) = 0$, then $\sin((m+1)\varphi) \neq 0$, and the equation $x_m = \frac{\sin((m+1)\varphi)}{\sin(m\varphi)}$ is considered wrong.

Proof of Theorem 1. In our following proof, we will only consider the case when φ is not an integer multiple of π, because we will not need the case when φ is a multiple of π in our later applications of Theorem 1. Besides, our following proof can be easily modified to work for the case of φ being a multiple of π as well (this modification is left to the reader).

We will establish Theorem 1 by induction over k:

For $k = 1$, we have to prove that the zero assertion holds if and only if $x_1 = \frac{\sin((1+1)\varphi)}{\sin(1\varphi)}$. Well, since the zero assertion always holds, we have to prove that the equation $x_1 = \frac{\sin((1+1)\varphi)}{\sin(1\varphi)}$ always holds. This is rather easy:

$$x_1 = 2\cos\varphi = \frac{2\sin\varphi\cos\varphi}{\sin\varphi} = \frac{\sin(2\varphi)}{\sin\varphi} = \frac{\sin((1+1)\varphi)}{\sin(1\varphi)}.$$

Thus, Theorem 1 is proven for $k = 1$.

Now we come to the induction step. Let $n \geq 1$ be an integer. Assume that Theorem 1 holds for $k = n$. This means that:

(*) If x_2, x_3, ..., x_n are $n-1$ complex numbers, then the chain of equations

$$x_1 = \frac{1}{x_1} + x_2 = \frac{1}{x_2} + x_3 = ... = \frac{1}{x_{n-1}} + x_n \tag{2}$$

holds if and only if the equation $x_m = \frac{\sin((m+1)\varphi)}{\sin(m\varphi)}$ holds for every $m \in \{1, 2, ..., n\}$.

We have to prove that Theorem 1 also holds for $k = n+1$. This means that we have to prove that:

(**) If x_2, x_3, ..., x_n, x_{n+1} are n complex numbers, then the chain of equations

$$x_1 = \frac{1}{x_1} + x_2 = \frac{1}{x_2} + x_3 = ... = \frac{1}{x_{n-1}} + x_n = \frac{1}{x_n} + x_{n+1} \tag{3}$$

holds if and only if the equation $x_m = \frac{\sin((m+1)\varphi)}{\sin(m\varphi)}$ holds for every $m \in \{1, 2, ..., n, n+1\}$.

So let's prove **(**)**. This requires verifying two assertions:

Assertion 1: If (3) holds, then $x_m = \frac{\sin((m+1)\varphi)}{\sin(m\varphi)}$ holds for every $m \in \{1, 2, ..., n, n+1\}$.

Assertion 2: If $x_m = \frac{\sin((m+1)\varphi)}{\sin(m\varphi)}$ holds for every $m \in \{1, 2, ..., n, n+1\}$, then (3) holds.

Before we step to the proofs of these assertions, we show that

$$x_1 = \frac{\sin(n\varphi)}{\sin((n+1)\varphi)} + \frac{\sin((n+2)\varphi)}{\sin((n+1)\varphi)}. \tag{4}$$

This is because

$$\frac{\sin(n\varphi)}{\sin((n+1)\varphi)} + \frac{\sin((n+2)\varphi)}{\sin((n+1)\varphi)} = \frac{\sin(n\varphi) + \sin((n+2)\varphi)}{\sin((n+1)\varphi)}$$
$$= \frac{\sin((n+1)\varphi - \varphi) + \sin((n+1)\varphi + \varphi)}{\sin((n+1)\varphi)}$$
$$= \frac{(\sin((n+1)\varphi)\cos\varphi - \cos((n+1)\varphi)\sin\varphi) + (\sin((n+1)\varphi)\cos\varphi + \cos((n+1)\varphi)\sin\varphi)}{\sin((n+1)\varphi)}$$
$$= \frac{2\sin((n+1)\varphi)\cos\varphi}{\sin((n+1)\varphi)} = 2\cos\varphi = x_1.$$

