The Neuberg-Mineur circle / Darij Grinberg

1. Introduction and geometrical prerequisites

The impetus for this note was given to me by a *Mathesis* article [6] by Thébault and Mineur from 1931 - more precisely, as I could not locate the article itself, by its review (JFM 57.0773.03) in the Jahrbuch für Mathematik [8], which only mentioned the result of the article without proof. The result - a theorem about quadrilaterals, which, in spite of its simplicity and elegance, has been apparently forgotten ever since - immediately attracted my attention. After some time I had found a proof and an extension of the result, which will make the subject of the following note. However, before we come to the formulation of these theorems, we have to discuss some prerequisites. Readers with sound experience in geometry can skip these and only pick up the notations (the ones defined in **1.** and the definition of P-zero circle in **5.**).

1. *Matters of notation.* I will use the abbreviation "circle $P_1P_2P_3$ " for the circle through three given points P_1, P_2, P_3. Similarly, I will write "circle $P_1P_2...P_n$ " for the circle through n given points P_1, P_2, ..., P_n, if such a circle exists. The distance between two points P and Q will be denoted by PQ.

2. *Directed lengths.* In the following, we will make use of directed lengths (also called signed lengths or directed segments). This means the following:

A *directed line* is defined as a pair $(g;\ \overrightarrow{v_g})$ of a line g and a vector $\overrightarrow{v_g}$ which is parallel to the line g and has the length 1. The line g will be called the *base line* of the directed line $(g;\ \overrightarrow{v_g})$, and the vector $\overrightarrow{v_g}$ will be called the *direction* of this directed line.

For each line g, there exist exactly two vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ which are parallel to this line g and have the length 1. Hence, for the line g, there exist exactly two directed lines which have g as their base line - namely $(g;\ \overrightarrow{v_1})$ and $(g;\ \overrightarrow{v_2})$. We say that we *direct* the line g when we choose one of these two directed lines.

Let $(g;\ \overrightarrow{v_g})$ be a directed line. For any two points A and B on its base line g, we denote by $\overline{AB}$ the real number λ which satisfies $\overrightarrow{AB} = \lambda \cdot \overrightarrow{v_g}$. This number λ is uniquely determined. We refer to $\overline{AB} = \lambda$ as the *directed length* of the segment AB. Hereby, of course, "directed length of the segment AB" is not the same as "directed length of the segment BA".

Obviously, this directed length depends on the direction of the directed line $(g;\ \overrightarrow{v_g})$. Thus, speaking of the directed length of the segment AB makes sense only if A and B are two points on a line which is *directed* (and not just two random points on the plane).

Directed lengths satisfy the following rules (that can be easily verified): Let $(g;\ \overrightarrow{v_g})$ be a directed line.

- For each point A on g, we have $\overline{AA} = 0$.
- For any two points A and B on g, we have $\overline{AB} + \overline{BA} = 0$, and thus $\overline{BA} = -\overline{AB}$.
- For any three points A, B, C on g, we have $\overline{AB} + \overline{BC} + \overline{CA} = 0$ and $\overline{AB} + \overline{BC} = \overline{AC}$.

- For any two points A and B on g, we have $\overline{AB} = AB$ or $\overline{AB} = -AB$. (In fact, by the definition of $\overline{AB}$, we have $\overrightarrow{AB} = \overline{AB} \cdot \overrightarrow{v_g}$, so that $AB = \left|\overrightarrow{AB}\right| = \left|\overline{AB} \cdot \overrightarrow{v_g}\right| = \left|\overline{AB}\right| \cdot \left|\overrightarrow{v_g}\right| = \left|\overline{AB}\right|$ (since $\left|\overrightarrow{v_g}\right| = 1$), and thus $\overline{AB} = AB$ or $\overline{AB} = -AB$.)

- For any two points A and B on g, we have $\overline{AB}^2 = AB^2$.

- For any three points A, B, C on g such that $B \neq C$, we have:
  $\frac{\overline{AB}}{\overline{BC}} = \frac{AB}{BC}$ if the point B lies inside the segment AC;
  $\frac{\overline{AB}}{\overline{BC}} = -\frac{AB}{BC}$ if the point B lies outside the segment AC;
  $\frac{\overline{AB}}{\overline{BC}} = 0$ if $B = A$.
  If $B = C$ and $A \neq C$, then one commonly writes $\frac{\overline{AB}}{\overline{BC}} = \infty$. (Hereby, no difference between $+\infty$ and $-\infty$ is made.)

- For any three points A, B, C on g, we have:
  $\overline{AB} \cdot \overline{AC} = -AB \cdot AC$ if the point A lies inside the segment BC;
  $\overline{AB} \cdot \overline{AC} = AB \cdot AC$ if the point A lies outside the segment BC;
  $\overline{AB} \cdot \overline{AC} = 0$ if $A = B$ or $A = C$.

The latter three rules have the following consequence: The values of $\overline{AB}^2$, $\frac{\overline{AB}}{\overline{BC}}$, and $\overline{AB} \cdot \overline{AC}$ depend on the points A, B, C only and not on the direction of the directed line $\left(g;\ \overrightarrow{v_g}\right)$. Hence, if A, B, C are three points on a line, then we can direct this line in two different ways, but both lead to the same value of $\overline{AB}^2$, to the same value of $\frac{\overline{AB}}{\overline{BC}}$, and to the same value of $\overline{AB} \cdot \overline{AC}$. Hence, we can speak of the terms $\overline{AB}^2$, $\frac{\overline{AB}}{\overline{BC}}$, and $\overline{AB} \cdot \overline{AC}$ for any three points A, B, C which lie on one line, without first having to direct this line (but of course, $\overline{AB}^2$ is just a complicated notation for AB^2).

A helpful property of directed lengths is the *uniqueness of the division ratio*: Let B_1 and B_2 be two points on a line AC. Then, $B_1 = B_2$ holds if and only if $\frac{\overline{AB_1}}{\overline{B_1C}} = \frac{\overline{AB_2}}{\overline{B_2C}}$.

What we will also use is the *intersecting chords theorem for directed lengths*: Let u and v be two lines which intersect at a point P. Let U and U' be two points on the line u, and let V and V' be two points on the line v. Then, $\overline{PU} \cdot \overline{PU'} = \overline{PV} \cdot \overline{PV'}$ holds if and only if there exists a circle which meets the line u at the points U and U' and meets the line v at the points V and V'. [1] This is an easy and basic fact (see [7],

[1] Hereby, we use the following convention:
If a circle k touches a line g at a point T, then we say that the circle k meets the line g at the points T and T.
Thus, the assertion "there exists a circle which meets the line u at the points U and U' and meets the line v at the points V and V' " is stronger than the assertion "the points U, U', V, V' lie on one circle". In fact, in the case when the points U and U' coincide, the latter assertion surely holds, while the former assertion is not necessary to hold (in fact, in this case it is equivalent to the existence of a circle which touches the line u at the point U and meets the line v at the points V and V').

Theorem 18, the equivalence of Assertions $\mathcal{D}_1$ and $\mathcal{D}_3$ for a proof[2]).

3. *Projective geometry and degenerate cases.* In order to understand what happens to geometrical assertions in some degenerate cases, it is helpful to have a concept of the projective plane. What I am going to explain now are some basics of this notion. Note that the projective plane is a fully abstract and strict concept of projective geometry - however, the following explanations aim at an intuitive understanding only, since we are going to use the projective plane as a means of insight and not as an exact theory in this paper.

The projective plane is the Euclidean plane, supplemented with the so-called infinite points and the so-called line at infinity. These have the following properites: Let g be a line (not the line at infinity). All lines parallel to the line g have a common "point", the so-called *infinite point* of the line g, which is, therefore, also the infinite point of each line parallel to g. All infinite points lie on one "line", the so-called *line at infinity*. Of course, infinite points are no points of the Euclidean plane, and the line at infinity is not a line of the Euclidean plane (thus, for instance, it does not make sense to speak of the parallel from a point to the line at infinity, or of the distance between two infinite points), but it is helpful for the intuition to imagine them as points and lines.

A useful convention: If B is the infinite point of a line AC, then we define the directed ratio $\frac{\overline{AB}}{\overline{BC}}$ as follows: $\frac{\overline{AB}}{\overline{BC}} = -1$. Of course, neither $\overline{AB}$ nor $\overline{BC}$ is defined, since B is not a point of the Euclidean plane.

Another helpful conception is that a circle can degenerate to the union of a line with the line at infinity. That is to say: For any line g, we can consider the union of the line g with the line at infinity[3] as a (degenerate) "circle". The center of this circle is an infinite point - namely, the common infinite point of all lines perpendicular to g. The radius of this circle is not defined.[4]

4. *Directed angles modulo* $180°$. Throughout this paper, we will use *directed angles modulo* $180°$, also called *crosses*. A good introduction into this kind of angles can be found in [1] (with [2] as a sequel) or in [3] (§1.7, where directed angles modulo $180°$ are simply referred to as directed angles). See also [4] and [5]. Here we will sketch a definition and basic properties of directed angles without proof:

An *Euclidean pair of lines* will mean a pair $(g;\ h)$, where g and h are two lines, none of which coincides with the line at infinity. Directed angles are equivalence classes of Euclidean pairs of lines, where two Euclidean pairs of lines $(g;\ h)$ and $(g';\ h')$ are said to be equivalent if and only if there exists a direct (i. e., orientation-preserving) congruence transformation that maps the line g to the line g' and maps the line h to the line h'. The equivalence class of an Euclidean pair of lines $(g;\ h)$ is denoted by

[2] [7], Theorem 18 additionally requires the condition that the points U, U', V and V' are distinct from P. However, the case when some of these points coincide with P is easy to handle.

[3] "Union" as in "union of sets", i. e. the set of all points lying on the line g or on the line at infinity.

[4] We note in passing that this conception of a circle degenerating to the union of a line with the line at infinity *makes more sense* than the usual conception that a circle can degenerate to a line. In fact, a circle intersects "many" lines (namely, all of its secants) in two points each, and thus one should also expect this from a degenerate circle. But a line intersects any other line in one point only. In constrast, the union of a line with the line at infinity intersects "most" lines (namely, all lines not passing through the point of intersection of the line with the line at infinity) in two points.

$\measuredangle(g;\ h)$ and called *directed angle between the lines g and h.* [5]

We define a directed angle $0°$ such that $\measuredangle(g;\ g) = 0°$ holds for each line g. We define addition and subtraction of directed angles such that directed angles form an Abelian group under addition (with $0°$ as the neutral element), such that $\measuredangle(g;\ h) = -\measuredangle(h;\ g)$ holds for any two lines g and h, and such that

$$\measuredangle(g_1;\ g_2) + \measuredangle(g_2;\ g_3) + ... + \measuredangle(g_{n-1};\ g_n) = \measuredangle(g_1;\ g_n) \quad \text{and}$$
$$\measuredangle(g_1;\ g_2) + \measuredangle(g_2;\ g_3) + ... + \measuredangle(g_{n-1};\ g_n) + \measuredangle(g_n;\ g_1) = 0°$$

holds for any n lines $g_1, g_2, ..., g_n$. It is easily seen that for two lines g and h, we have $\measuredangle(g;\ h) = 0°$ if and only if $g \parallel h$.

A directed angle $90°$ is defined in a way that two lines g and h satisfy $\measuredangle(g;\ h) = 90°$ if and only if $g \perp h$. We note that this angle $90°$ satisfies $90° + 90° = 0°$, that is, $90° = -90°$.

For each directed angle φ and each positive integer n, we define the angle $n\varphi$ as $\underbrace{\varphi + \varphi + ... + \varphi}_{n \text{ terms}}$, the angle 0φ as $0°$, and the angle $(-n)\varphi$ as $-(n\varphi)$.

For any three points A, B, C in the plane which satisfy $A \neq B$ and $B \neq C$, we define $\measuredangle ABC$ as an abbreviation for $\measuredangle(AB;\ BC)$. This angle $\measuredangle ABC$ is well-defined even if some of the points A, B, C are infinite points, *as long as* none of the two lines AB and BC coincides with the line at infinity! Some important properties of directed angles are:

- Let u and v be two lines through a point P. Let U and U' be two points on the line u different from P, and let V and V' be two points on the line v different from P. Then,

$$\measuredangle UPV = \measuredangle U'PV' = -\measuredangle VPU = -\measuredangle V'PU' = \measuredangle(u;\ v) = -\measuredangle(v;\ u).$$

- Let A, B, C be three points such that $A \neq B$ and $B \neq C$, and such that B is not an infinite point. The points A, B, C lie on one line if and only if $\measuredangle ABC = 0°$.

- Let A, B, C_1, C_2 be four points such that $A \neq B$, $B \neq C_1$ and $B \neq C_2$, and such that B is not an infinite point. The points B, C_1, C_2 lie on one line if and only if $\measuredangle ABC_1 = \measuredangle ABC_2$.

- *Difference exchange formula.* For any four lines a, b, c, d, we have $\measuredangle(a;\ b) - \measuredangle(c;\ d) = \measuredangle(a;\ c) - \measuredangle(b;\ d)$.

- Let g, h, g', h' be four lines with $g \parallel g'$. Then, $\measuredangle(g;\ h) = \measuredangle(g';\ h')$ holds if and only if $h \parallel h'$.

- *Chordal angle theorem.* Let A, B, C, D be four points such that the directed angles $\measuredangle ACB$ and $\measuredangle ADB$ are well-defined. Then, $\measuredangle ACB = \measuredangle ADB$ holds if and only if the four points A, B, C, D lie on one circle.

[5] Of course, the order of the lines is important here - the directed angle between the lines g and h is not the same as the directed angle between the lines h and g.

- *Tangent-chordal angle theorem.* Let A, B, C be three points on a circle, and t the tangent to this circle at the point A. Then, $\measuredangle(t;\ AB) = \measuredangle ACB$ and $\measuredangle(AB;\ t) = \measuredangle BCA$.

- *Central angle theorem.* Let A, B, C be three points on a circle, and let O be the center of this circle. Then, $\measuredangle OBA = \measuredangle BAO = 90° - \measuredangle ACB$.

- *Similitude of triangles.* Let ABC and $A'B'C'$ be two non-degenerate triangles, none of whose vertices are infinite points.[6]
  The triangles ABC and $A'B'C'$ are directly similar (i. e. one can be obtained from the other one through an orientation-preserving similitude) if and only if $\measuredangle ABC = \measuredangle A'B'C'$ and $\measuredangle ACB = \measuredangle A'C'B'$.
  The triangles ABC and $A'B'C'$ are oppositely similar (i. e. one can be obtained from the other one through an orientation-reversing similitude) if and only if $\measuredangle ABC = -\measuredangle A'B'C'$ and $\measuredangle ACB = -\measuredangle A'C'B'$.

