Problem proposal for "Mathematical Reflections": A line through the centroid

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Darij Grinberg, *Problem S64*, Mathematical Reflections 5/2007.

Problem. Let G be the centroid of a triangle ABC, and let g be a line through the point G.

The line g intersects the line BC at a point X.

The parallel to the line BG through A intersects the line g at a point X_b.

The parallel to the line CG through A intersects the line g at a point X_c.

Prove that $\frac{1}{GX} + \frac{1}{GX_b} + \frac{1}{GX_c} = 0$, where the segments are directed. (See Fig. 1.)

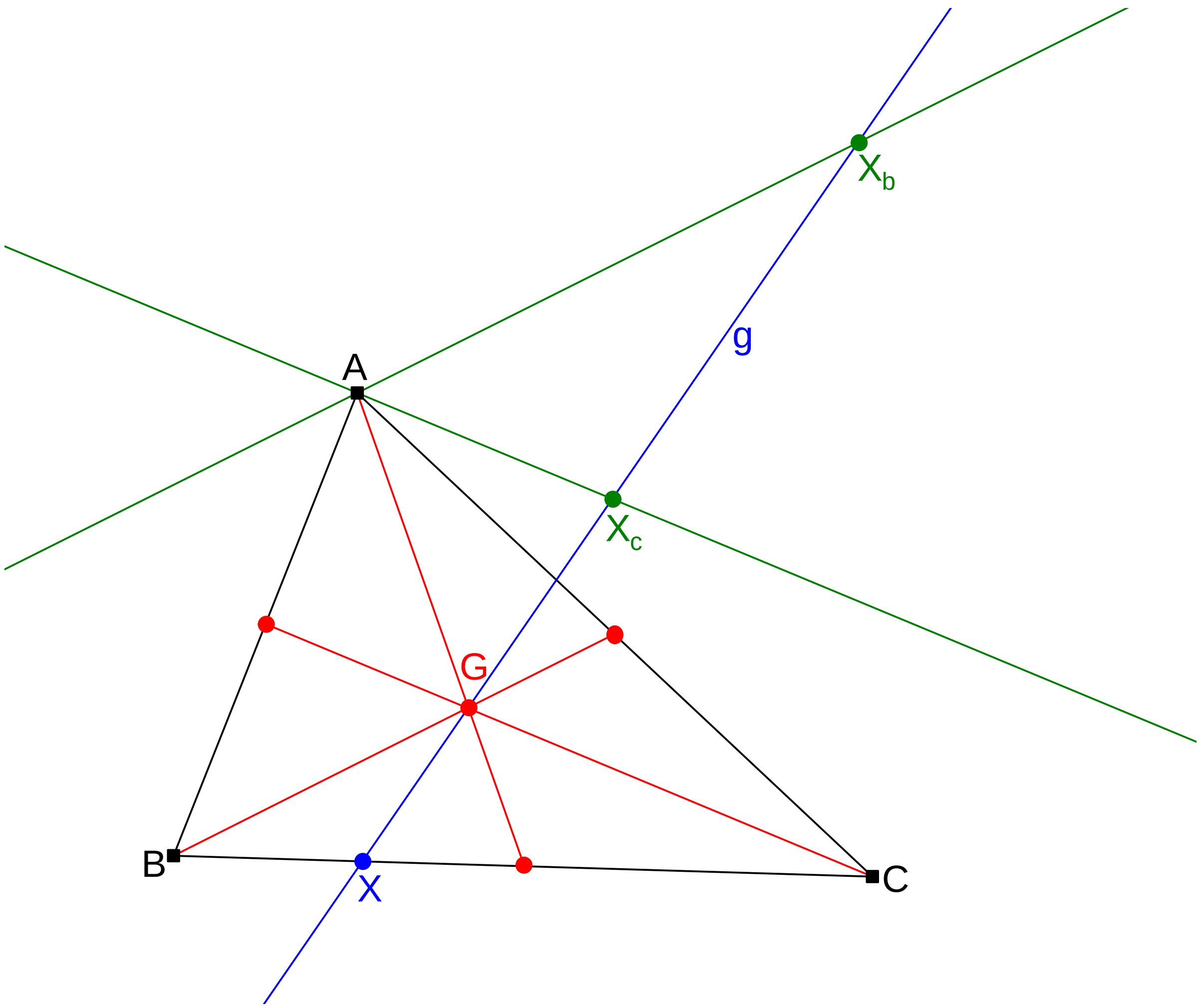

Fig. 1

Solution. (See Fig. 2.) Let C' be the midpoint of the segment AB. Then, the line CC' is a median of triangle ABC, and thus passes through the centroid G of this triangle.

Let the parallel to the line CG through B intersect the line g at a point Y_c.

Then, $AX_c \parallel CG$ and $BY_c \parallel CG$, so that $AX_c \parallel BY_c \parallel CG$. Thus, the points G, X_c, Y_c are the images of the points C', A, B under a parallel projection from the line AB onto the line g. Since parallel projection preserves ratios of signed lengths, we thus have $\frac{GX_c}{GY_c} = \frac{C'A}{C'B}$. But C' is the midpoint of AB, so that $C'A = -C'B$, and thus $\frac{C'A}{C'B} = -1$. Hence, $\frac{GX_c}{GY_c} = -1$, so that $GX_c = -GY_c$.