Now, let's prove Assertion 1: We assume that (3) holds. We have to prove that $x_m = \frac{\sin((m+1)\varphi)}{\sin(m\varphi)}$ holds for every $m \in \{1, 2, ..., n, n+1\}$. In fact, since (3) yields (2), we can conclude from **(*)** that the equation $x_m = \frac{\sin((m+1)\varphi)}{\sin(m\varphi)}$ holds for every $m \in \{1, 2, ..., n\}$. It remains to prove this equation for $m = n+1$; in other words, it remains to prove that $x_{n+1} = \frac{\sin((n+2)\varphi)}{\sin((n+1)\varphi)}$. In order to prove this, we note that the equation $x_m = \frac{\sin((m+1)\varphi)}{\sin(m\varphi)}$, which holds for every $m \in \{1, 2, ..., n\}$, particularly yields $x_n = \frac{\sin((n+1)\varphi)}{\sin(n\varphi)}$. Hence, $\frac{1}{x_n} = \frac{\sin(n\varphi)}{\sin((n+1)\varphi)}$. Now, (3) yields $x_1 = \frac{1}{x_n} + x_{n+1}$, so that $x_1 = \frac{\sin(n\varphi)}{\sin((n+1)\varphi)} + x_{n+1}$. Comparing this with (4), we obtain $x_{n+1} = \frac{\sin((n+2)\varphi)}{\sin((n+1)\varphi)}$, qed.. Thus, Assertion 1 is proven.

Now we will show Assertion 2. To this end, we assume that $x_m = \frac{\sin((m+1)\varphi)}{\sin(m\varphi)}$ holds for every $m \in \{1, 2, ..., n, n+1\}$, and we want to show that (3) holds.

We have assumed that the equation $x_m = \frac{\sin((m+1)\varphi)}{\sin(m\varphi)}$ holds for every $m \in \{1, 2, ..., n, n+1\}$, so that in particular, it holds for every $m \in \{1, 2, ..., n\}$. Hence, according to **(*)**, the equation (2) must hold. Now, we are going to prove the equation $x_1 = \frac{1}{x_n} + x_{n+1}$.

Since $x_m = \frac{\sin((m+1)\varphi)}{\sin(m\varphi)}$ holds for every $m \in \{1, 2, ..., n, n+1\}$, we have $x_n = \frac{\sin((n+1)\varphi)}{\sin(n\varphi)}$ and $x_{n+1} = \frac{\sin((n+2)\varphi)}{\sin((n+1)\varphi)}$. The former of these two equations yields $\frac{1}{x_n} = \frac{\sin(n\varphi)}{\sin((n+1)\varphi)}$. Thus, the equation (4) results in

$$x_1 = \frac{\sin(n\varphi)}{\sin((n+1)\varphi)} + \frac{\sin((n+2)\varphi)}{\sin((n+1)\varphi)} = \frac{1}{x_n} + x_{n+1}.$$

Thus, the equation $x_1 = \frac{1}{x_n} + x_{n+1}$ is proven. Combining this equation with (2), we get (3), and this completes the proof of Assertion 2.

As both Assertions 1 and 2 are now verified, the induction step is done, so that the proof of Theorem 1 is complete.

The first consequence of Theorem 1 will be:

Theorem 2. Let $n \geq 1$ be an integer, and let x_1, x_2, ..., x_n be n nonzero complex numbers such that

$$x_1 = \frac{1}{x_1} + x_2 = \frac{1}{x_2} + x_3 = ... = \frac{1}{x_{n-1}} + x_n = \frac{1}{x_n}. \tag{5}$$

Then, there exists some integer $j \in \{1, 2, ..., n+1\}$ such that $x_1 = 2\cos\frac{j\pi}{n+2}$ and $x_m = \frac{\sin\left((m+1)\frac{j\pi}{n+2}\right)}{\sin\left(m\frac{j\pi}{n+2}\right)}$ for every $m \in \{1, 2, ..., n\}$.

Proof of Theorem 2. We need two auxiliary assertions:
Assertion 1: We have $x_1 \neq 2$.

Assertion 2: We have $x_1 \neq -2$.