The main advantage of directed angles is - as with directed lengths - their help in simplifying proofs: Using directed angles, one can avoid much casework due to the arrangement of the points.

5. *Zero circles.* In the second proof of Theorem 4, we are going to work with so-called "zero circles". A *zero circle* is simply a circle with radius 0, and all that is required from the reader is not to fear dealing with such circles just as with usual circles. Concretely, zero circles have the following properties:

For each point P (which is not an infinite point), there exists exactly one circle with center P and radius 0. This circle will be called the *P-zero circle*. This circle passes through one point only, namely through the point P. Tangents to this circle are all lines through the point P. The power of a point Q with respect to the P-zero circle is QP^2.

2. The main results

We start with two facts belonging to common knowledge among triangle geometers. The first one will be used as a lemma in our later arguments (Fig. 1):

Theorem 1. Let ABC be a triangle. The tangent to the circle ABC at the point B intersects the line CA at a point Y. Then,

$$\frac{\overline{CY}}{\overline{YA}} = -\frac{BC^2}{AB^2}.$$

In words: The tangent to the circumcircle of a triangle at a vertex of this triangle divides the opposite side externally in the ratio of the squares of the adjacent sides.

[6] A triangle is called *degenerate* if there exists a line passing through all its three vertices. We have to exclude degenerate triangles here, since there are no similitude criteria for degenerate triangles which rely on angles only.

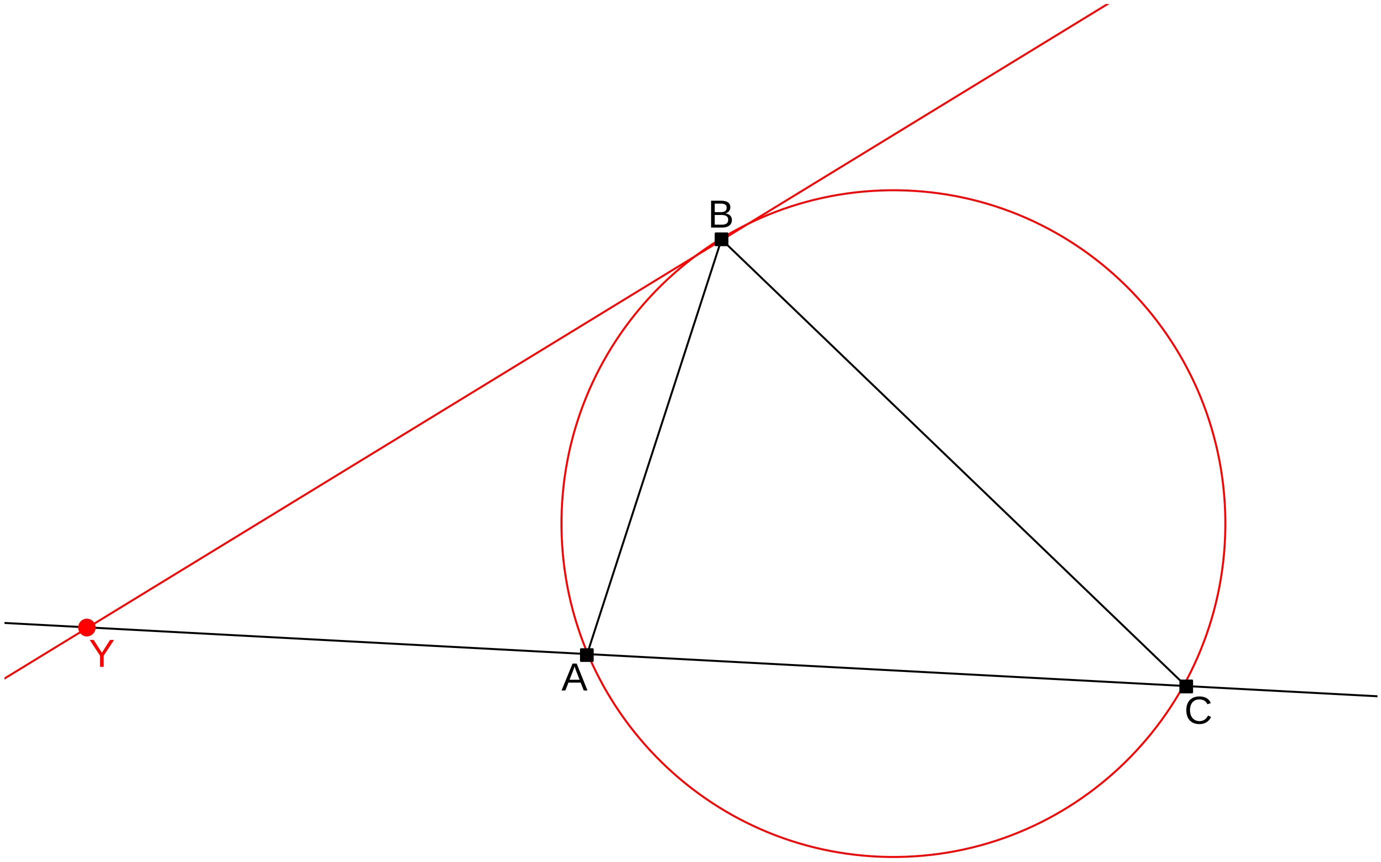

Fig. 1

Proof. Since the line BY is the tangent to the circle ABC at B, the tangent-chordal angle theorem yields $\measuredangle(AB;\ BY) = \measuredangle ACB$. Since $\measuredangle(AB;\ BY) = \measuredangle ABY$ and $\measuredangle ACB = -\measuredangle BCY$, this becomes $\measuredangle ABY = -\measuredangle BCY$. Further, obviously $\measuredangle AYB = -\measuredangle BYC$. Hence, the triangles ABY and BCY are oppositely similar, and thus

$$\frac{YC}{YB} = \frac{BC}{AB} \qquad \text{and} \qquad \frac{YB}{YA} = \frac{BC}{AB}.$$

Multiplication of these two equations yields

$$\frac{YC}{YB} \cdot \frac{YB}{YA} = \left(\frac{BC}{AB}\right)^2, \qquad \text{hence} \qquad \frac{YC}{YA} = \left(\frac{BC}{AB}\right)^2, \qquad \text{and thus}$$

$$\frac{CY}{YA} = \frac{YC}{YA} = \left(\frac{BC}{AB}\right)^2 = \frac{BC^2}{AB^2}.$$

Now, since the point Y lies outside the segment CA (else, Y would lie inside the circle ABC, but this is impossible since the line BY is tangent to this circle), we have $\frac{\overline{CY}}{\overline{YA}} = -\frac{CY}{YA}$, so that $\frac{\overline{CY}}{\overline{YA}} = -\frac{BC^2}{AB^2}$. This proves Theorem 1.

Now, considering not just the tangent to the circle ABC at B, but also the similar tangents at C and A, we come to the following result (Fig. 2):

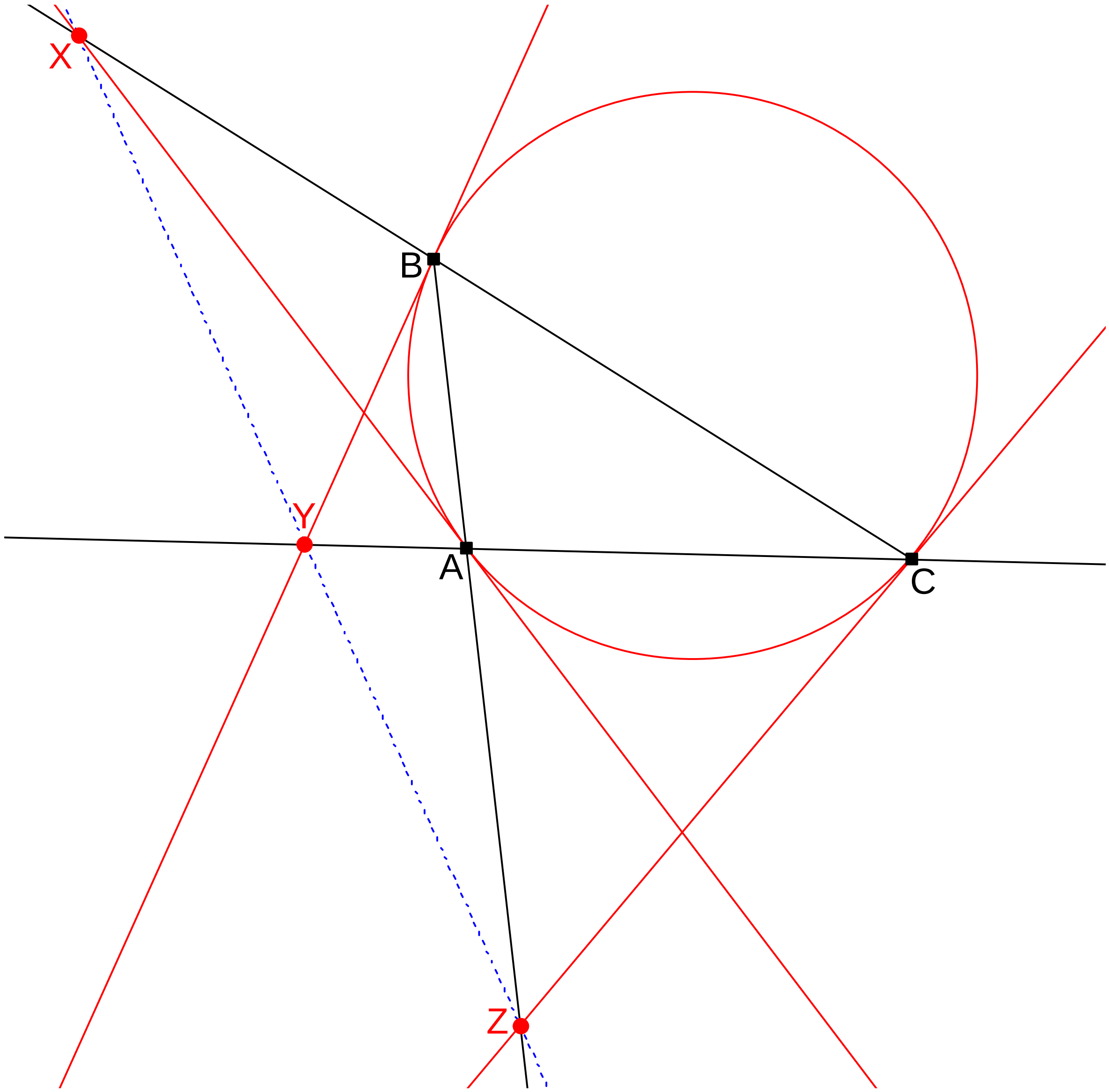

Fig. 2

Theorem 2. Let ABC be a triangle. The tangents to the circle ABC at the points A, B, C intersect the lines BC, CA, AB at the points X, Y, Z, respectively.

a) We have

$$\frac{\overline{BX}}{\overline{XC}} = -\frac{AB^2}{CA^2}; \qquad \frac{\overline{CY}}{\overline{YA}} = -\frac{BC^2}{AB^2}; \qquad \frac{\overline{AZ}}{\overline{ZB}} = -\frac{CA^2}{BC^2}.$$

b) The points X, Y, Z lie on one line.

This line is called the **Lemoine axis** of triangle ABC.

Proof of Theorem 2. Theorem 1 yields $\frac{\overline{CY}}{\overline{YA}} = -\frac{BC^2}{AB^2}$, and similarly we have

$\dfrac{\overline{BX}}{\overline{XC}} = -\dfrac{AB^2}{CA^2}$ and $\dfrac{\overline{AZ}}{\overline{ZB}} = -\dfrac{CA^2}{BC^2}$. This proves Theorem 2 **a)**. From

$$\frac{\overline{BX}}{\overline{XC}} \cdot \frac{\overline{CY}}{\overline{YA}} \cdot \frac{\overline{AZ}}{\overline{ZB}} = \left(-\frac{AB^2}{CA^2}\right) \cdot \left(-\frac{BC^2}{AB^2}\right) \cdot \left(-\frac{CA^2}{BC^2}\right) = -1,$$

it follows using the Menelaos theorem (applied to the triangle ABC and the points X, Y, Z on its sidelines BC, CA, AB) that the points X, Y, Z lie on one line. Thus, Theorem 2 **b)** is proven as well, and hence our proof of Theorem 2 is complete.

The above proof of Theorem 2 **b)** is not the only one imaginable; literature abounds in proofs using polars, the Pascal theorem and other methods.