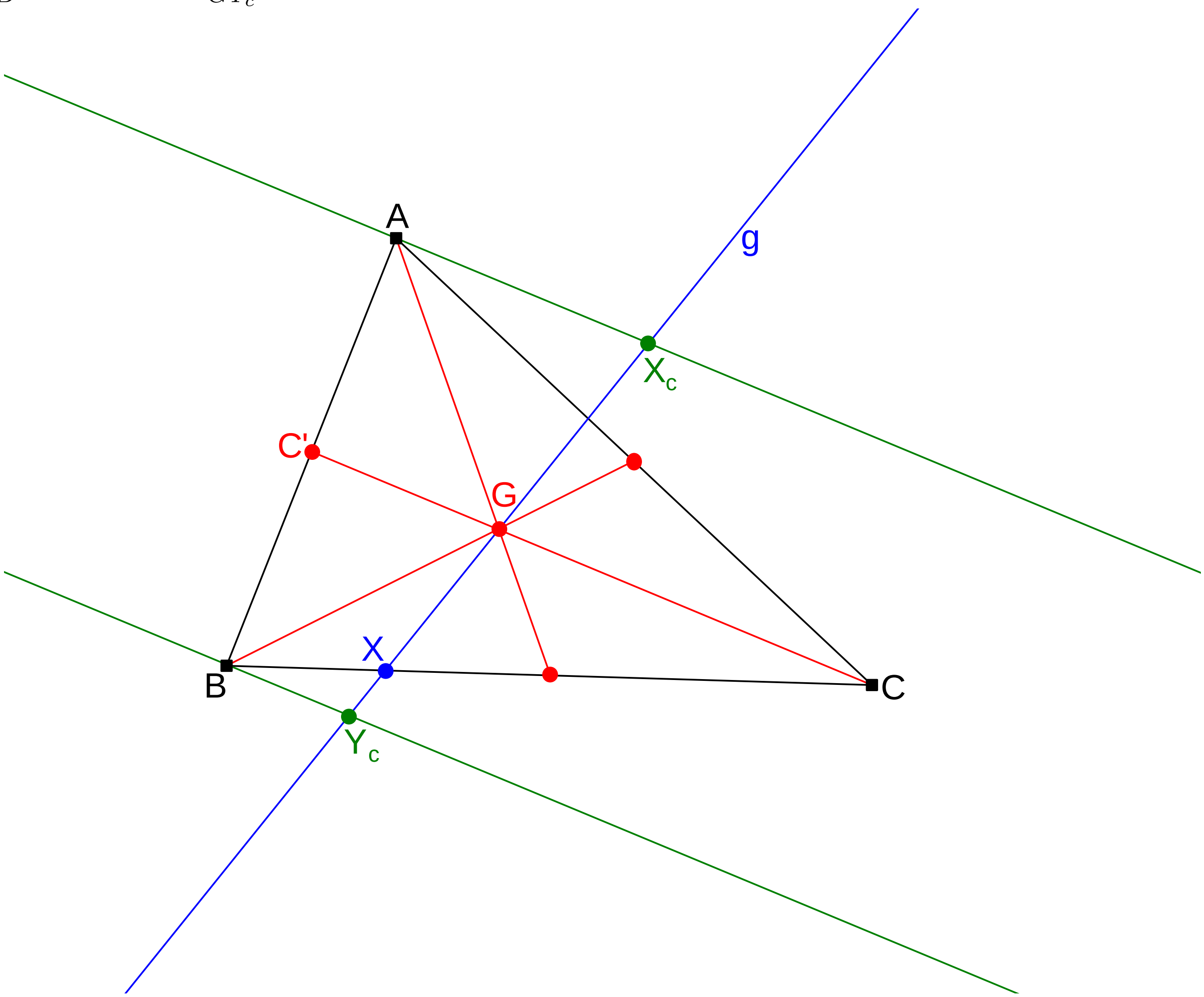

Fig. 2

Now, $BY_c \parallel CG$, so that the Thales theorem yields $\frac{GX}{GY_c} = \frac{CX}{CB}$. Hence, $\frac{GX}{GX_c} = \frac{GX}{-GY_c} = -\frac{GX}{GY_c} = -\frac{CX}{CB} = \frac{CX}{-CB} = \frac{CX}{BC}$. Similarly, $\frac{GX}{GX_b} = \frac{BX}{CB}$. Thus,

$$\frac{GX}{GX_b} + \frac{GX}{GX_c} = \frac{BX}{CB} + \frac{CX}{BC} = \frac{XB}{BC} + \frac{CX}{BC} = \frac{CX + XB}{BC} = \frac{CB}{BC} = \frac{-BC}{BC} = -1.$$

Dividing this equation by GX, we obtain $\frac{1}{GX_b} + \frac{1}{GX_c} = \frac{-1}{GX}$, so that $\frac{1}{GX} + \frac{1}{GX_b} + \frac{1}{GX_c} = 0$, and the problem is solved.

Remark. Using the above problem and its solution, we can give a new proof to the following fact ([1], §2.1, problem 8):

> **Theorem 1.** Let G be the centroid of a triangle ABC, and let g be a line through the point G.
>
> Let the line g intersect the lines BC, CA, AB at three points X, Y, Z.
>
> Then, $\frac{1}{GX} + \frac{1}{GY} + \frac{1}{GZ} = 0$, where the segments are directed.