Proof of Assertion 1. Assume the contrary. Then, $x_1 = 2$. Now, we can prove by induction over m that $x_m = 1 + \frac{1}{m}$ for every $m \in \{1, 2, ..., n\}$. (In fact: For $m = 1$, we have to show that $x_1 = 1 + \frac{1}{1}$, what rewrites as $x_1 = 2$ and this was our assumption. Now, assume that $x_m = 1 + \frac{1}{m}$ holds for some $m \in \{1, 2, ..., n-1\}$. We want to prove that $x_{m+1} = 1 + \frac{1}{m+1}$ holds as well. Well, the equation (5) yields $x_1 = \frac{1}{x_m} + x_{m+1}$, so that $x_{m+1} = x_1 - \frac{1}{x_m}$. Since $x_1 = 2$ and $x_m = 1 + \frac{1}{m}$, we thus have $x_{m+1} = 2 - \frac{1}{1 + \frac{1}{m}} = \frac{m+2}{m+1} = 1 + \frac{1}{m+1}$. Hence, the induction proof is complete.)

Now, since we have shown that $x_m = 1 + \frac{1}{m}$ holds for every $m \in \{1, 2, ..., n\}$, we have $x_n = 1 + \frac{1}{n}$ in particular. But (5) yields $x_1 = \frac{1}{x_n}$, so that $1 = x_1 \cdot x_n = 2 \cdot \left(1 + \frac{1}{n}\right)$, what is obviously wrong since $2 \cdot \left(1 + \frac{1}{n}\right) > 2 \cdot 1 > 1$. Hence, we obtain a contradiction, and thus our assumption that Assertion 1 doesn't hold was wrong. This proves Assertion 1.

The *proof of Assertion 2* is similar (this time we have to show that if $x_1 = -2$, then $x_m = -\left(1 + \frac{1}{m}\right)$ for every $m \in \{1, 2, ..., n\}$).

Now, since the function $\cos : \mathbb{C} \to \mathbb{C}$ is surjective, there must exist a complex number φ such that $\frac{x_1}{2} = \cos \varphi$. Hereby, if $\frac{x_1}{2}$ is real and satisfies $-1 \leq \frac{x_1}{2} \leq 1$, then we take this φ such that φ is real and satisfies $\varphi \in [0, \pi]$ (this is possible since $\cos : [0, \pi] \to [-1, 1]$ is surjective).

Assertions 1 and 2 state that $x_1 \neq 2$ and $x_1 \neq -2$. Hence, $\frac{x_1}{2} \neq 1$ and $\frac{x_1}{2} \neq -1$. Since $\frac{x_1}{2} = \cos \varphi$, this yields $\cos \varphi \neq 1$ and $\cos \varphi \neq -1$, and thus φ is not an integer multiple of π.

Define another complex number x_{n+1} by $x_{n+1} = 0$. Then, (5) rewrites as

$$x_1 = \frac{1}{x_1} + x_2 = \frac{1}{x_2} + x_3 = ... = \frac{1}{x_{n-1}} + x_n = \frac{1}{x_n} + x_{n+1}. \tag{6}$$

Since $\frac{x_1}{2} = \cos \varphi$, we have $x_1 = 2 \cos \varphi$, so that we can apply Theorem 1 to the n complex numbers x_2, x_3, ..., x_{n+1}, and from the chain of equations (6) we conclude that $x_m = \frac{\sin((m+1)\varphi)}{\sin(m\varphi)}$ holds for every $m \in \{1, 2, ..., n+1\}$.

Thus, in particular, $x_{n+1} = \frac{\sin((n+2)\varphi)}{\sin((n+1)\varphi)}$. Since $x_{n+1} = 0$, we thus must have $\frac{\sin((n+2)\varphi)}{\sin((n+1)\varphi)} = 0$. This yields $\sin((n+2)\varphi) = 0$. Thus, $(n+2)\varphi$ is an integer multiple of π. Let $j \in \mathbb{Z}$ be such that $(n+2)\varphi = j\pi$. Then, $\varphi = \frac{j\pi}{n+2}$. Thus,

$x_1 = 2\cos\varphi$ becomes $x_1 = 2\cos\dfrac{j\pi}{n+2}$, and $x_m = \dfrac{\sin\left(\left(m+1\right)\varphi\right)}{\sin\left(m\varphi\right)}$ becomes $x_m = \dfrac{\sin\left(\left(m+1\right)\dfrac{j\pi}{n+2}\right)}{\sin\left(m\dfrac{j\pi}{n+2}\right)}$. It remains to show that $j \in \{1, 2, ..., n+1\}$.