We can interpret Theorem 2 as a characterization of the points on the sidelines of a triangle ABC which divide the respective sides externally in the ratios of the squares of the adjacent sides - these points are the points X, Y, Z and lie on one line. What can be said about points that divide the sides of a *quadrilateral* externally in the ratios of the squares of the adjacent sides? Here is where the result of [6] emerges (Fig. 3):

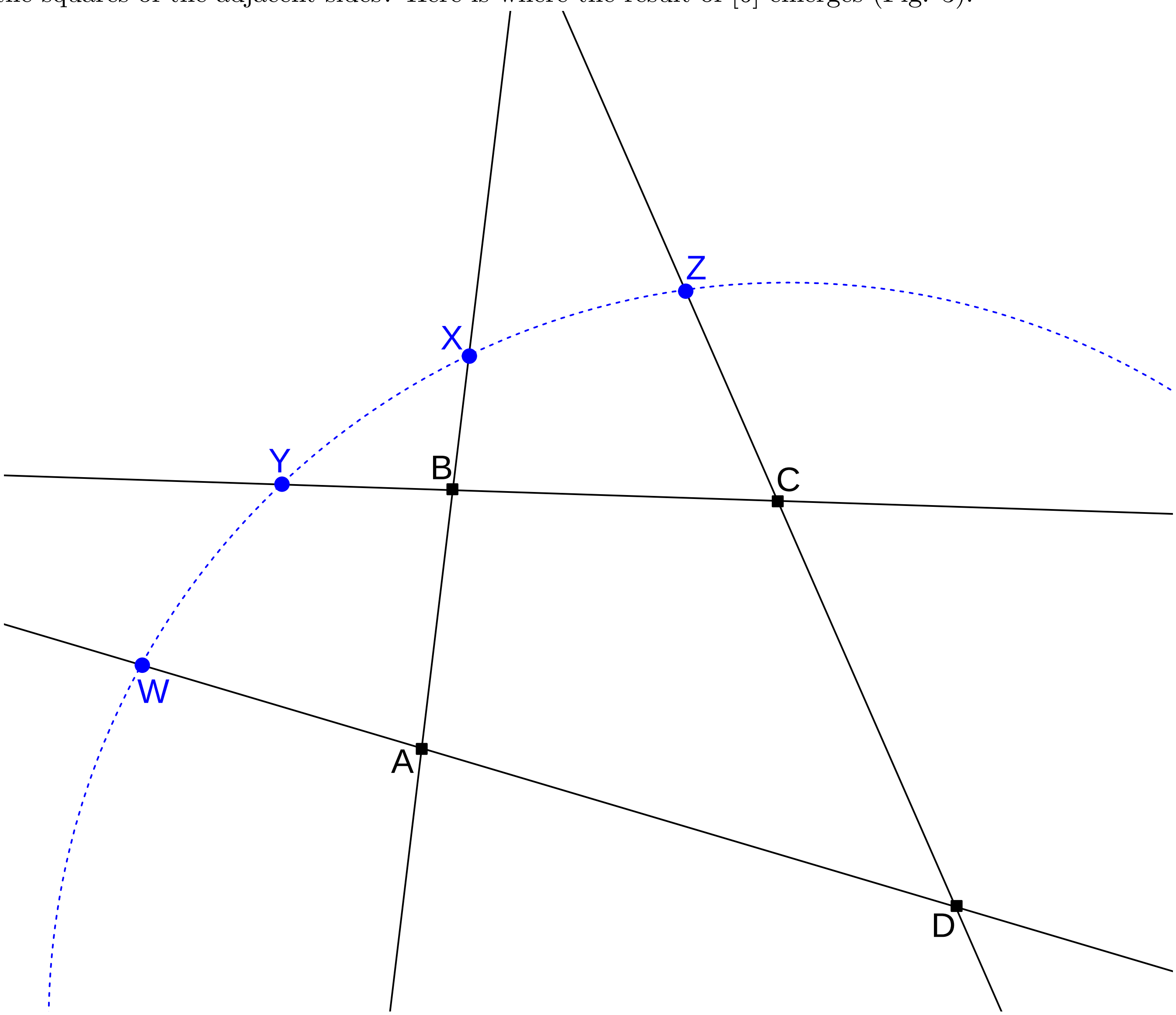

Fig. 3

Theorem 3, the Neuberg-Mineur theorem. Let $ABCD$ be a quadrilateral, and let X, Y, Z, W be the points on the lines AB, BC, CD, DA which divide the sides AB, BC, CD, DA externally in the ratios of the squares of the adjacent sides, i. e. which satisfy

$$\frac{\overline{AX}}{\overline{XB}} = -\frac{DA^2}{BC^2}; \qquad \frac{\overline{BY}}{\overline{YC}} = -\frac{AB^2}{CD^2}; \qquad \frac{\overline{CZ}}{\overline{ZD}} = -\frac{BC^2}{DA^2}; \qquad \frac{\overline{DW}}{\overline{WA}} = -\frac{CD^2}{AB^2}.$$

Then, the points X, Y, Z, W lie on one circle.

We will refer to this circle as the **Neuberg-Mineur circle** of the quadrilateral $ABCD$.

What was not mentioned in the review of [6], but reveals itself upon experimentation (Fig. 4):

Theorem 4. In the configuration of Theorem 3, we have: If $ABCD$ is a cyclic quadrilateral, then the points X, Y, Z, W lie on one line.

That is, the Neuberg-Mineur circle of a cyclic quadrilateral degenerates to the union of a line with the line at infinity.

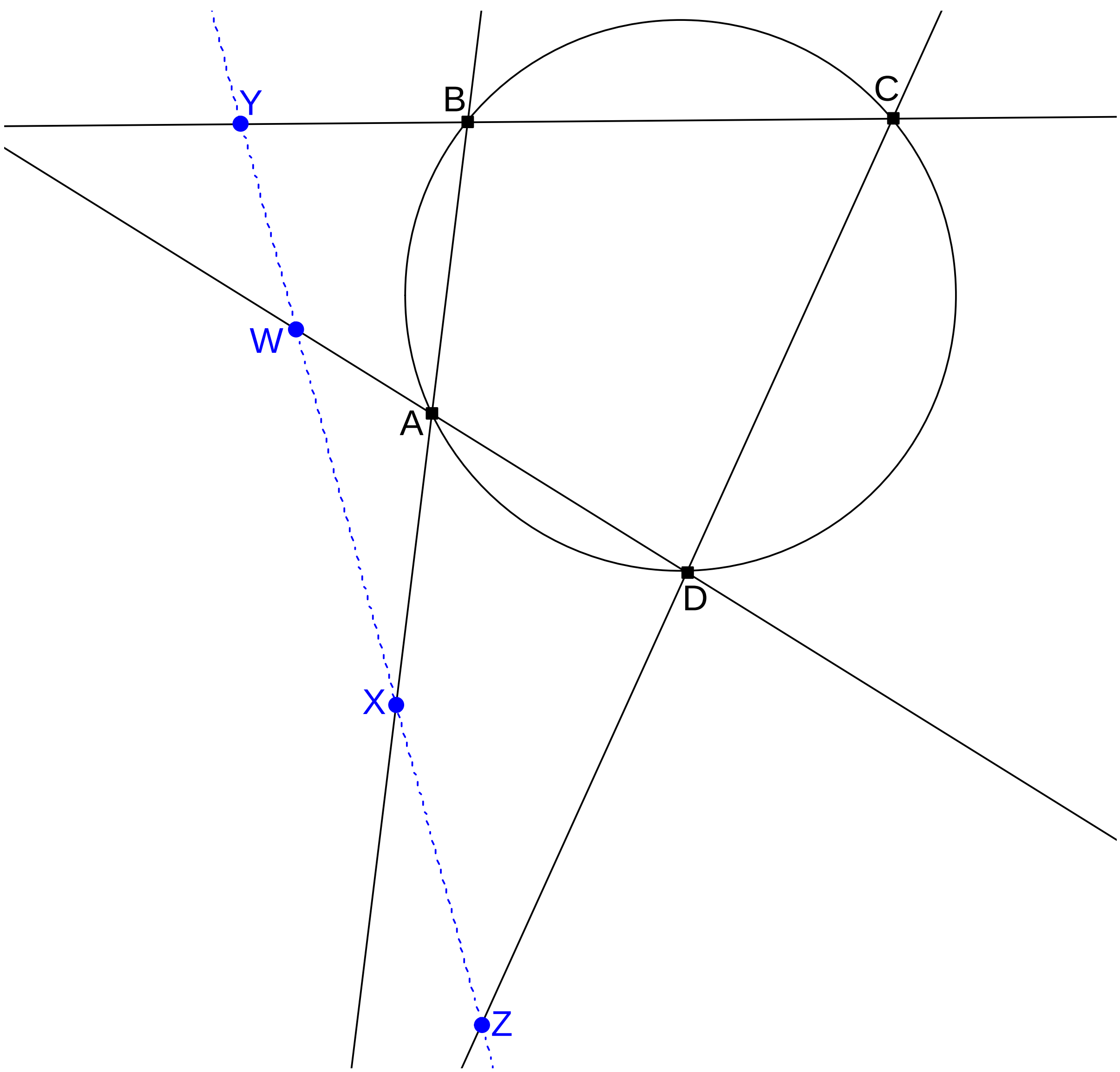

Fig. 4

3. A lemma on directly similar triangles

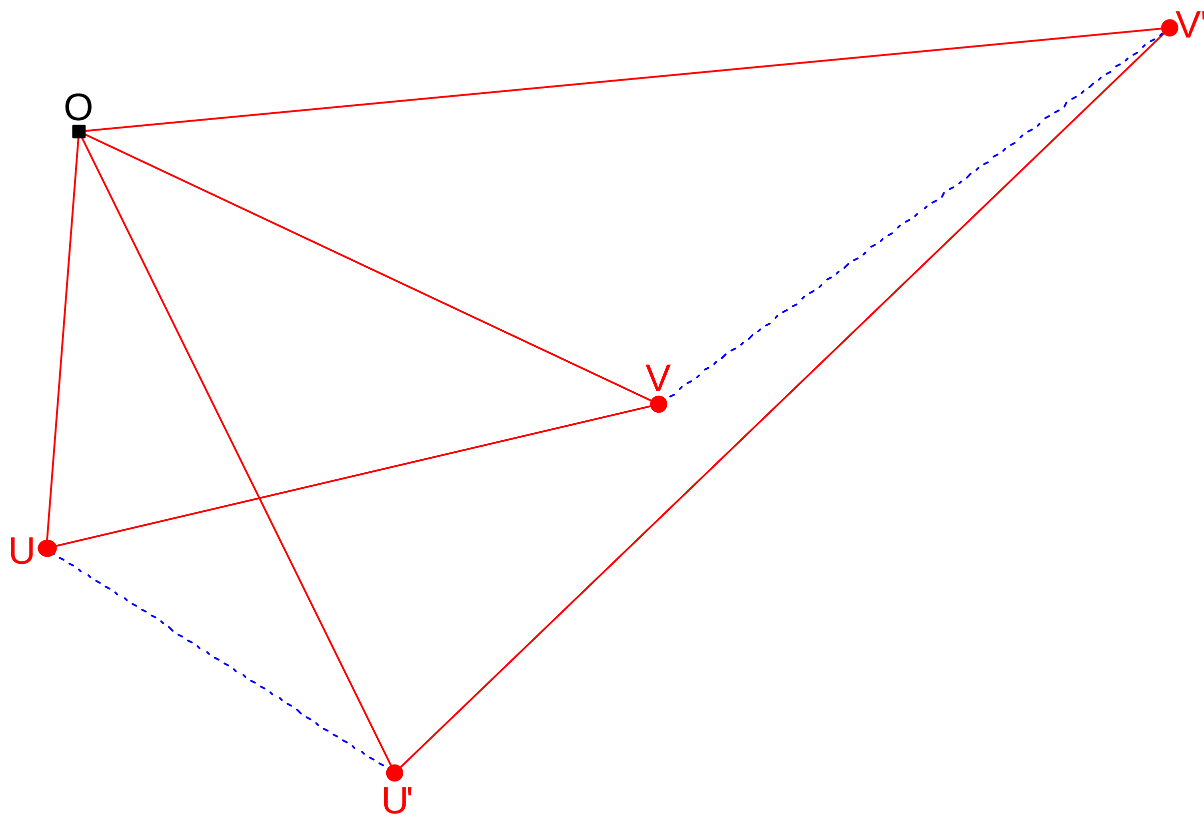

Fig. 5

Before we come to proving Theorems 3 and 4, we establish a very simple lemma (Fig. 5):

> **Theorem 5.** Let O, U, V, U', V' be five points such that the triangles OUV and $OU'V'$ are directly similar. Then, the triangles OUU' and OVV' are directly similar.

Proof of Theorem 5. Theorem 5 can be easily proven with the aid of non-directed angles or directed angles modulo 360°. However, so as to stay consequent, we are going to use directed angles modulo 180°. With this kind of angles, Theorem 5 is slightly harder to prove. Here is one such proof:

(See Fig. 6.) We assume that the triangles OUV, $OU'V'$, OUU', OVV' are all non-degenerate.[7] Let P be the point of intersection of the lines UV and $U'V'$. Since the triangles OUV and $OU'V'$ are directly similar, we have $\measuredangle OUV = \measuredangle OU'V'$. In other words, $\measuredangle OUP = \measuredangle OU'P$. Thus, the points O, P, U, U' lie on one circle. Thus, $\measuredangle OUU' = \measuredangle OPU'$. Similarly, $\measuredangle OVV' = \measuredangle OPV'$. Now, $\measuredangle OPU' = \measuredangle OPV'$. Hence, $\measuredangle OUU' = \measuredangle OVV'$. Similarly, $\measuredangle OU'U = \measuredangle OV'V$. Thus, the triangles OUU' and OVV' are directly similar, and Theorem 5 is proven.

[7] The case of degenerate triangles is left to the reader. Note that the similitude of degenerate triangles cannot be proven using angles alone - here you need to consider ratios of lengths (and directed lengths can be of good use).

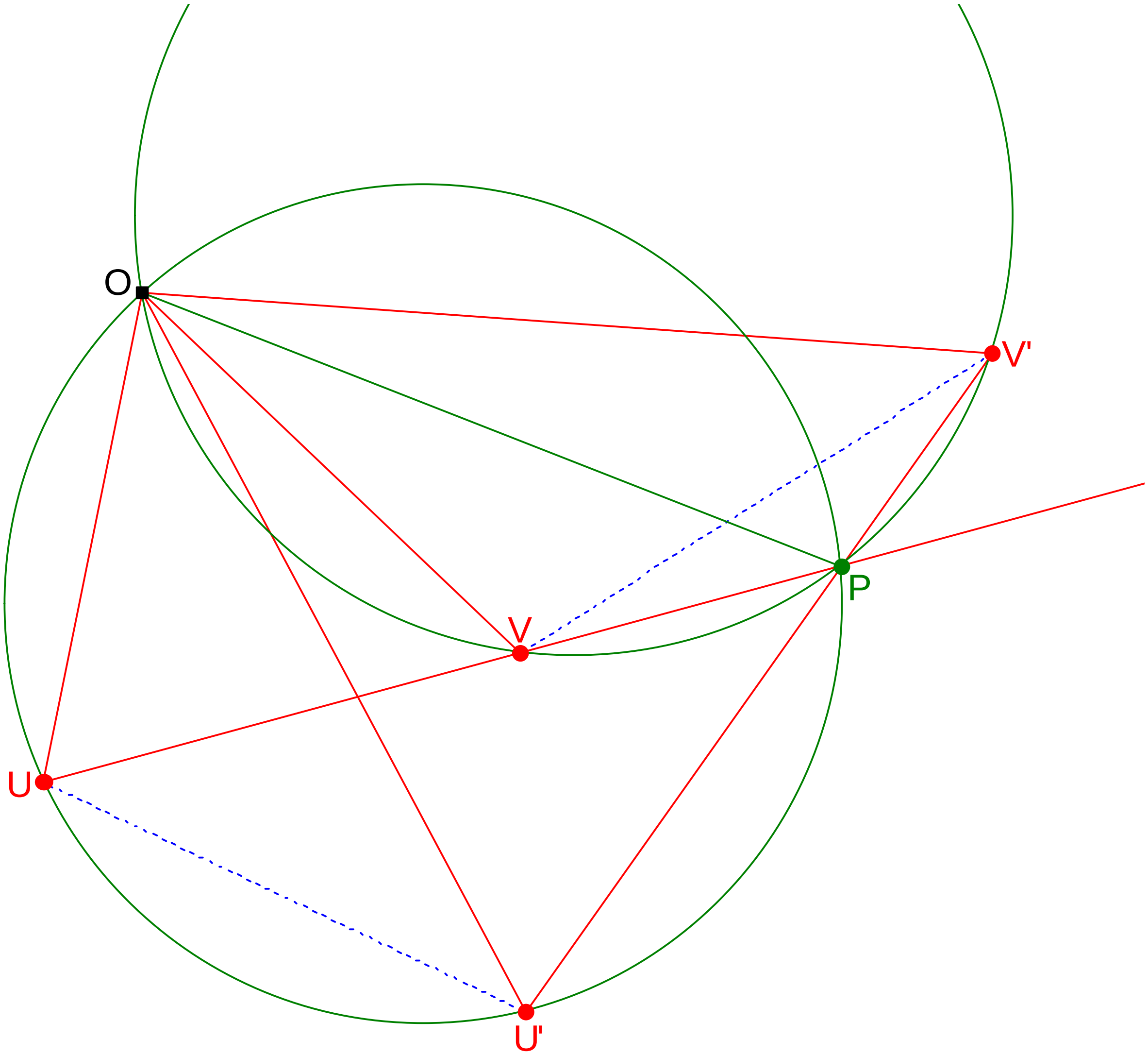

Fig. 6

4. Proof of Theorem 3

Now we can start with the *proof of Theorem 3*. In the configuration of Theorem 3, we consider the points of intersection of opposite sidelines of the quadrilateral $ABCD$. Let E be the point of intersection of the lines AB and CD, and let F be the point of intersection of the lines BC and DA.