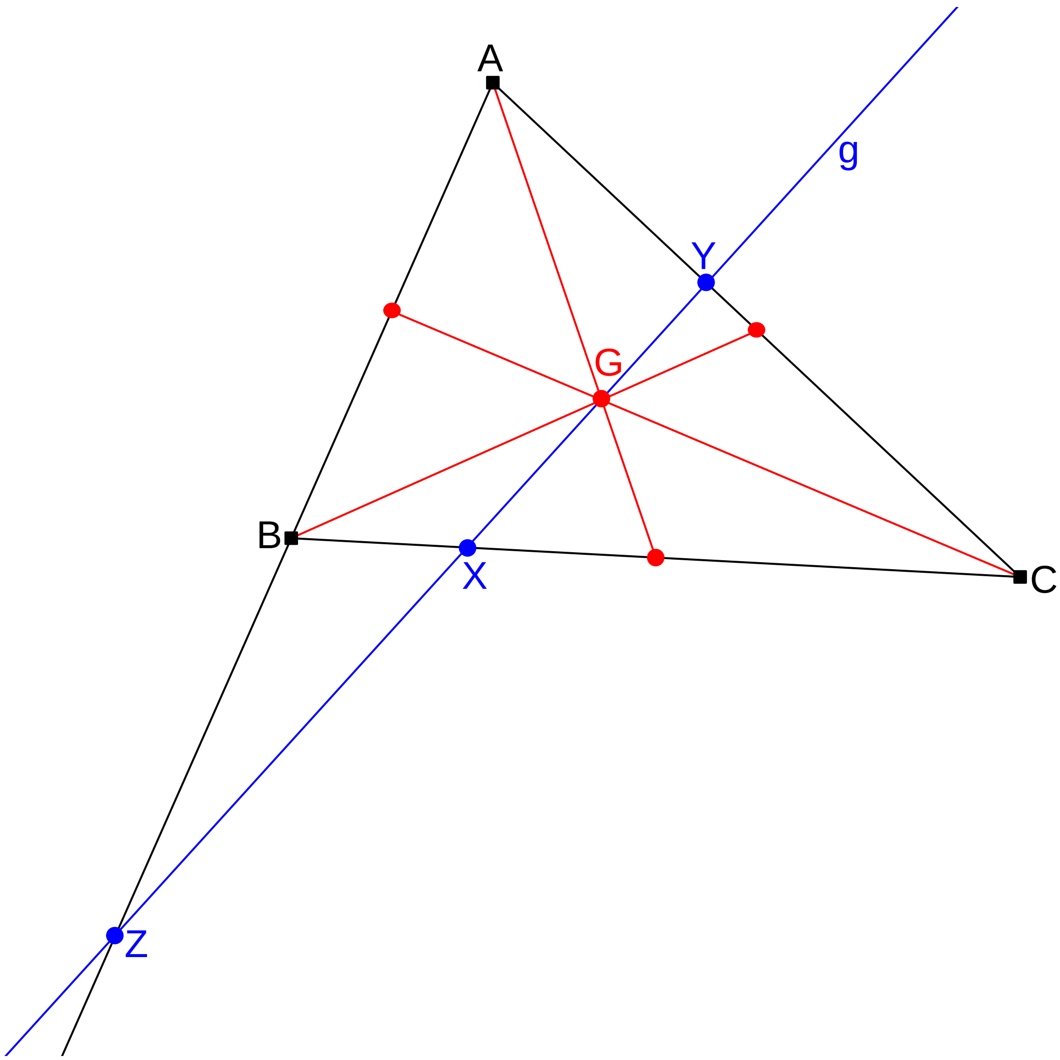

Fig. 3

Proof of Theorem 1. (See Fig. 4.) Let the parallel to the line CG through A meet the line g at a point X_c.

Let the parallel to the line BG through A meet the line g at a point X_b.

Let the parallel to the line CG through B meet the line g at a point Y_c.

Let the parallel to the line AG through C meet the line g at a point Z_a.

According to the problem, we have $\frac{1}{GX} + \frac{1}{GX_b} + \frac{1}{GX_c} = 0$, so that $\frac{1}{GX} = -\left(\frac{1}{GX_b} + \frac{1}{GX_c}\right)$. But during the solution of the problem, we have also shown that

$GX_c = -GY_c$. Thus,

$$\frac{1}{GX} = -\left(\frac{1}{GX_b} + \frac{1}{GX_c}\right) = -\left(\frac{1}{GX_b} + \frac{1}{-GY_c}\right) = -\left(\frac{1}{GX_b} - \frac{1}{GY_c}\right) = \frac{1}{GY_c} - \frac{1}{GX_b}.$$

Similarly,

$$\frac{1}{GY} = \frac{1}{GZ_a} - \frac{1}{GY_c} \qquad \text{and} \qquad \frac{1}{GZ} = \frac{1}{GX_b} - \frac{1}{GZ_a}.$$

Thus,

$$\frac{1}{GX} + \frac{1}{GY} + \frac{1}{GZ} = \left(\frac{1}{GY_c} - \frac{1}{GX_b}\right) + \left(\frac{1}{GZ_a} - \frac{1}{GY_c}\right) + \left(\frac{1}{GX_b} - \frac{1}{GZ_a}\right) = 0,$$

and Theorem 1 is proven.

