Problem O49, Mathematical Reflections 3/2007

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Problem. Let A_1, B_1, C_1 be points on the sides BC, CA, AB of a triangle ABC. The lines AA_1, BB_1, CC_1 intersect the circumcircle of triangle ABC at the points A_2, B_2, C_2, apart from A, B, C, respectively. Show that

$$\frac{AA_1}{A_1A_2} + \frac{BB_1}{B_1B_2} + \frac{CC_1}{C_1C_2} \geq \frac{3s^2}{r\,(4R+r)},$$

where $s = \dfrac{a+b+c}{2}$ is the semiperimeter, r is the inradius, and R is the circumradius of triangle ABC. (See Fig. 1.)

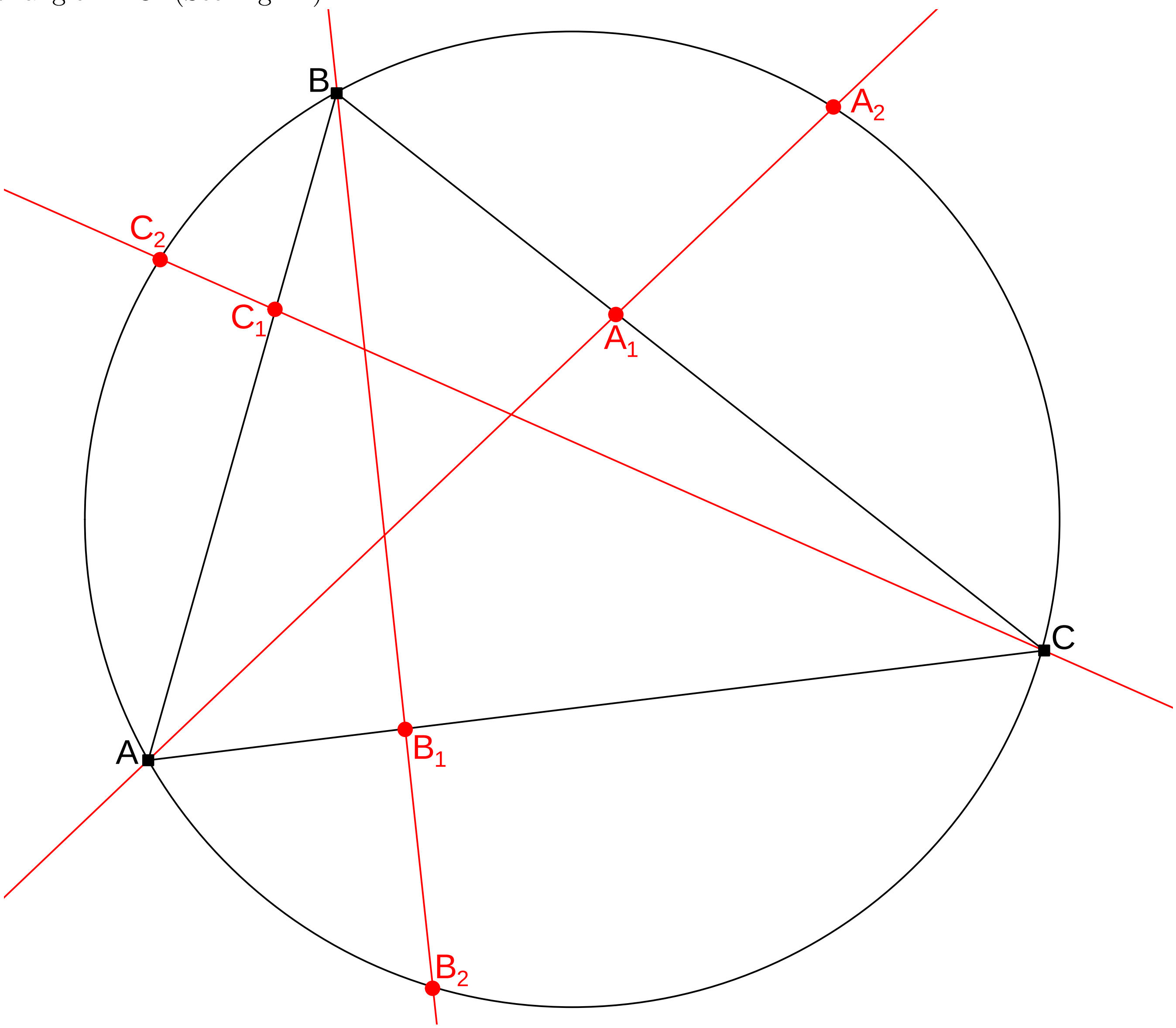

Fig. 1

Solution. Using the well-known identity $4R + r = r_a + r_b + r_c$ (which is more well-known in the form $4R = r_a + r_b + r_c - r$), we have

$$r(4R+r) = r(r_a + r_b + r_c) = rr_a + rr_b + rr_c$$
$$= (s-b)(s-c) + (s-c)(s-a) + (s-a)(s-b)$$

(where we use the simple and known formulas $rr_a = (s-b)(s-c)$, $rr_b = (s-c)(s-a)$ and $rr_c = (s-a)(s-b)$). Hence, the inequality in question,

$$\frac{AA_1}{A_1A_2} + \frac{BB_1}{B_1B_2} + \frac{CC_1}{C_1C_2} \geq \frac{3s^2}{r(4R+r)},$$

becomes

$$\frac{AA_1}{A_1A_2} + \frac{BB_1}{B_1B_2} + \frac{CC_1}{C_1C_2} \geq \frac{3s^2}{(s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b)}. \qquad (1)$$

Thus, in order to solve the problem, it remains to prove the inequality (1).