(See Fig. 7.) The circles EBC and EDA intersect at the point E and at a second point; we denote this second point by P. [8]. Then, $\measuredangle PBC = \measuredangle PEC$ (chordal angle theorem in the circle $EBCP$) and $\measuredangle PED = \measuredangle PAD$ (chordal angle theorem in the circle $EDAP$). Hence, $\measuredangle PBC = \measuredangle PEC = \measuredangle PED = \measuredangle PAD$, so that $\measuredangle PBF = \measuredangle PBC = \measuredangle PAD = \measuredangle PAF$. Consequently, the point P lies on the circle FAB. Similarly, the point P lies on the circle FCD. Altogether, the point P thus lies on the circles EBC, EDA, FAB, FCD. Thus we have shown:

[8] If the circles EBC and EDA touch each other at the point E, then we set $P = E$.

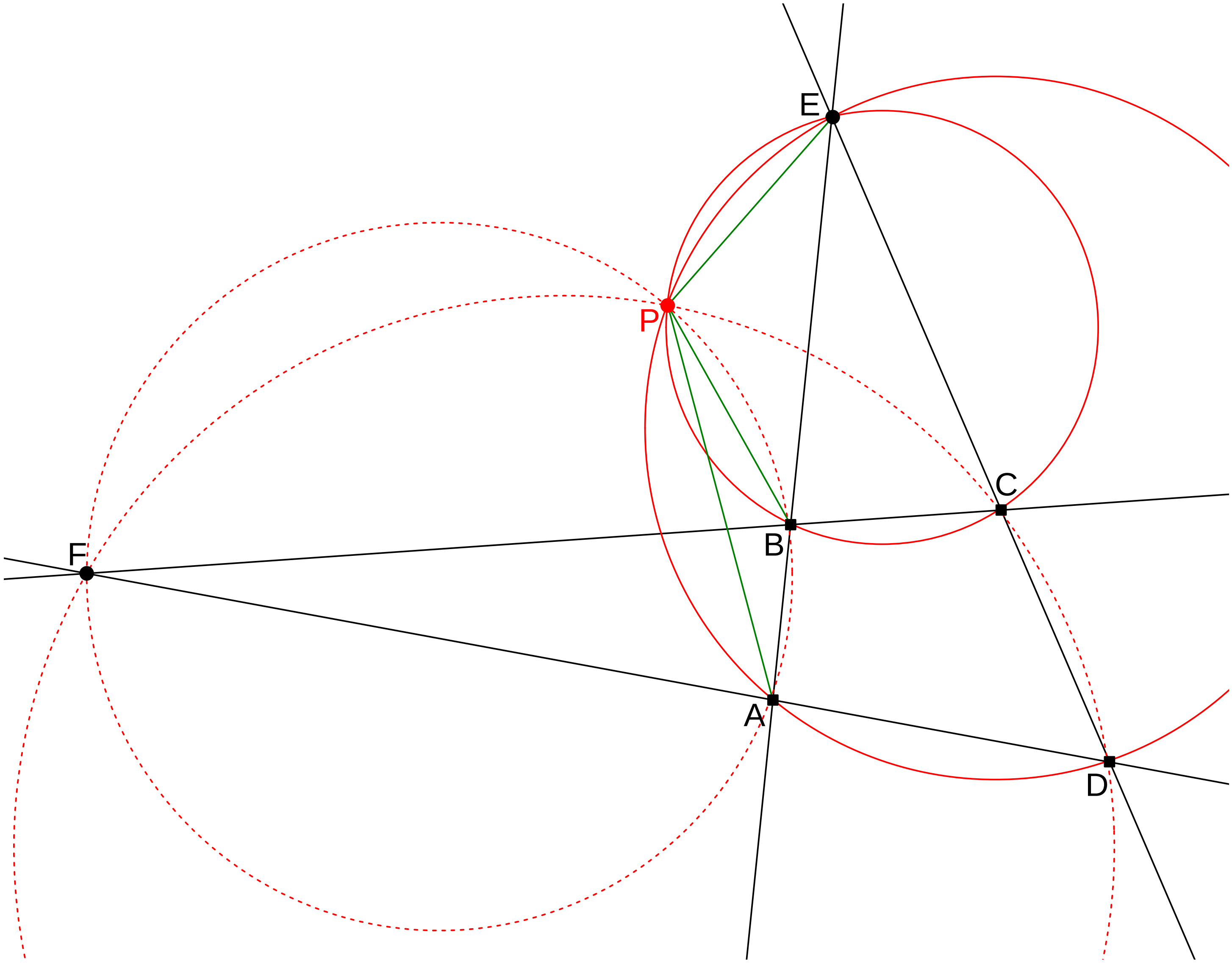

Fig. 7

Theorem 6, the Steiner-Miquel theorem. Let $ABCD$ be a quadrilateral. Let E be the point of intersection of the lines AB and CD, and let F be the point of intersection of the lines BC and DA. Then, the circles EBC, EDA, FAB, FCD have a common point P.

This point P is called the **Miquel point** of the quadrilateral $ABCD$ (or, equivalently, the Miquel point of the four lines AB, BC, CD, DA). (See Fig. 8.)

In other words: Four arbitrary lines in the plane form four triangles (which are obtained if one considers each group of three out of the four lines). The circumcircles of these triangles have a common point (the so-called Miquel point of the four lines).

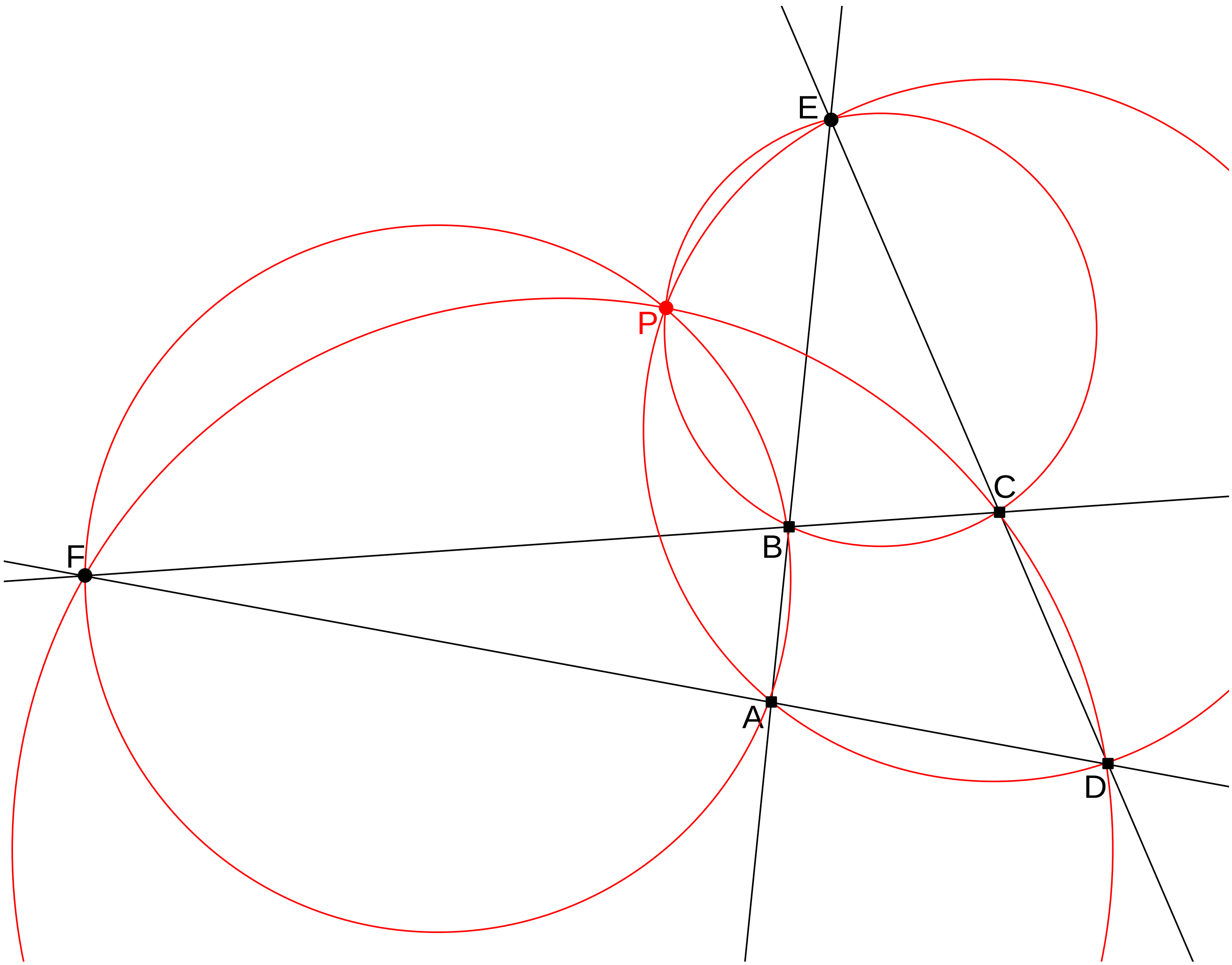

Fig. 8

(See Fig. 9.) During the proof of Theorem 6, we have showed that $\measuredangle PBC = \measuredangle PAD$. Similarly, $\measuredangle PCB = \measuredangle PDA$. Thus, the triangles PBC and PAD are directly similar. Similarly, the triangles PCD and PBA are directly similar (sorry for the pun). We formulate this as a theorem for further use:

Theorem 7. In the configuration of Theorem 6, we have: The triangles PBC and PAD are directly similar; the triangles PCD and PBA are directly similar.