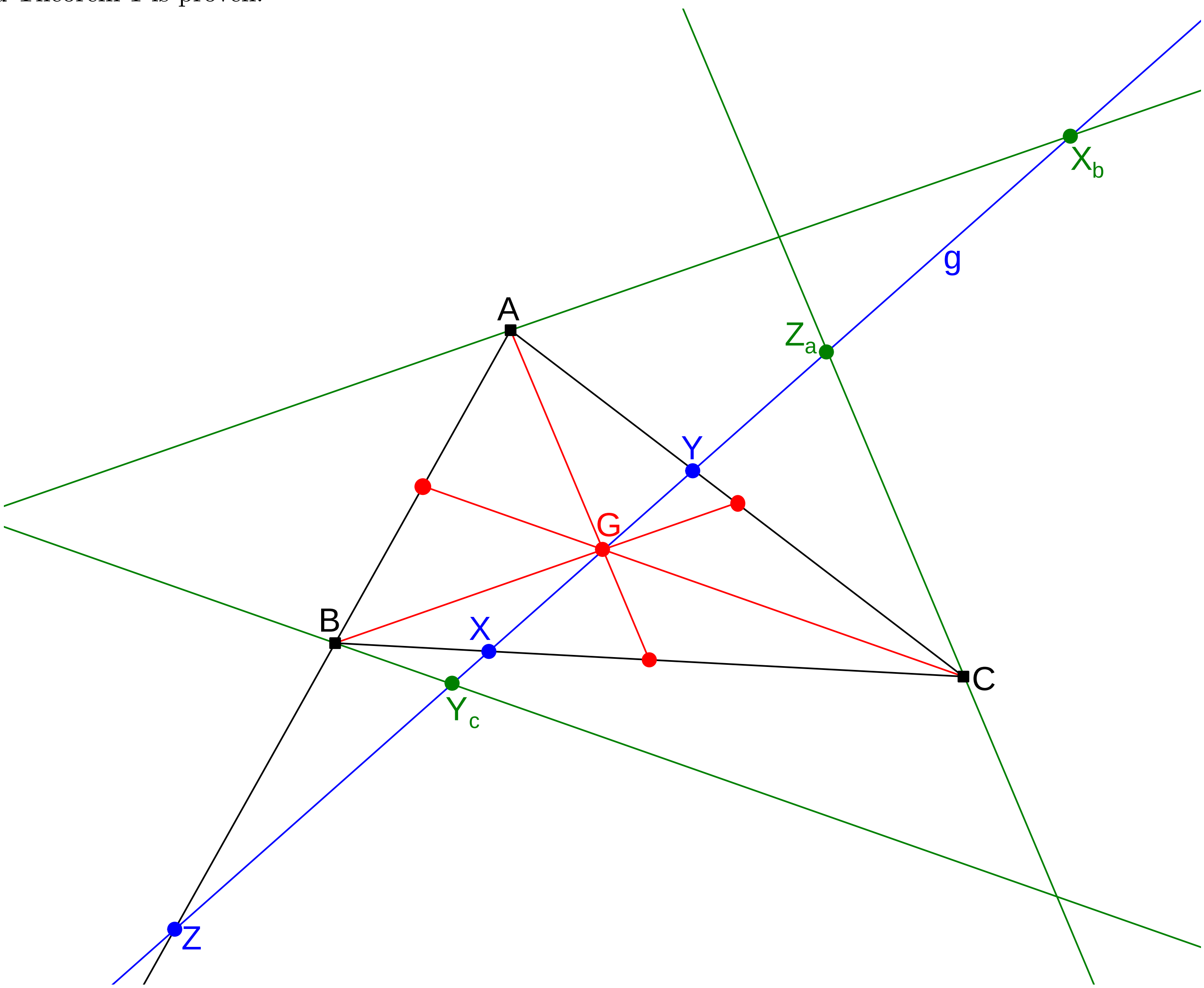

Fig. 4

References

[1] H. S. M. Coxeter, S. L. Greitzer, *Geometry Revisited*, Mathematical Association of America: New Mathematical Library, volume 19.