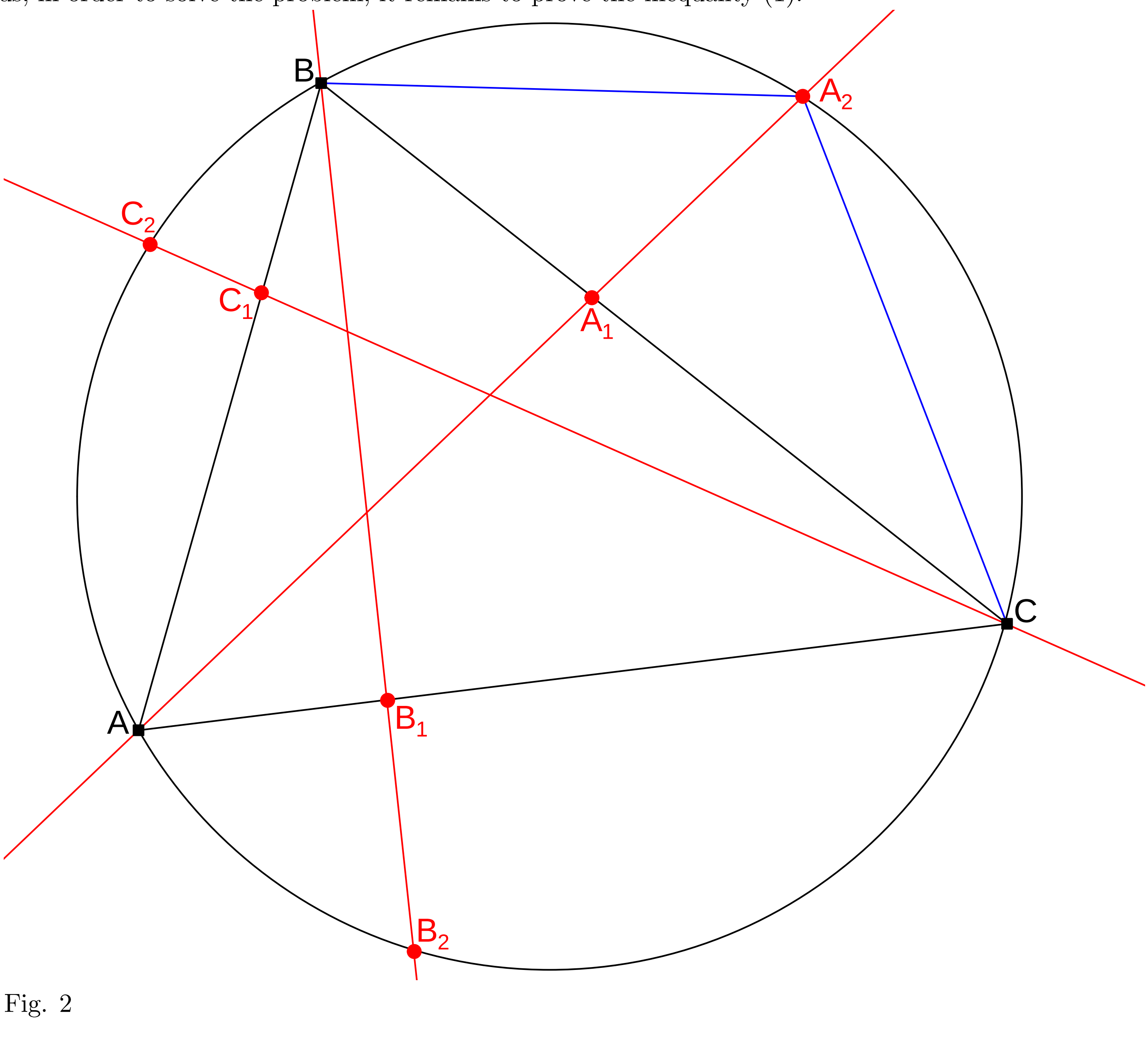

Fig. 2

(See Fig. 2.) Since the point A_2 lies on the circumcircle of triangle ABC, we have $\measuredangle A_2BC = \measuredangle A_2AC$, or, equivalently, $\measuredangle A_1BA_2 = \measuredangle CAA_1$. Also, since the point A_2 lies on the circumcircle of triangle ABC, we get $\measuredangle BA_2A = \measuredangle BCA$, what rewrites as $\measuredangle BA_2A_1 = C$. Now, by the Sine Law in triangles ABA_1 and A_2BA_1, we have $\dfrac{AA_1}{BA_1} = \dfrac{\sin\measuredangle ABA_1}{\sin\measuredangle BAA_1}$ and $\dfrac{A_1A_2}{BA_1} = \dfrac{\sin\measuredangle A_1BA_2}{\sin\measuredangle BA_2A_1}$, so that