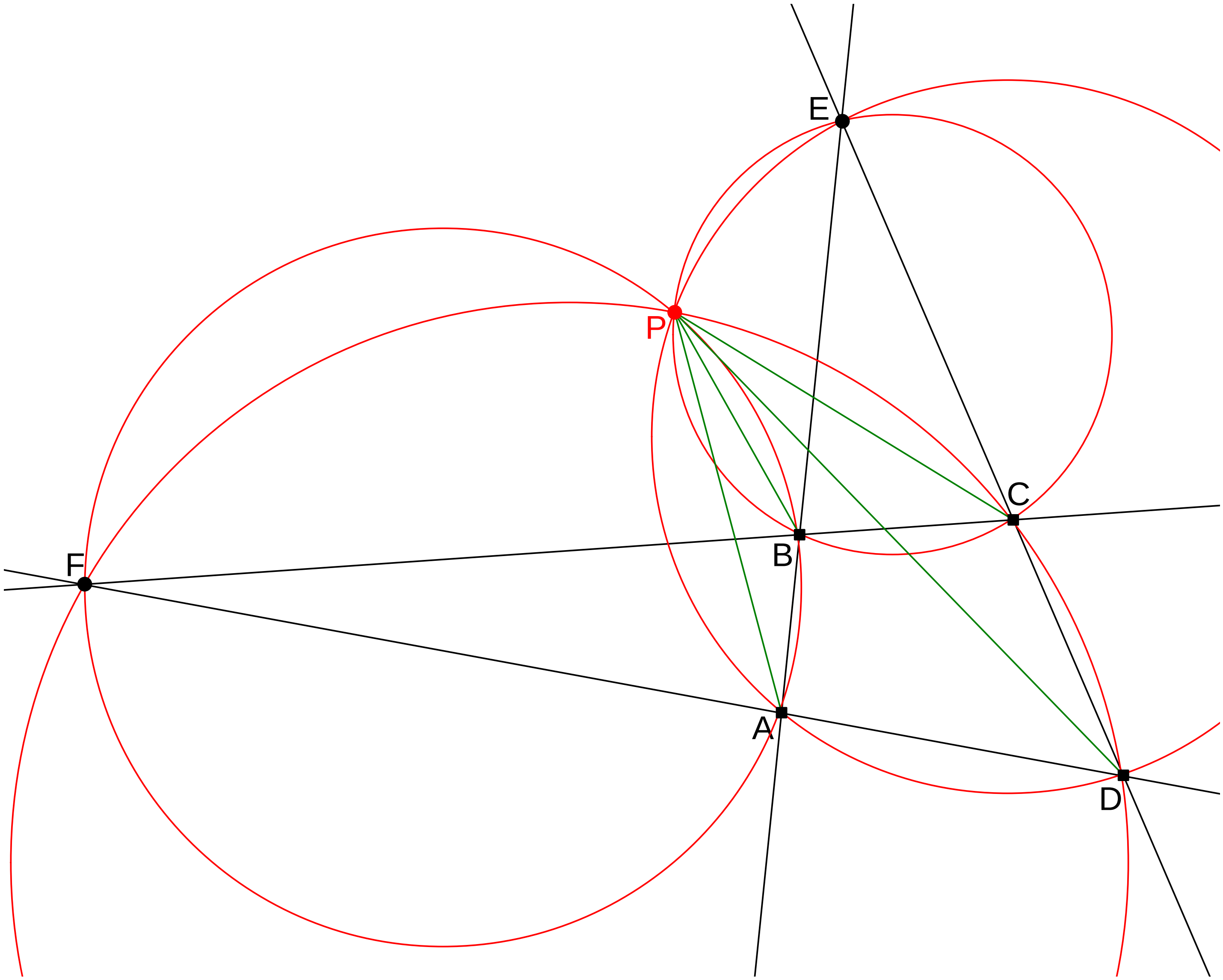

Fig. 9

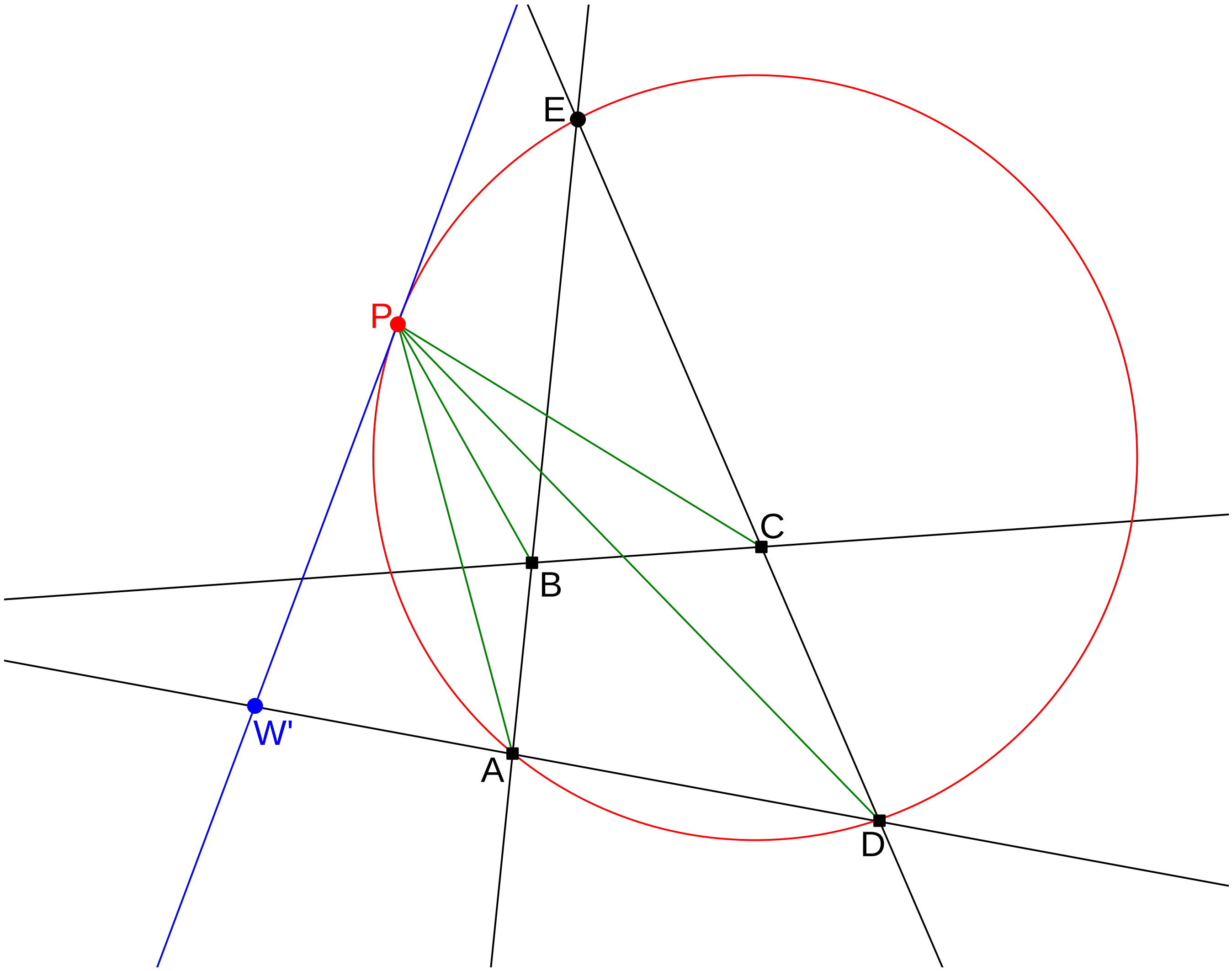

Fig. 10

(See Fig. 10.) Since the triangles PCD and PBA are directly similar, we have $\frac{PD}{PA} = \frac{CD}{BA}$, so that $\frac{PD}{AP} = \frac{CD}{AB}$. Let the tangent to the circle $EDAP$ at the point P intersect the line DA at a point W'. Then, W' is the point of intersection of the tangent to the circle PDA at the point P with the line DA; hence, Theorem 1 yields

$$\frac{\overline{DW'}}{\overline{W'A}} = -\frac{PD^2}{AP^2} = -\left(\frac{PD}{AP}\right)^2 = -\left(\frac{CD}{AB}\right)^2 = -\frac{CD^2}{AB^2} = \frac{\overline{DW}}{\overline{WA}}.$$

Hence, the points W' and W coincide. Since W' was defined as the point of intersection of the tangent to the circle $EDAP$ at the point P with the line DA, we can thus conclude that the point W is the point of intersection of the tangent to the circle $EDAP$ at the point P with the line DA. Similar results can be inferred about the points X, Y, Z. We combine:

Theorem 8. In the configuration of Theorems 3 and 6[9], we have: The points X, Y, Z, W are the points of intersection of the tangents to the

[9] This is the configuration consisting of the quadrilateral $ABCD$, the points X, Y, Z, W defined in Theorem 3, and the points E, F, P defined in Theorem 6.

circles $FABP$, $EBCP$, $FCDP$, $EDAP$ at the point P with the lines AB, BC, CD, DA, respectively. (See Fig. 11.)

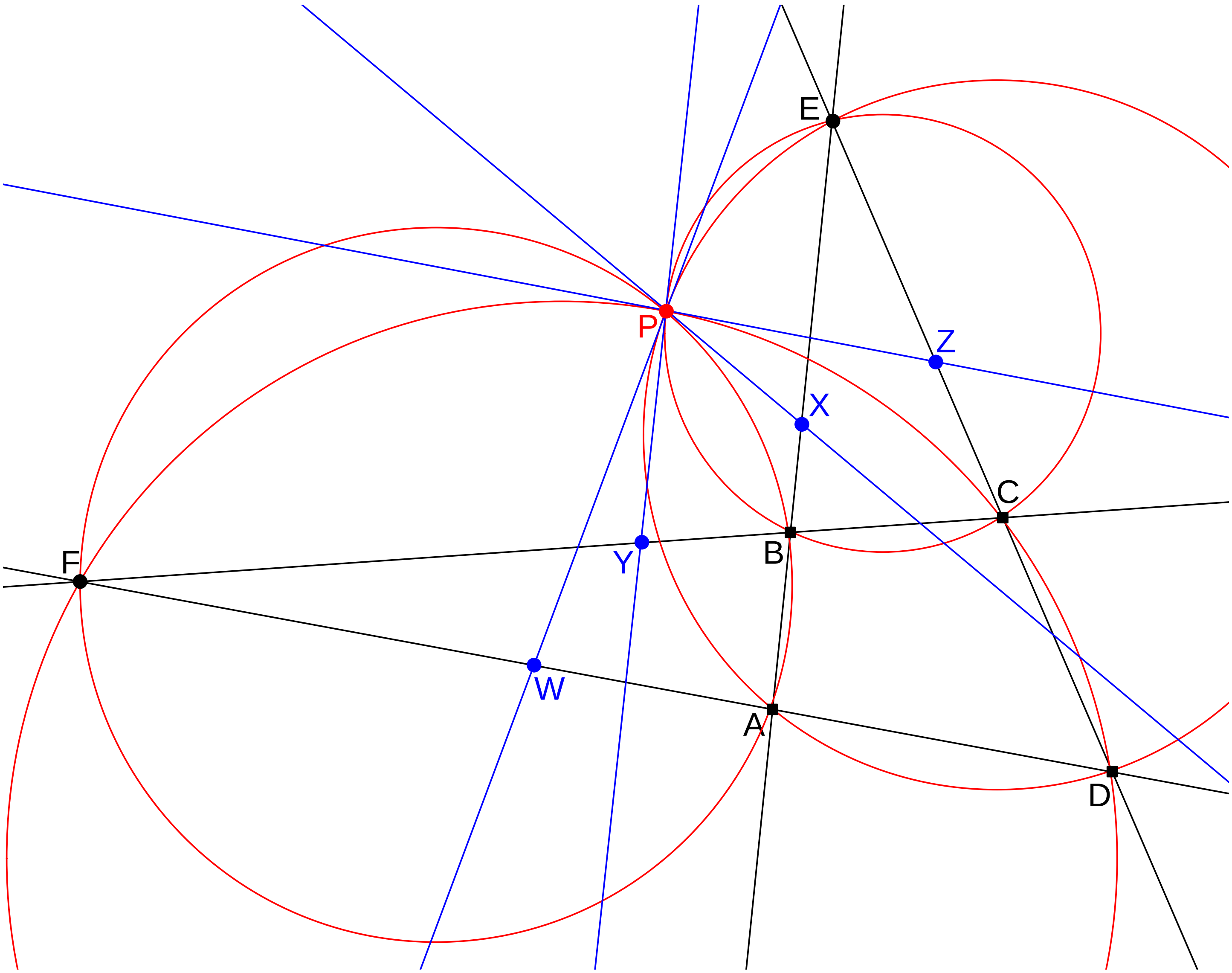

Fig. 11

(See Fig. 12.) The triangles PCD and PBA are directly similar (after Theorem 7). But, according to Theorem 8, the point Z is the point of intersection of the tangent to the circle PCD at P with the line CD, and the point X is the point of intersection of the tangent to the circle PBA at P with the line BA. Hence, the points Z and X are corresponding points in the triangles PCD and PBA. Corresponding points in similar triangles form similar triangles themselves; since the triangles PCD and PBA are directly similar, this yields that the triangles PCZ and PBX are directly similar. Hence, $\measuredangle CZP = \measuredangle BXP$.

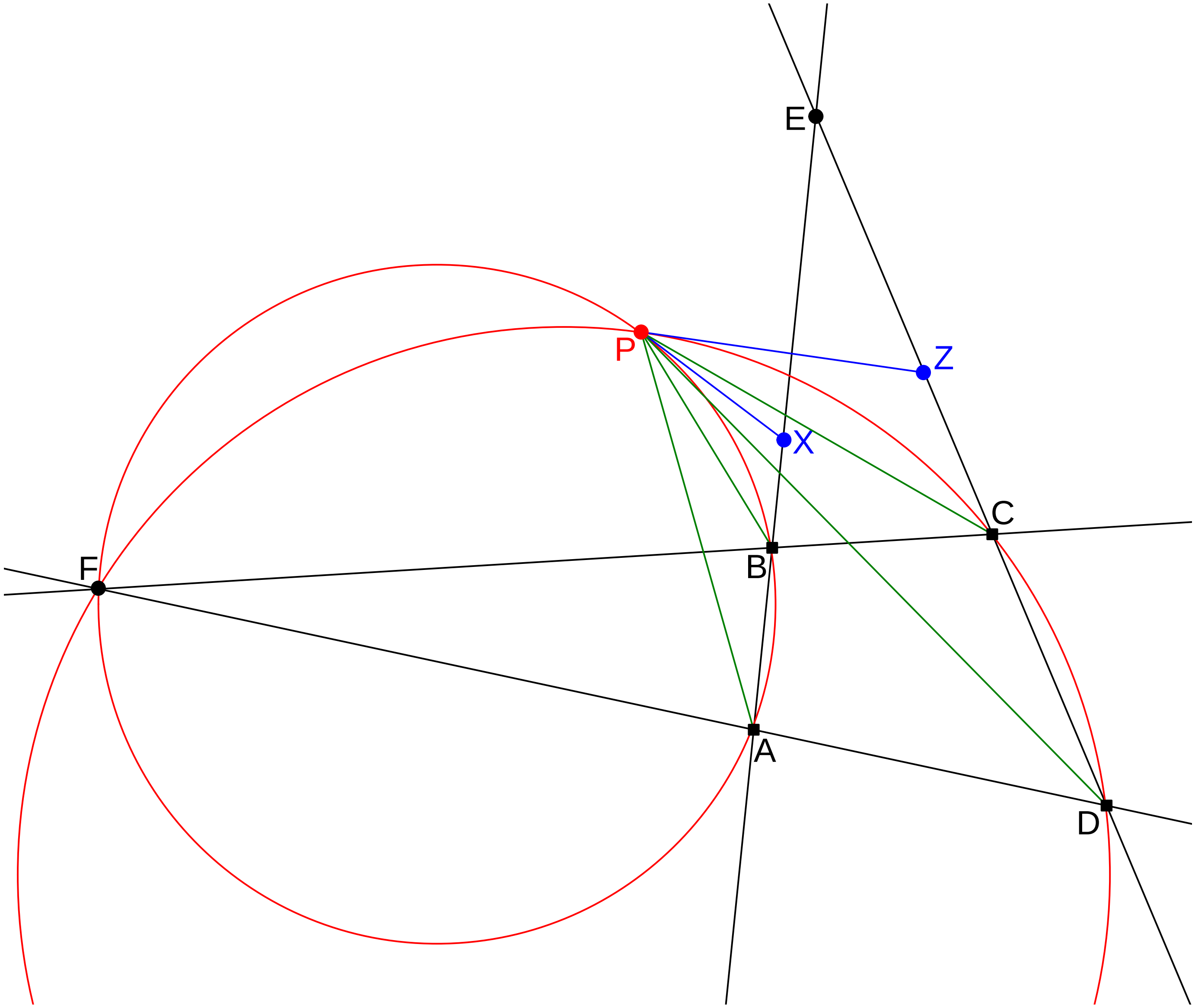

Fig. 12

The equation $\measuredangle CZP = \measuredangle BXP$ rewrites as $\measuredangle EZP = \measuredangle EXP$; hence, the point P lies on the circle EZX. Similarly, the point P lies on the circle FYW. We combine:

Theorem 9. In the configuration of Theorems 3 and 6, we have: The point P lies on the circles EZX and FYW. (See Fig. 13.)

