Three properties of the symmedian point / Darij Grinberg

1. On isogonal conjugates

The purpose of this note is to synthetically establish three results about the symmedian point of a triangle. Two of these don't seem to have received synthetic proofs hitherto. Before formulating the results, we remind about some fundamentals which we will later use, starting with the notion of isogonal conjugates.

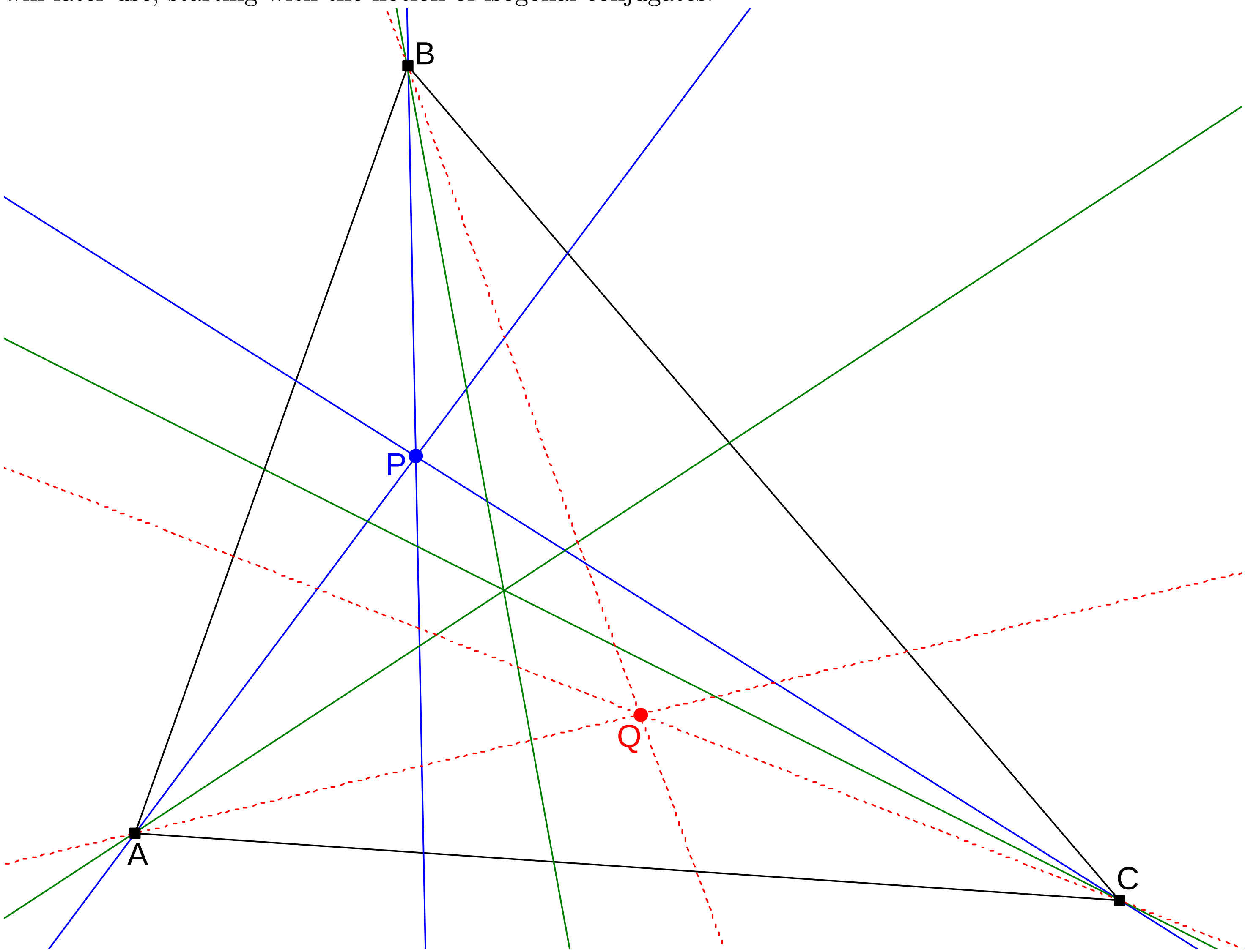

Fig. 1

The definition of isogonal conjugates is based on the following theorem (Fig. 1):

Theorem 1. Let ABC be a triangle and P a point in its plane. Then, the reflections of the lines AP, BP, CP in the angle bisectors of the angles CAB, ABC, BCA concur at one point.