$$\frac{AA_1}{A_1A_2} = \frac{AA_1}{BA_1} : \frac{A_1A_2}{BA_1} = \frac{\sin\measuredangle ABA_1}{\sin\measuredangle BAA_1} : \frac{\sin\measuredangle A_1BA_2}{\sin\measuredangle BA_2A_1} = \frac{\sin B}{\sin\measuredangle BAA_1} : \frac{\sin\measuredangle CAA_1}{\sin C}$$
$$= \frac{\sin B\cdot\sin C}{\sin\measuredangle BAA_1\cdot\sin\measuredangle CAA_1}.$$

Now, $\measuredangle BAA_1 + \measuredangle CAA_1 = A$, so that

$$\sin\measuredangle BAA_1\cdot\sin\measuredangle CAA_1 = \frac{1}{2}\left(\underbrace{\cos(\measuredangle BAA_1 - \measuredangle CAA_1)}_{\leq 1,\text{ since every cosine is }\leq 1} - \cos(\measuredangle BAA_1 + \measuredangle CAA_1)\right)$$
$$\leq \frac{1}{2}\left(1 - \cos(\measuredangle BAA_1 + \measuredangle CAA_1)\right) = \frac{1}{2}(1-\cos A) = \sin^2\frac{A}{2}.$$