Actually, more can be said:

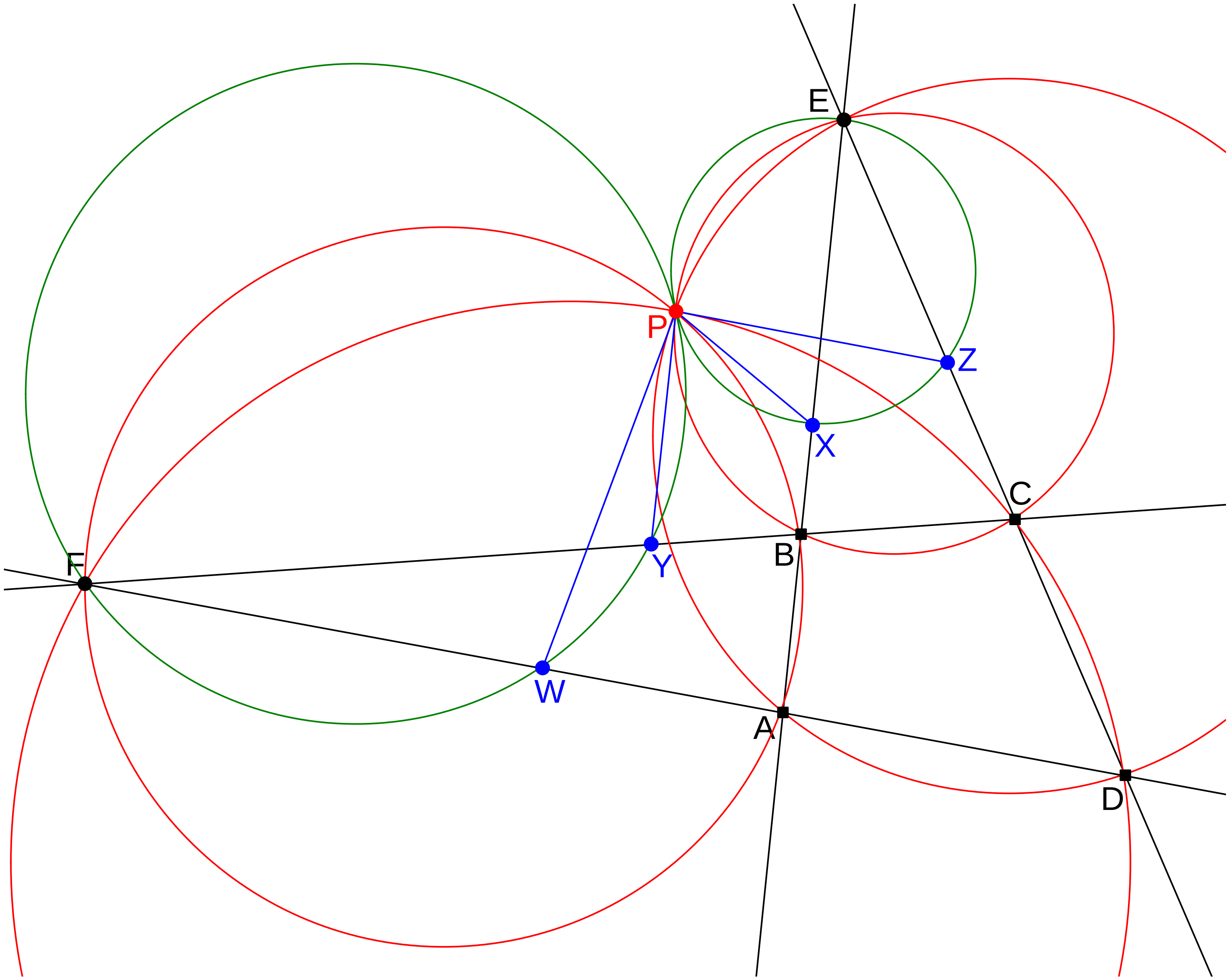

Fig. 13

Theorem 10. In the configuration of Theorems 3 and 6, we have: The circles EZX and FYW touch each other at the point P.

Proof. Let t_1 and t_2 be the tangents to the circles EZX and FYW at the point P. We want to show that these two tangents t_1 and t_2 coincide.

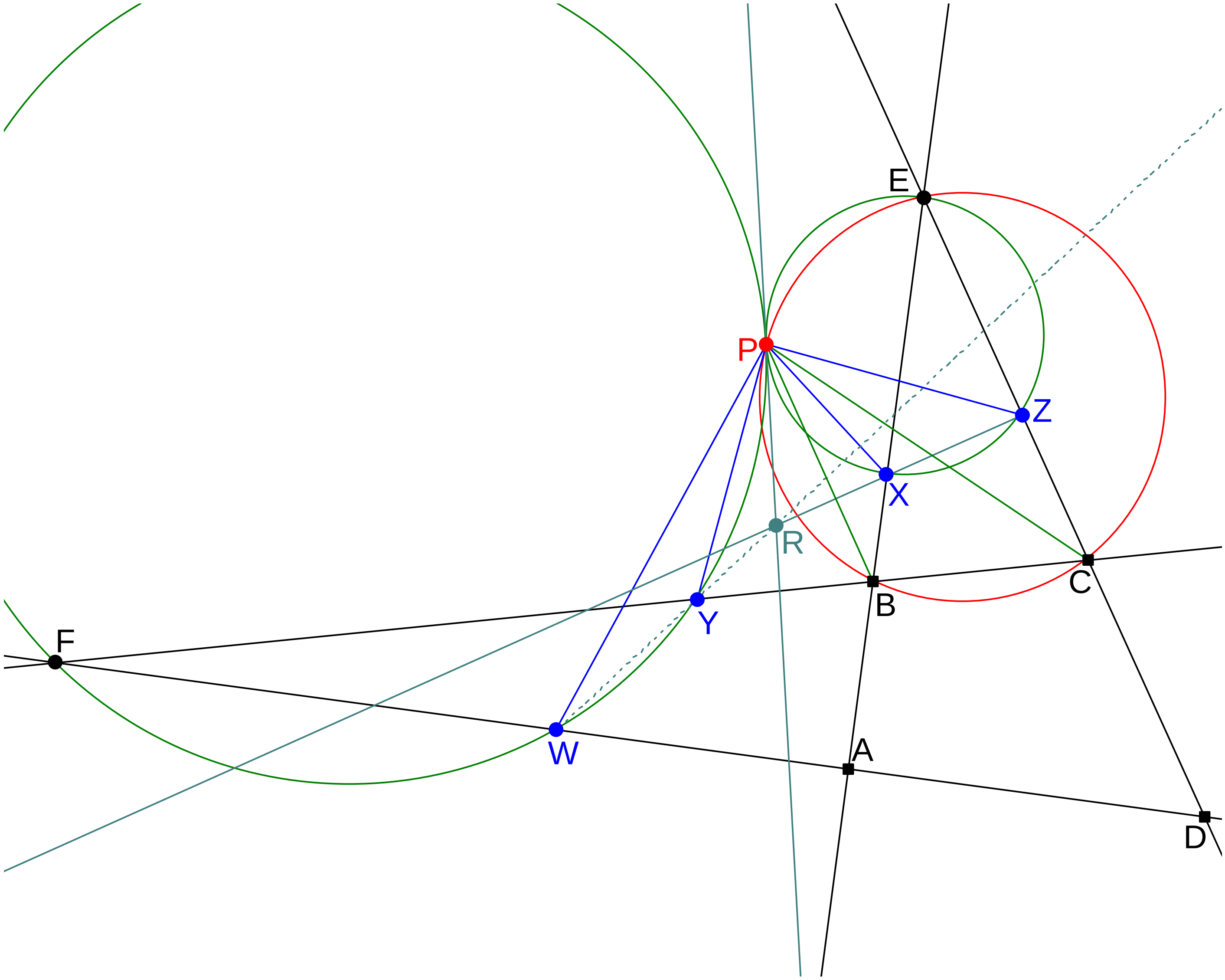

Fig. 14

Since t_1 is the tangent to the circle EZX at P, the tangent-chordal angle theorem yields $\measuredangle(t_1;\ EP) = \measuredangle PXE$. Since t_2 is the tangent to the circle FYW at P, the tangent-chordal angle theorem yields $\measuredangle(FP;\ t_2) = \measuredangle FYP$. Hence,

$$\begin{aligned}\measuredangle(t_1;\ t_2) &= \measuredangle(t_1;\ EP) + \measuredangle(EP;\ FP) + \measuredangle(FP;\ t_2) \\ &= \measuredangle PXE + \measuredangle EPF + \measuredangle FYP \\ &= \measuredangle PXE + \measuredangle EPF - \measuredangle PYF. \qquad (1)\end{aligned}$$

But

$$\measuredangle PXE = \measuredangle(PX;\ AB) = \measuredangle(PX;\ BP) + \measuredangle(BP;\ AB). \qquad (2)$$

Since - according to Theorem 8 - the line PX is the tangent to the circle $FABP$ at the point P, the tangent-chordal angle theorem entails $\measuredangle(PX;\ BP) = \measuredangle PFB$, and obviously $\measuredangle(BP;\ AB) = \measuredangle PBA$. Hence, (2) becomes $\measuredangle PXE = \measuredangle PFB + \measuredangle PBA$.

Similarly, $\measuredangle PYF = \measuredangle PEB + \measuredangle PBC$, and thus

$$\begin{aligned}
&\measuredangle PXE + \measuredangle EPF - \measuredangle PYF \\
&= (\measuredangle PFB + \measuredangle PBA) + \measuredangle EPF - (\measuredangle PEB + \measuredangle PBC) \\
&= \measuredangle PFB + \measuredangle PBA + \measuredangle EPF + (-\measuredangle PEB) + (-\measuredangle PBC) \\
&= \measuredangle PFB + \measuredangle PBA + \measuredangle EPF + \measuredangle BEP + \measuredangle CBP \\
&= \measuredangle PFB + \measuredangle CBP + \measuredangle PBA + \measuredangle BEP + \measuredangle EPF \\
&= \measuredangle (PF;\ BC) + \measuredangle (BC;\ BP) + \measuredangle (BP;\ AB) + \measuredangle (AB;\ PE) + \measuredangle (PE;\ PF) \\
&= 0^\circ. \qquad (3)
\end{aligned}$$

Combining this with (1), we obtain $\measuredangle (t_1;\ t_2) = 0^\circ$. Thus, the lines t_1 and t_2 are parallel. Since these lines t_1 and t_2 both pass through the point P, they must therefore coincide. Since these lines t_1 and t_2 were defined as the tangents to the circles EZX and FYW at the point P, this yields: The circles EZX and FYW have a common tangent at their point of intersection P. Hence, these two circles must touch each other at the point P. This proves Theorem 10.

(See Fig. 14.) Let the common tangent of the circles EZX and FYW at their point of tangency P intersect the line XZ at a point R. Then, the point R is the point of intersection of the tangent to the circle PZX at P with the line XZ.

Since the triangles PCZ and PBX are directly similar, Theorem 5 yields that the triangles PCB and PZX are directly similar. The point Y is the point of intersection of the tangent to the circle PCB at P with the line BC (after Theorem 8); the point R is the point of intersection of the tangent to the circle PZX at P with the line XZ. Hence, the points Y and R are corresponding points in the triangles PCB and PZX. Corresponding points in similar triangles form similar triangles themselves. Since the triangles PCB and PZX are directly similar, this yields that the triangles PBY and PXR are directly similar. After Theorem 5, this entails that the triangles PBX and PYR are directly similar. Hence, $\measuredangle PYR = \measuredangle PBX$.

We have shown above that the triangles PCB and PZX are directly similar. Similarly, the triangles PAB and PWY are directly similar, and this yields $\measuredangle WYP = \measuredangle ABP$. Hence, $\measuredangle WYR = \measuredangle WYP + \measuredangle PYR = \measuredangle ABP + \measuredangle PBX = \measuredangle ABX = 0^\circ$ (since the points A, B and X lie on one line). Therefore, the points W, Y and R lie on one line, i. e. the point R lies on the line YW. But we have defined the point R as the point of intersection of the common tangent of the circles EZX and FYW at their point of tangency P with the line XZ. Thus, we obtain:

Theorem 11. In the configuration of Theorems 3 and 6, we have: The common tangent of the circles EZX and FYW at their point of tangency P and the lines XZ and YW intersect each other at one point R. (See Fig. 15.)