This point is called **the isogonal conjugate of the point** P **with respect to triangle** ABC. We denote this point by Q.

Note that we work in the projective plane; this means that in Theorem 1, both the point P and the point of concurrence of the reflections of the lines AP, BP, CP in the angle bisectors of the angles CAB, ABC, BCA can be infinite points.

We are not going to prove Theorem 1 here, since it is pretty well-known and was showed e. g. in [5], Remark to Corollary 5. Instead, we show a property of isogonal conjugates.

At first, we meet a convention: Throughout the whole paper, we will make use of directed angles modulo 180°. An introduction into this type of angles was given in [4] (in German).

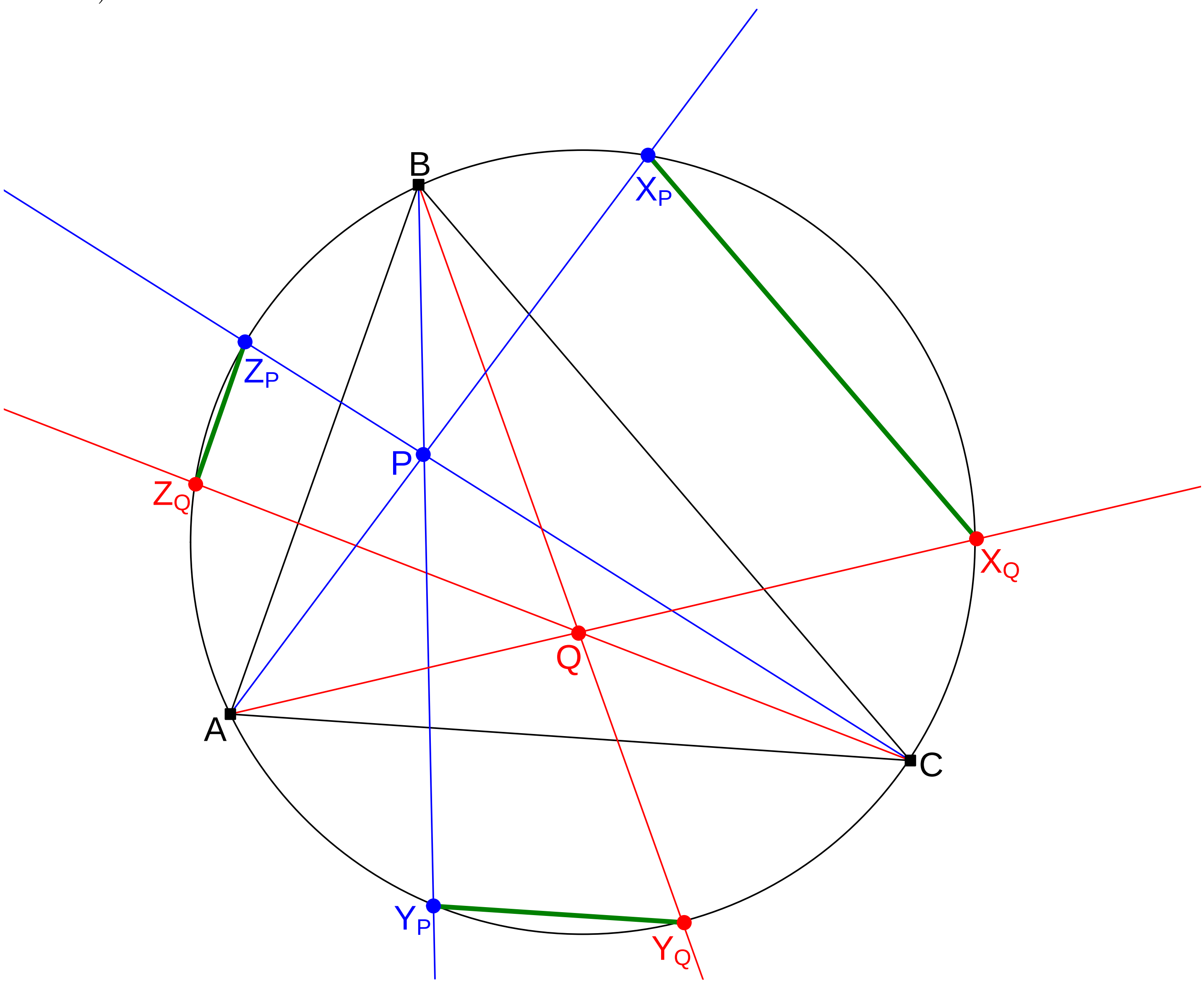

Fig. 2

Theorem 2. Let P be a point in the plane of a triangle ABC, and let Q be the isogonal conjugate of the point P with respect to triangle ABC. Then:

a) We have $\measuredangle BAQ = -\measuredangle CAP$, $\measuredangle CAQ = -\measuredangle BAP$, $\measuredangle CBQ = -\measuredangle ABP$, $\measuredangle ABQ = -\measuredangle CBP$, $\measuredangle ACQ = -\measuredangle BCP$ and $\measuredangle BCQ = -\measuredangle ACP$. (See Fig. 1.)

b) Let X_P, Y_P, Z_P be the points of intersection of the lines AP, BP, CP with the circumcircle of triangle ABC (different from A, B, C). Let X_Q, Y_Q, Z_Q be the points of intersection of the lines AQ, BQ, CQ with the circumcircle of triangle ABC (different from A, B, C). Then, $X_PX_Q \parallel BC$, $Y_PY_Q \parallel CA$ and $Z_PZ_Q \parallel AB$. (See Fig. 2.)

c) The perpendicular bisectors of the segments BC, CA, AB are simultaneously the perpendicular bisectors of the segments $X_P X_Q$, $Y_P Y_Q$, $Z_P Z_Q$. (See Fig. 3.)