Now, $\dfrac{x_1}{2} = \cos\varphi = \cos\dfrac{j\pi}{n+2}$ must be real and satisfy $-1 \leq \dfrac{x_1}{2} \leq 1$ (since cosines of real angles are real and lie between -1 and 1). Therefore, according to the definition of φ, we have $\varphi \in [0, \pi]$. Since φ is not a multiple of π, this becomes $\varphi \in \left]0, \pi\right[$. Since $\varphi = \dfrac{j\pi}{n+2}$, this yields $j \in \left]0, n+2\right[$. Since j is an integer, this results in $j \in \{1, 2, ..., n+1\}$. Hence, Theorem 2 is proven.

The first problem from the MathLinks thread asks us to show:

> **Theorem 3.** Let $n \geq 1$ be an integer, and let x_1, x_2, ..., x_n be n positive real numbers such that
>
> $$x_1 = \frac{1}{x_1} + x_2 = \frac{1}{x_2} + x_3 = ... = \frac{1}{x_{n-1}} + x_n = \frac{1}{x_n}.$$
>
> Then, $x_1 = 2\cos\dfrac{\pi}{n+2}$ and $x_m = \dfrac{\sin\left(\left(m+1\right)\dfrac{\pi}{n+2}\right)}{\sin\left(m\dfrac{\pi}{n+2}\right)}$ for every $m \in \{1, 2, ..., n\}$.

Proof of Theorem 3. According to Theorem 2, there exists some integer $j \in \{1, 2, ..., n+1\}$ such that $x_1 = 2\cos\dfrac{j\pi}{n+2}$ and $x_m = \dfrac{\sin\left(\left(m+1\right)\dfrac{j\pi}{n+2}\right)}{\sin\left(m\dfrac{j\pi}{n+2}\right)}$ for every $m \in \{1, 2, ..., n\}$. For every $m \in \{1, 2, ..., n, n+1\}$, we thus have

$$\prod_{s=1}^{m-1} x_s = \prod_{s=1}^{m-1} \frac{\sin\left(\left(s+1\right)\frac{j\pi}{n+2}\right)}{\sin\left(s\frac{j\pi}{n+2}\right)} = \frac{\prod\limits_{s=1}^{m-1} \sin\left(\left(s+1\right)\frac{j\pi}{n+2}\right)}{\prod\limits_{s=1}^{m-1} \sin\left(s\frac{j\pi}{n+2}\right)}$$
$$= \frac{\prod\limits_{s=2}^{m} \sin\left(s\frac{j\pi}{n+2}\right)}{\prod\limits_{s=1}^{m-1} \sin\left(s\frac{j\pi}{n+2}\right)} = \frac{\sin\left(m\frac{j\pi}{n+2}\right)}{\sin\left(1\frac{j\pi}{n+2}\right)} = \frac{\sin\left(m\frac{j\pi}{n+2}\right)}{\sin\frac{j\pi}{n+2}}.$$

Since the reals x_1, x_2, ..., x_{m-1} are all positive, their product $\prod\limits_{s=1}^{m-1} x_s$ is positive, and this yields that $\dfrac{\sin\left(m\dfrac{j\pi}{n+2}\right)}{\sin\dfrac{j\pi}{n+2}}$ is positive. But since $j \in \{1, 2, ..., n+1\}$, the term

$\sin \frac{j\pi}{n+2}$ is positive (since $0 < \frac{j\pi}{n+2} < \pi$), and thus it follows that $\sin \left(m \frac{j\pi}{n+2} \right)$ is positive. Since this holds for every $m \in \{1, 2, ..., n, n+1\}$, this means that the numbers $\sin \left(m \frac{j\pi}{n+2} \right)$ are positive for all $m \in \{1, 2, ..., n, n+1\}$. Since $j \in \{1, 2, ..., n+1\}$, this yields $j = 1$ [1]. Hence, $x_1 = 2 \cos \frac{j\pi}{n+2}$ becomes $x_1 = 2 \cos \frac{\pi}{n+2}$, and $x_m = \frac{\sin \left((m+1) \frac{j\pi}{n+2} \right)}{\sin \left(m \frac{j\pi}{n+2} \right)}$ becomes $x_m = \frac{\sin \left((m+1) \frac{\pi}{n+2} \right)}{\sin \left(m \frac{\pi}{n+2} \right)}$. This proves Theorem 3.