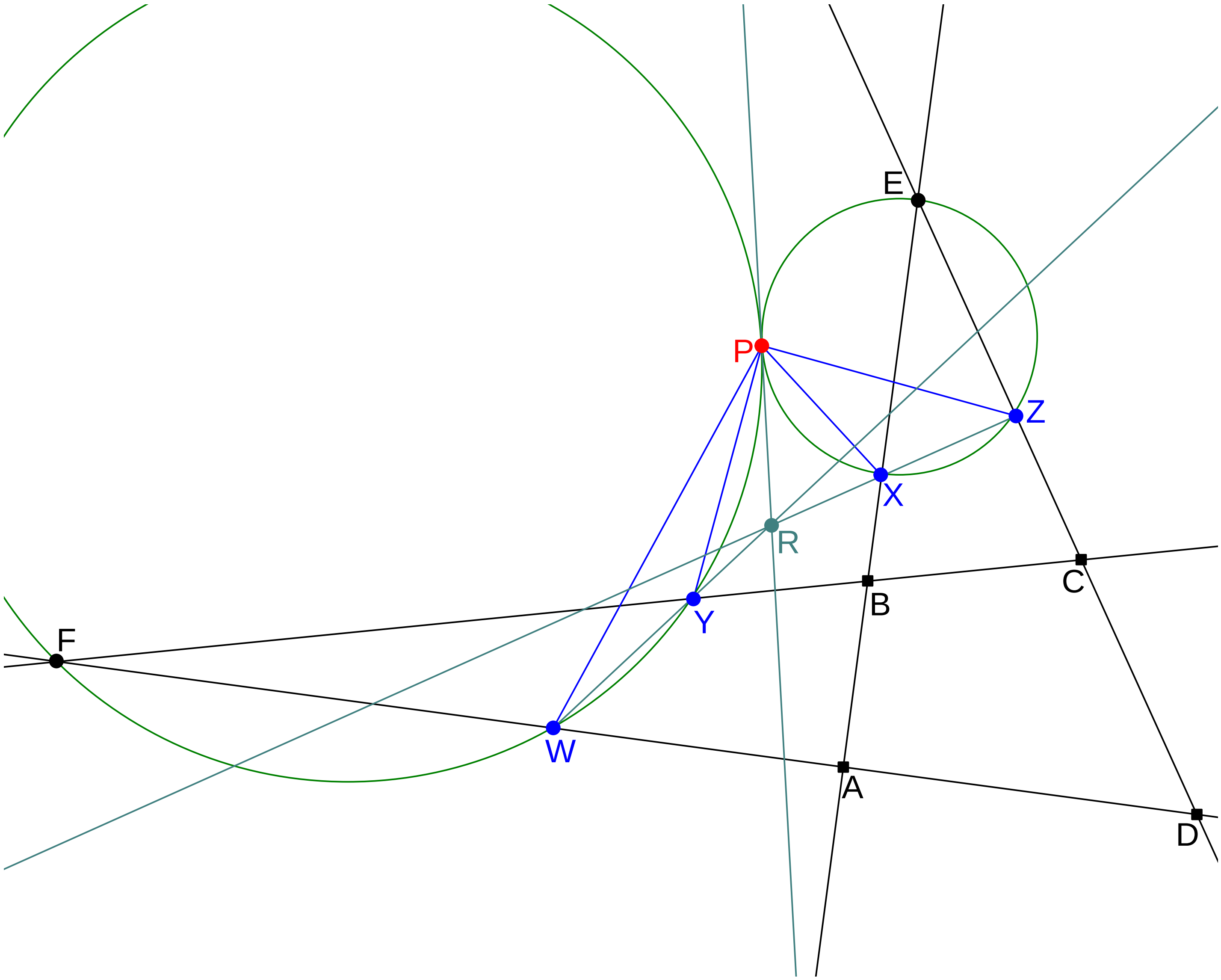

Fig. 15

The circle EZX meets the line XZ at the points X and Z, and meets the line RP at the points P and P (since it touches the line RP at the point P). The two lines XZ and RP intersect at R. Thus, the intersecting chords theorem for directed lengths yields $\overline{RX} \cdot \overline{RZ} = \overline{RP} \cdot \overline{RP}$. Similarly, $\overline{RY} \cdot \overline{RW} = \overline{RP} \cdot \overline{RP}$. Hence, $\overline{RX} \cdot \overline{RZ} = \overline{RY} \cdot \overline{RW}$. Since the two lines XZ and YW intersect at R, this yields, according to the intersecting chords theorem for directed lengths, that there exists a circle which meets the line XZ at the points X and Z and meets the line YW at the points Y and W. Thus, the points X, Y, Z, W lie on one circle. Theorem 3 is thus proven.

5. Two proofs of Theorem 4

(See Fig. 16.) Now it remains to prove Theorem 4, and we will do this in two different ways.

First proof of Theorem 4. The following proof of Theorem 4 continues our proof of Theorem 3: We will use some of our above auxiliary results that we have obtained on our way to the proof of Theorem 3, but we will also use the additional condition of Theorem 4 that the quadrilateral $ABCD$ is cyclic. Consequently, in the following, we are considering the configuration of Theorems 3 and 6 in the particular case when the quadrilateral $ABCD$ is cyclic. This means that we are considering a cyclic quadrilateral $ABCD$, the points X, Y, Z, W, the point of intersection E of the lines AB and CD, the point of intersection F of the lines BC and DA, and the Miquel point P of the quadrilateral $ABCD$.

Since $ABCD$ is a cyclic quadrilateral, the points A, B, C, D lie on one circle. Thus, after the chordal angle theorem, $\measuredangle DCB = \measuredangle DAB$. But the chordal angle theorem for the circle $EBCP$ yields $\measuredangle EPB = \measuredangle ECB$, and the chordal angle theorem for the circle $FABP$ yields $\measuredangle BPF = \measuredangle BAF$. Therefore,

$$\begin{aligned}\measuredangle EPF &= \measuredangle EPB + \measuredangle BPF = \measuredangle ECB + \measuredangle BAF = \measuredangle DCB + \measuredangle BAD \\ &= \measuredangle DAB + \measuredangle BAD = \measuredangle DAD = 0^\circ.\end{aligned}$$

This signifies that the point P lies on the line EF. We have thus shown:

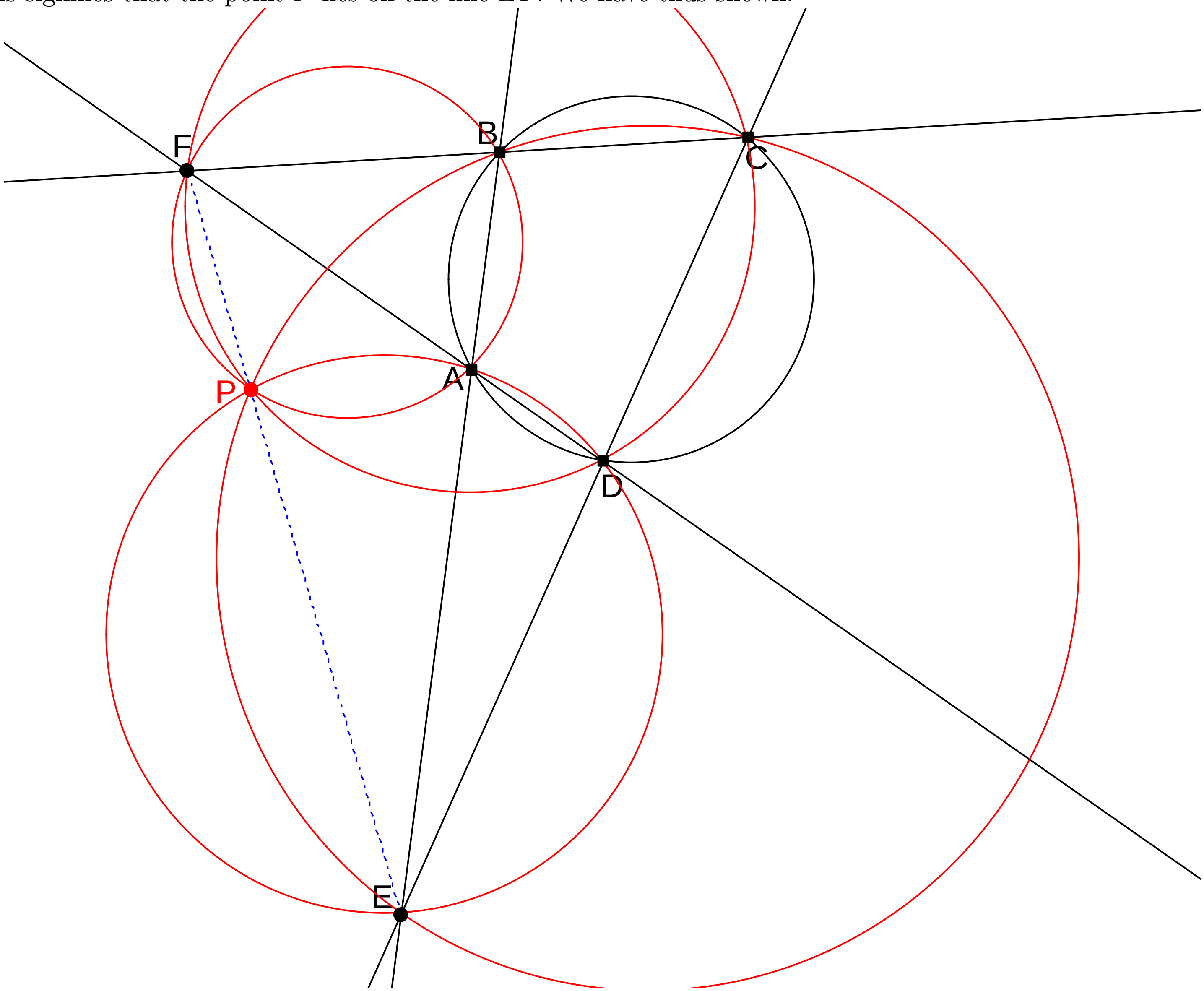

Fig. 16

Theorem 12. In the configuration of Theorem 6, we have: If $ABCD$ is a cyclic quadrilateral, then the Miquel point P lies on the line EF.

That is, the Miquel point of a cyclic quadrilateral lies on the line joining the points of intersection of opposite sides of the quadrilateral. (See Fig. 16.)

(See Fig. 17.) From (3), we have $\measuredangle PXE + \measuredangle EPF - \measuredangle PYF = 0°$. In view of $\measuredangle EPF = 0°$, this simplifies to $\measuredangle PXE - \measuredangle PYF = 0°$, and thus $\measuredangle PXE = \measuredangle PYF$. In other words, $\measuredangle PXB = \measuredangle PYB$. Thus, the points P, B, X, Y lie on one circle, so that the chordal angle theorem yields $\measuredangle PXY = \measuredangle PBY$. Similarly, the points P, A, W, X lie on one circle, and therefore the chordal angle theorem yields $\measuredangle PAW = \measuredangle PXW$. But we know for a longer time that $\measuredangle PBC = \measuredangle PAD$. Hence, $\measuredangle PXY = \measuredangle PBY = \measuredangle PBC = \measuredangle PAD = \measuredangle PAW = \measuredangle PXW$. This entails that the points W, X, Y lie on one line. Similarly, the points X, Y, Z lie on one line. Thus, all four points X, Y, Z, W lie on one line, and Theorem 4 is proven.

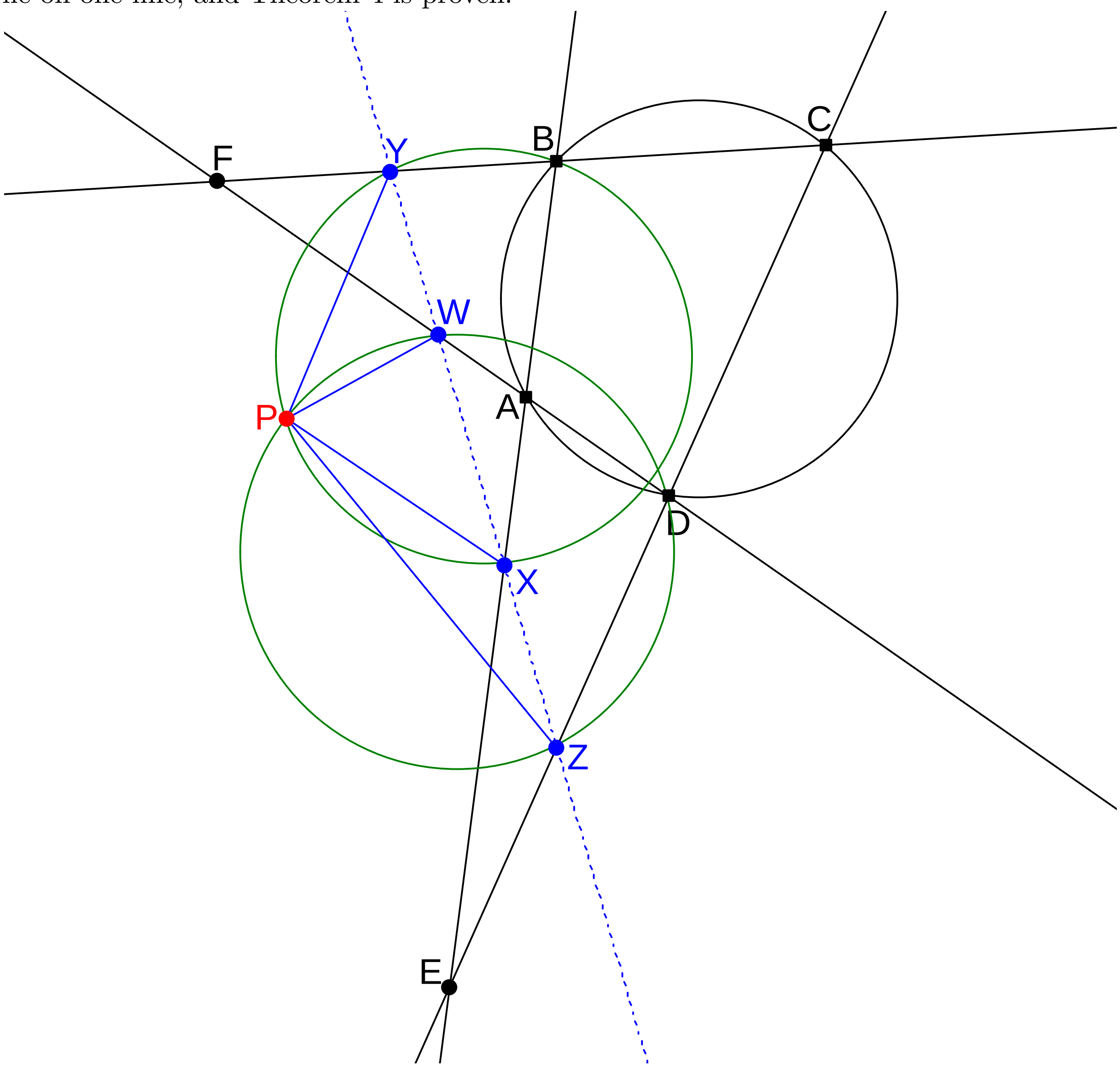

Fig. 17

Second proof of Theorem 4. The following proof of Theorem 4 is independent of our above proof of Theorem 3. Instead, this proof relies on the notion of the radical axis of two circles. We work in the configuration of Theorem 3 under the additional assumption that $ABCD$ is a cyclic quadrilateral. We first show an auxiliary fact:

Theorem 13. In the configuration of Theorem 3, we have: Let G be the point of intersection of the lines AC and BD.

If $ABCD$ is a cyclic quadrilateral, then the points X, Y, Z, W are the points of intersection of the tangents to the circles GAB, GBC, GCD, GDA at the point G with the lines AB, BC, CD, DA, respectively.