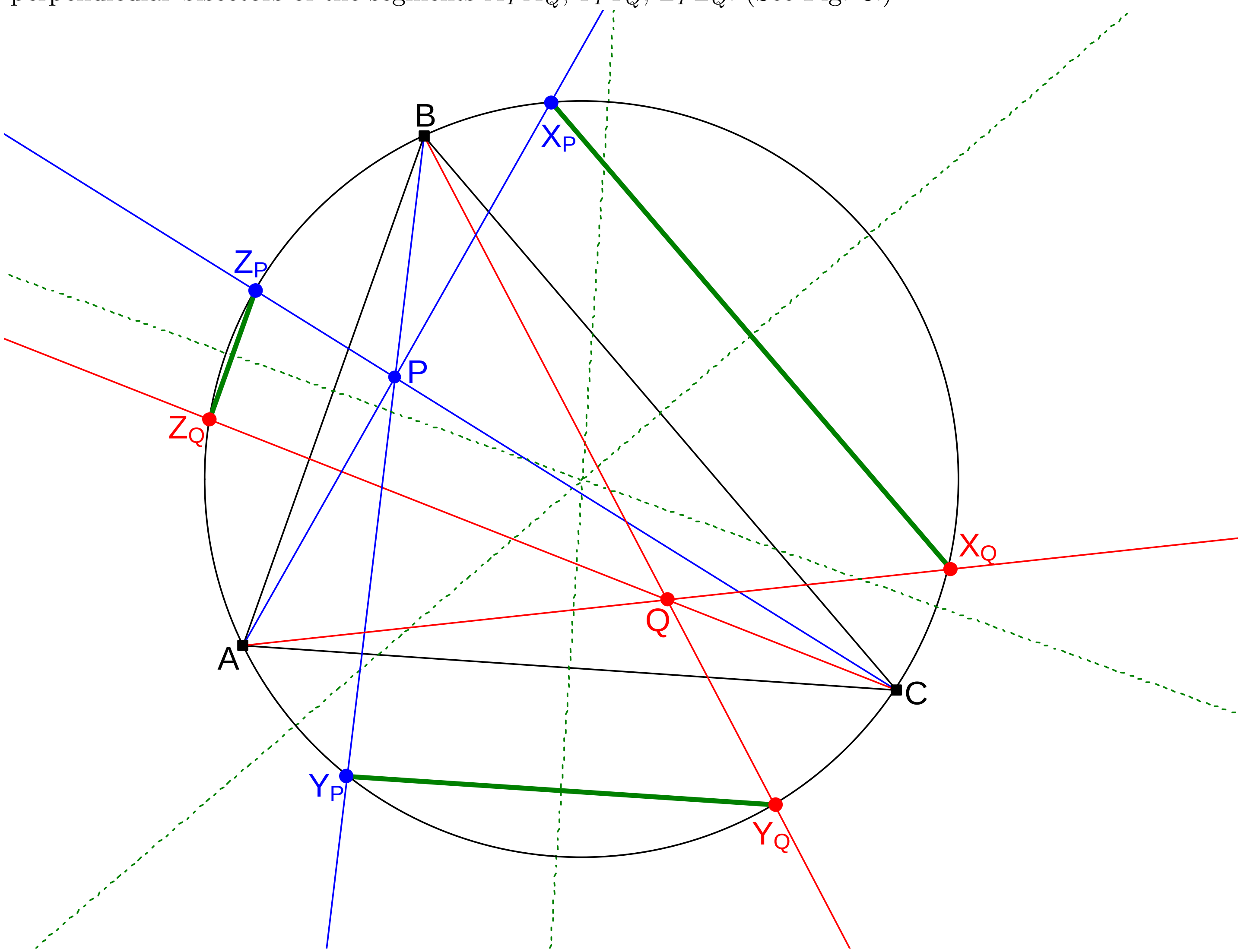

Fig. 3

Here is a *proof of Theorem 2.* Skip it if you find the theorem trivial.

a) The point Q lies on the reflection of the line BP in the angle bisector of the angle ABC. In other words, the reflection in the angle bisector of the angle ABC maps the line BP to the line BQ. On the other hand, this reflection maps the line AB to the line BC (since the axis of reflection is the angle bisector of the angle ABC). Since reflection in a line leaves directed angles invariant in their absolute value, but changes their sign, we thus have $\measuredangle (BC;\ BQ) = -\measuredangle (AB;\ BP)$. Equivalently, $\measuredangle CBQ = -\measuredangle ABP$. Similarly, $\measuredangle ABQ = -\measuredangle CBP$, $\measuredangle ACQ = -\measuredangle BCP$, $\measuredangle BCQ = -\measuredangle ACP$, $\measuredangle BAQ = -\measuredangle CAP$ and $\measuredangle CAQ = -\measuredangle BAP$. This proves Theorem 2 **a)**.

b) (See Fig. 4.) Theorem 2 **a)** yields $\measuredangle CBQ = -\measuredangle ABP$. In other words, $\measuredangle CBY_Q = \measuredangle Y_P BA$. But since the points Y_P and Y_Q lie on the circumcircle of triangle