Hence,

$$\frac{AA_1}{A_1A_2} = \frac{\sin B\cdot\sin C}{\sin\measuredangle BAA_1\cdot\sin\measuredangle CAA_1} \geq \frac{\sin B\cdot\sin C}{\sin^2\dfrac{A}{2}} = \frac{2\sin\dfrac{B}{2}\cos\dfrac{B}{2}\cdot 2\sin\dfrac{C}{2}\cos\dfrac{C}{2}}{\sin^2\dfrac{A}{2}}$$
$$= \frac{2\cdot\sqrt{\dfrac{(s-c)(s-a)}{ca}}\cdot\sqrt{\dfrac{s(s-b)}{ca}}\cdot 2\cdot\sqrt{\dfrac{(s-a)(s-b)}{ab}}\cdot\sqrt{\dfrac{s(s-c)}{ab}}}{\left(\dfrac{(s-b)(s-c)}{bc}\right)}$$
$$\text{(by the half-angle formulas)}$$
$$= \frac{4s(s-a)}{a^2},$$

and similarly $\dfrac{BB_1}{B_1B_2} \geq \dfrac{4s(s-b)}{b^2}$ and $\dfrac{CC_1}{C_1C_2} \geq \dfrac{4s(s-c)}{c^2}$. Hence, in order to prove the inequality (1), it is enough to show that

$$\frac{4s(s-a)}{a^2} + \frac{4s(s-b)}{b^2} + \frac{4s(s-c)}{c^2} \geq \frac{3s^2}{(s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b)}.$$

This simplifies to

$$\frac{s-a}{a^2} + \frac{s-b}{b^2} + \frac{s-c}{c^2} \geq \frac{3s}{4((s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b))}. \qquad (2)$$

Hence, in order to solve the problem, it is enough to prove this inequality (2).

Now, denote $x = s-a$, $y = s-b$, $z = s-c$. Then, it is known that x, y, z are positive reals, and we have

$$x+y+z = (s-a)+(s-b)+(s-c) = 3s-(a+b+c) = 3s-2s = s$$

and thus $y+z=(x+y+z)-x=s-(s-a)=a$ and similarly $z+x=b$ and $x+y=c$. Hence, the inequality (2) is equivalent to

$$\frac{x}{(y+z)^2}+\frac{y}{(z+x)^2}+\frac{z}{(x+y)^2}\geq\frac{3(x+y+z)}{4(yz+zx+xy)}.$$

Now, this inequality can be proven as follows: It rewrites as

$$\frac{x^2}{x(y+z)^2}+\frac{y^2}{y(z+x)^2}+\frac{z^2}{z(x+y)^2}\geq\frac{3(x+y+z)}{4(yz+zx+xy)}.$$

But the Cauchy-Schwarz inequality in Engel form yields

$$\frac{x^2}{x(y+z)^2}+\frac{y^2}{y(z+x)^2}+\frac{z^2}{z(x+y)^2}\geq\frac{(x+y+z)^2}{x(y+z)^2+y(z+x)^2+z(x+y)^2},$$

so that it only remains to prove that

$$\frac{(x+y+z)^2}{x(y+z)^2+y(z+x)^2+z(x+y)^2}\geq\frac{3(x+y+z)}{4(yz+zx+xy)},\qquad\text{what is equivalent to}$$
$$4(x+y+z)(yz+zx+xy)\geq 3\left(x(y+z)^2+y(z+x)^2+z(x+y)^2\right).$$

But this follows from

$$\begin{aligned}&4(x+y+z)(yz+zx+xy)-3\left(x(y+z)^2+y(z+x)^2+z(x+y)^2\right)\\&=x(y-z)^2+y(z-x)^2+z(x-y)^2\geq 0\qquad\text{(since squares are }\geq 0).\end{aligned}$$

Thus, the inequality (2) is proven, and the problem is solved.