A converse of Theorem 3 is:

Theorem 4. Let $n \geq 1$ be an integer, and define n reals $x_1, x_2, ..., x_n$ by $x_m = \frac{\sin \left((m+1) \frac{\pi}{n+2} \right)}{\sin \left(m \frac{\pi}{n+2} \right)}$ for every $m \in \{1, 2, ..., n\}$. Then, the reals $x_1, x_2, ..., x_n$ are positive. Besides, $x_1 = 2 \cos \frac{\pi}{n+2}$, and the reals $x_1, x_2, ..., x_n$ satisfy the equation (5).

Proof of Theorem 4. At first, it is clear that the reals $x_1, x_2, ..., x_n$ are positive, because, for every $m \in \{1, 2, ..., n\}$, we have $\sin \left((m+1) \frac{\pi}{n+2} \right) > 0$ and $\sin \left(m \frac{\pi}{n+2} \right) > 0$ (since $0 < (m+1) \frac{\pi}{n+2} < \pi$ and $0 < m \frac{\pi}{n+2} < \pi$) and thus

$$x_m = \frac{\sin \left((m+1) \frac{\pi}{n+2} \right)}{\sin \left(m \frac{\pi}{n+2} \right)} > 0.$$

[1] *Proof.* Assume the contrary - that is, assume that $j \geq 2$.

Then, the smallest of the angles $m \frac{j\pi}{n+2}$ for $m \in \{1, 2, ..., n, n+1\}$ is $1 \frac{j\pi}{n+2} = \frac{j\pi}{n+2} < \pi$ (since $j < n+2$), and the largest one is $(n+1) \frac{j\pi}{n+2} \geq (n+1) \frac{2\pi}{n+2} = \frac{2(n+1)}{n+2}\pi = \pi + \frac{n}{n+2}\pi \geq \pi$. Thus, some but not all of the numbers $m \in \{1, 2, ..., n, n+1\}$ satisfy $m \frac{j\pi}{n+2} \geq \pi$. Let μ be the smallest $m \in \{1, 2, ..., n, n+1\}$ satisfying $m \frac{j\pi}{n+2} \geq \pi$. Then, $\mu \frac{j\pi}{n+2} \geq \pi$, but $(\mu - 1) \frac{j\pi}{n+2} < \pi$. Hence,

$$\begin{aligned} \mu \frac{j\pi}{n+2} &= \frac{j\pi}{n+2} + (\mu - 1) \frac{j\pi}{n+2} < \frac{(n+2)\pi}{n+2} + \pi \qquad \left(\text{since } j < n+2 \text{ and } (\mu - 1) \frac{j\pi}{n+2} < \pi\right) \\ &= 2\pi, \end{aligned}$$

what, together with $\mu \frac{j\pi}{n+2} \geq \pi$, yields $\pi \leq \mu \frac{j\pi}{n+2} < 2\pi$. Thus, $\sin \left(\mu \frac{j\pi}{n+2} \right) \leq 0$. But this contradicts to the fact that $\sin \left(m \frac{j\pi}{n+2} \right)$ is positive for all $m \in \{1, 2, ..., n, n+1\}$. Hence, we get a contradiction, so that our assumption that $j \geq 2$ was wrong. Hence, j must be 1.

The equation $x_1 = 2\cos\frac{\pi}{n+2}$ is pretty obvious:

$$x_1 = \frac{\sin\left((1+1)\frac{\pi}{n+2}\right)}{\sin\left(1\frac{\pi}{n+2}\right)} = \frac{\sin\left(2\frac{\pi}{n+2}\right)}{\sin\frac{\pi}{n+2}} = \frac{2\sin\frac{\pi}{n+2}\cos\frac{\pi}{n+2}}{\sin\frac{\pi}{n+2}} = 2\cos\frac{\pi}{n+2}.$$

Remains to prove the equation (5). In order to do this, define a real $x_{n+1} = 0$. Then,

$$x_{n+1} = 0 = \frac{0}{\sin\left((n+1)\frac{\pi}{n+2}\right)} = \frac{\sin\pi}{\sin\left((n+1)\frac{\pi}{n+2}\right)} = \frac{\sin\left((n+2)\frac{\pi}{n+2}\right)}{\sin\left((n+1)\frac{\pi}{n+2}\right)}.$$