ABC, we have $\measuredangle CBY_Q = \measuredangle CY_P Y_Q$ and $\measuredangle Y_P BA = \measuredangle Y_P CA$. Thus, $\measuredangle CY_P Y_Q = \measuredangle Y_P CA$. In other words, $\measuredangle (CY_P;\ Y_P Y_Q) = \measuredangle (CY_P;\ CA)$. This yields $Y_P Y_Q \parallel CA$, and analogous reasoning leads to $Z_P Z_Q \parallel AB$ and $X_P X_Q \parallel BC$. Hence, Theorem 2 **b)** is proven.

c) After Theorem 2 **b)**, the segments $Y_P Y_Q$ and CA are parallel. Hence, the perpendicular bisectors of these segments $Y_P Y_Q$ and CA are also parallel. But these perpendicular bisectors have a common point, namely the center of the circumcircle of triangle ABC (since the segments Y_PY_Q and CA are chords in this circumcircle, and the perpendicular bisector of a chord in a circle always passes through the center of the circle). So, the perpendicular bisectors of the segments Y_PY_Q and CA are parallel and have a common point; thus, they must coincide. In other words, the perpendicular bisector of the segment CA is simultaneously the perpendicular bisector of the segment Y_PY_Q. Similarly for AB and Z_PZ_Q and for BC and X_PX_Q. This proves Theorem 2 **c)**.