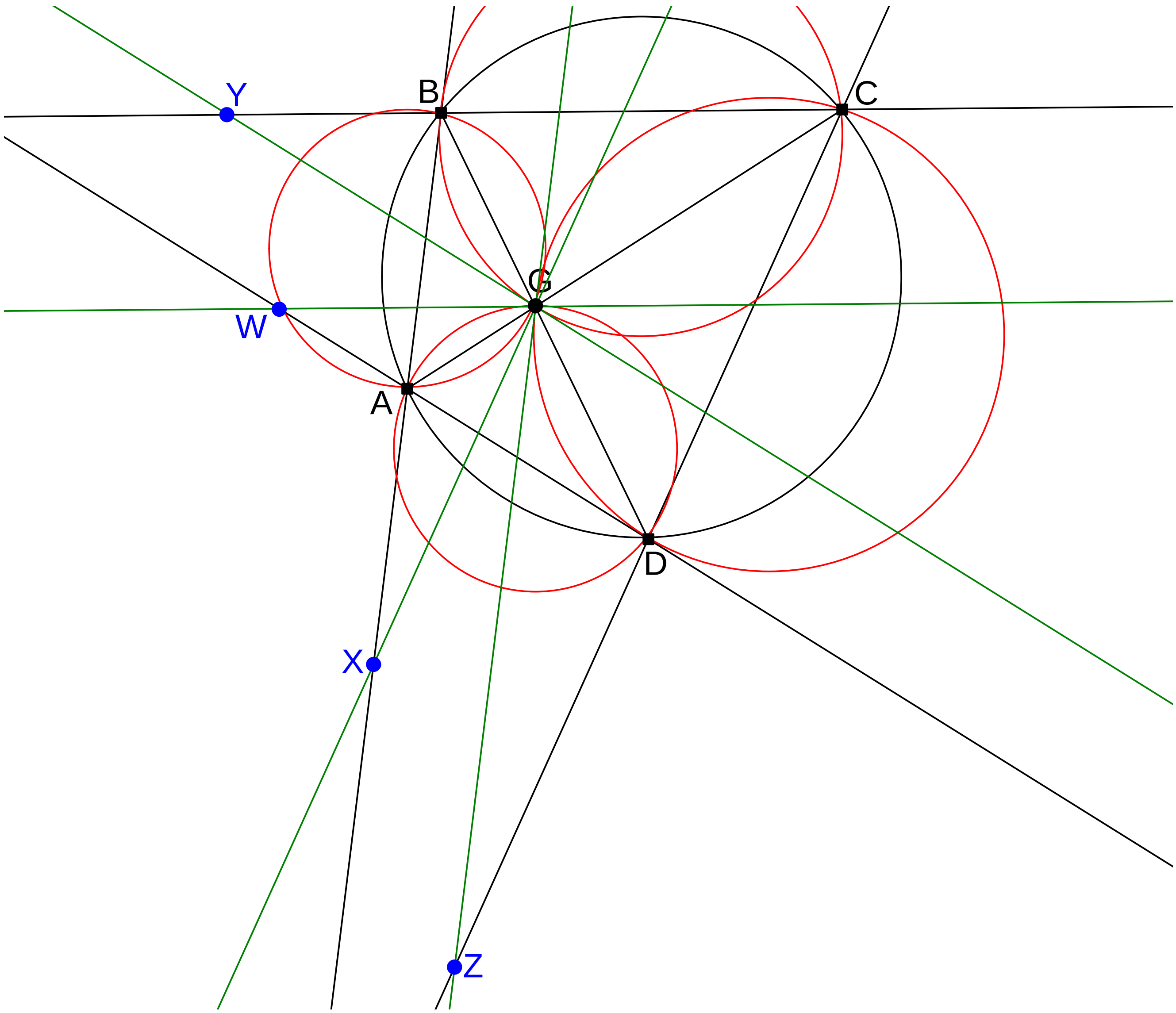

Fig. 18

Proof of Theorem 13. (See Fig. 19.) We work under the assumption that $ABCD$ is a cyclic quadrilateral.

This assumption yields that the points A, B, C, D lie on one circle. According to the chordal angle theorem, we therefore have $\measuredangle CAD = \measuredangle CBD$. Since $\measuredangle CAD = \measuredangle GAD$ and $\measuredangle CBD = -\measuredangle GBC$, this becomes $\measuredangle GAD = -\measuredangle GBC$. Similarly, $\measuredangle GDA = -\measuredangle GCB$. Thus, the triangles GAD and GBC are oppositely similar. Hence, $\dfrac{GA}{GB} = \dfrac{AD}{BC}$. In other words, $\dfrac{GA}{BG} = \dfrac{DA}{BC}$.

Let the tangent to the circle GAB at the point G intersect the line AB at a point X'. Then, Theorem 1 yields

$$\frac{\overline{AX'}}{\overline{X'B}} = -\frac{GA^2}{BG^2} = -\left(\frac{GA}{BG}\right)^2 = -\left(\frac{DA}{BC}\right)^2 = -\frac{DA^2}{BC^2} = \frac{\overline{AX}}{\overline{XB}}.$$

Hence, the points X and X' coincide. Since X' was defined as the point of intersection of the tangent to the circle GAB at the point G with the line AB, this yields: The point X is the point of intersection of the tangent to the circle GAB at the point G with the line AB. Similarly we can prove analogous characterizations of the points Y, Z, W. Thus, Theorem 13 is proven.

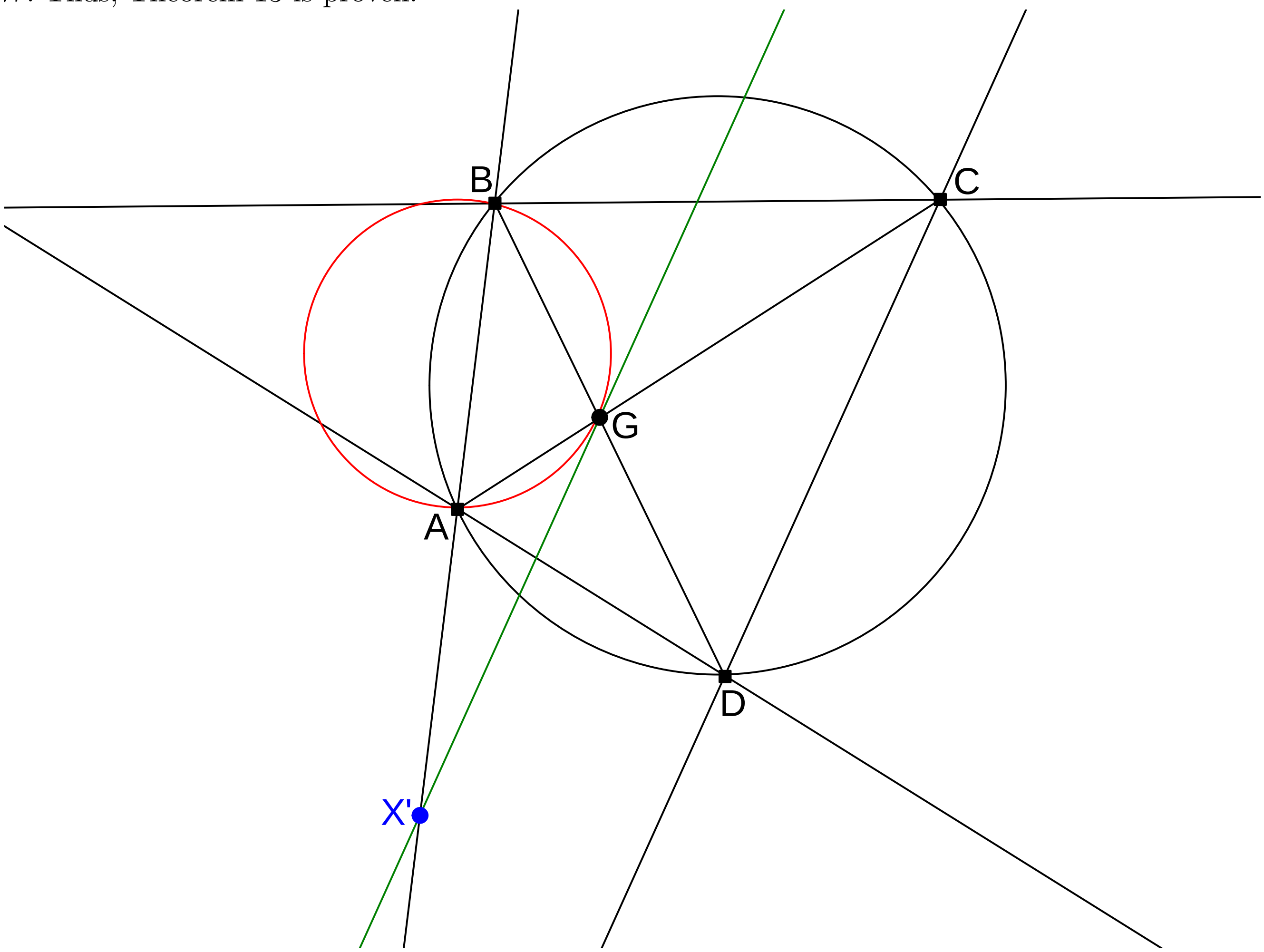

Fig. 19

According to Theorem 13, the point X is the point of intersection of the tangent to the circle GAB at the point G with the line AB. Consequently, the circle GAB meets the line AB at the points A and B, and meets the line GX at the points G and G (since it touches the line GX at the point G). Therefore, after the intersecting chords theorem for directed lengths, $\overline{XA} \cdot \overline{XB} = \overline{XG} \cdot \overline{XG}$ (since X is the point of intersection of the lines AB and GX). Now, $\overline{XA} \cdot \overline{XB}$ is the power of the point X with respect to the circumcircle of the cyclic quadrilateral $ABCD$, whereas $\overline{XG} \cdot \overline{XG} = \overline{XG}^2 = XG^2$ is the power of the point X with respect to the G-zero circle. Hence, the point X has equal powers with respect to the circumcircle of the cyclic quadrilateral $ABCD$ and the G-zero circle. Thus, the point X lies on the radical axis of the circumcircle of the cyclic quadrilateral $ABCD$ and the G-zero circle. Similarly, the points Y, Z, W lie on this radical axis as well. Hence, the four points X, Y, Z, W lie on one line - namely, on the radical axis of the circumcircle of the cyclic quadrilateral $ABCD$ and the G-zero circle. This proves Theorem 4.

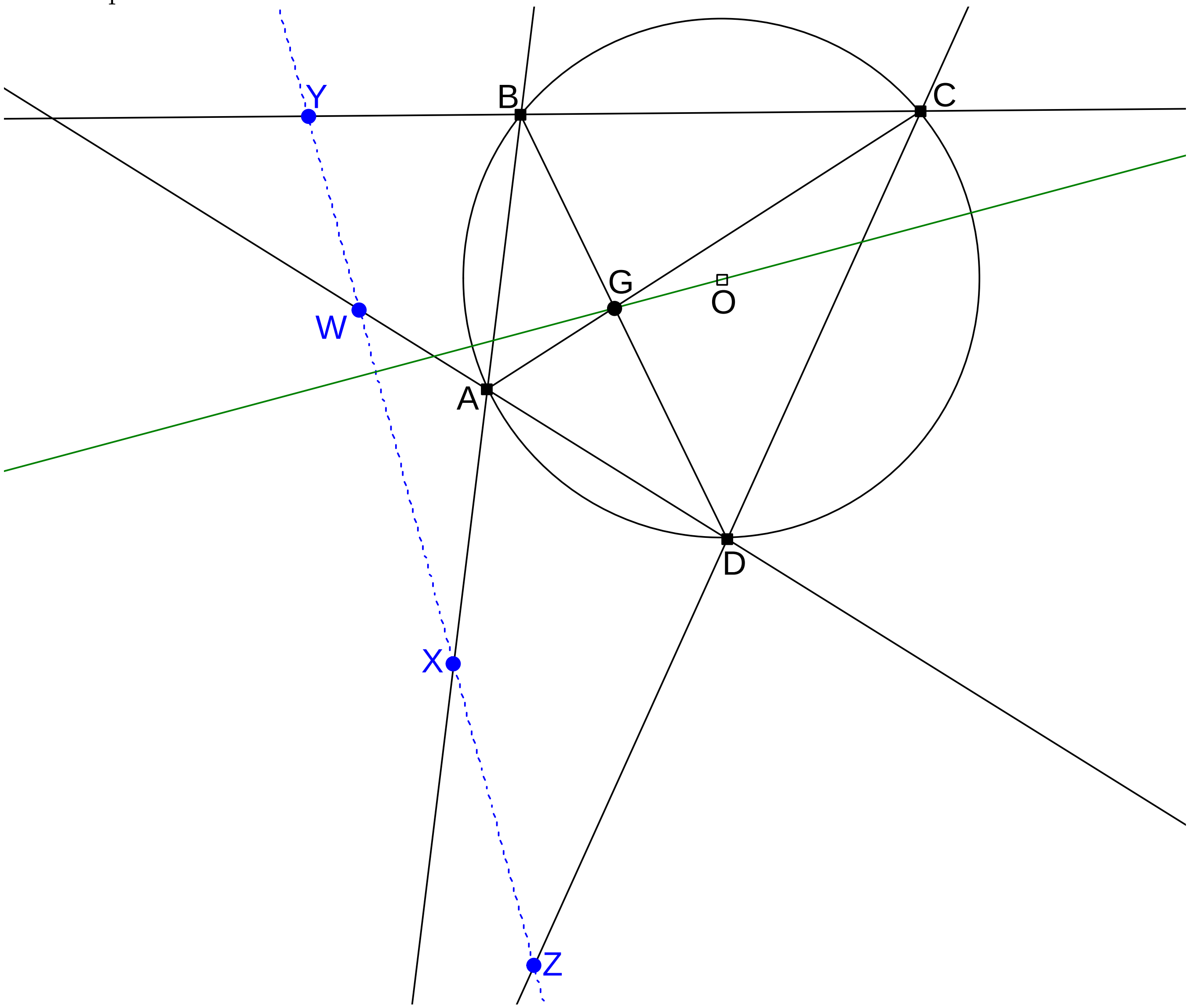

Fig. 20

This second proof of Theorem 4 actually entails an additional assertion: Let O be the center of the circumcircle of the cyclic quadrilateral $ABCD$. Since the radical axis of two circles is always perpendicular to the line joining their centers, it is clear that the radical axis of the circumcircle of the cyclic quadrilateral $ABCD$ and the G-zero circle is perpendicular to the line joining the center of the circumcircle of the cyclic quadrilateral $ABCD$ (i. e., the point O) with the center of the G-zero circle (i. e., the point G). That is, this radical axis is perpendicular to the line OG. Hence we have shown:

Theorem 14. In the configuration of Theorem 3, we have: Let G be the point of intersection of the lines AC and BD.

We assume that $ABCD$ is a cyclic quadrilateral. Then, the points X, Y,

Z, W lie on one line - namely, on the radical axis of the circumcircle of the cyclic quadrilateral $ABCD$ and the G-zero circle. This radical axis is perpendicular to the line OG, where O is the center of the circumcircle of the cyclic quadrilateral $ABCD$. (See Fig. 20.)

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