Hence, the equation $x_m = \frac{\sin\left((m+1)\frac{\pi}{n+2}\right)}{\sin\left(m\frac{\pi}{n+2}\right)}$ holds not only for every $m \in \{1, 2, ..., n\}$, but also for $m = n+1$. Thus, altogether, it holds for every $m \in \{1, 2, ..., n, n+1\}$. Consequently, according to Theorem 1 (for $\varphi = \frac{\pi}{n+2}$ and $k = n+1$), we have

$$x_1 = \frac{1}{x_1} + x_2 = \frac{1}{x_2} + x_3 = ... = \frac{1}{x_{n-1}} + x_n = \frac{1}{x_n} + x_{n+1}.$$

Using $x_{n+1} = 0$, this simplifies to (5). Thus, Theorem 4 is proven.

Now we are ready to solve the second MathLinks problem:

Theorem 5. Let $n \geq 1$ be an integer, and let y_1, y_2, ..., y_n be n positive reals. Then,

$$\min\left\{y_1, \frac{1}{y_1} + y_2, \frac{1}{y_2} + y_3, ..., \frac{1}{y_{n-1}} + y_n, \frac{1}{y_n}\right\} \leq 2\cos\frac{\pi}{n+2}. \qquad (7)$$

Proof of Theorem 5. We will prove Theorem 5 by contradiction: Assume that (7) is not valid. Then,

$$\min\left\{y_1, \frac{1}{y_1} + y_2, \frac{1}{y_2} + y_3, ..., \frac{1}{y_{n-1}} + y_n, \frac{1}{y_n}\right\} > 2\cos\frac{\pi}{n+2}. \qquad (8)$$

Define n reals $x_1, x_2, ..., x_n$ by $x_m = \frac{\sin\left((m+1)\frac{\pi}{n+2}\right)}{\sin\left(m\frac{\pi}{n+2}\right)}$ for every $m \in \{1, 2, ..., n\}$.

Then, according to Theorem 4, the reals x_1, x_2, ..., x_n are positive. Besides, $x_1 = 2\cos\frac{\pi}{n+2}$, and the reals x_1, x_2, ..., x_n satisfy the equation (5).

Now we will prove that $y_j > x_j$ for every $j \in \{1, 2, ..., n\}$. This we will prove by induction over j: For $j = 1$, we have to show that $y_1 > x_1$. This, in view of

$x_1 = 2\cos\frac{\pi}{n+2}$, becomes $y_1 > 2\cos\frac{\pi}{n+2}$, what follows from (8). Thus, $y_j > x_j$ is proven for $j = 1$.

Now, for the induction step, we assume that $y_j > x_j$ is proven for some $j \in \{1, 2, ..., n-1\}$. We want to show that we also have $y_{j+1} > x_{j+1}$.

In fact, according to (5), we have $x_1 = \frac{1}{x_j} + x_{j+1}$, what, because of $x_1 = 2\cos\frac{\pi}{n+2}$, comes down to $2\cos\frac{\pi}{n+2} = \frac{1}{x_j} + x_{j+1}$. Since $y_j > x_j$, we have $\frac{1}{x_j} > \frac{1}{y_j}$, so this yields $2\cos\frac{\pi}{n+2} > \frac{1}{y_j} + x_{j+1}$. On the other hand, (8) yields $\frac{1}{y_j} + y_{j+1} > 2\cos\frac{\pi}{n+2}$. Thus, $\frac{1}{y_j} + y_{j+1} > \frac{1}{y_j} + x_{j+1}$, and thus $y_{j+1} > x_{j+1}$ is proven. This completes the induction proof of $y_j > x_j$ for every $j \in \{1, 2, ..., n\}$.

This, in particular, yields $y_n > x_n$, so that $\frac{1}{x_n} > \frac{1}{y_n}$. On the other hand, after (8), we have $\frac{1}{y_n} > 2\cos\frac{\pi}{n+2}$. But $2\cos\frac{\pi}{n+2} = x_1$, and (5) yields $x_1 = \frac{1}{x_n}$. Thus, we get the following chain of inequalities:

$$\frac{1}{x_n} > \frac{1}{y_n} > 2\cos\frac{\pi}{n+2} = x_1 = \frac{1}{x_n}.$$

This chain is impossible to hold. Therefore we get a contradiction, so that our assumption was wrong, and Theorem 5 is proven.