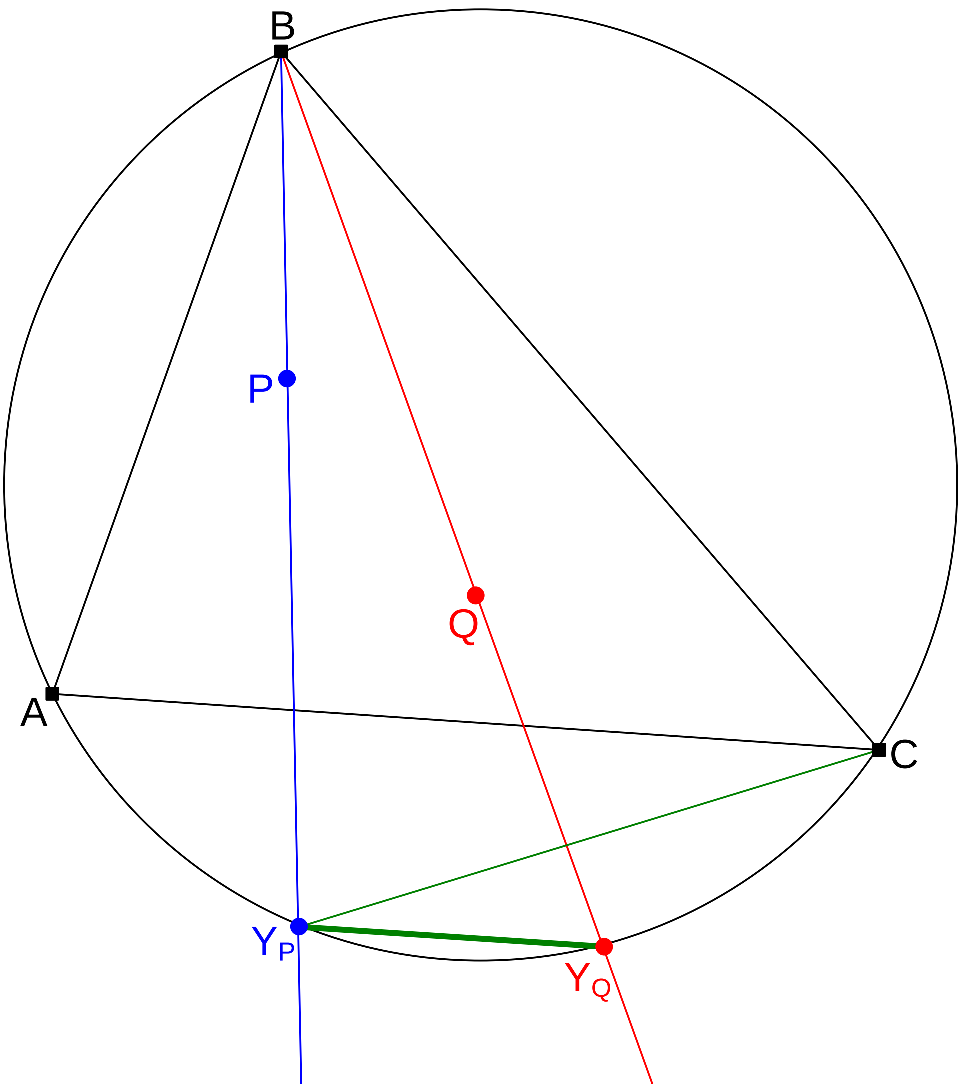

Fig. 4

2. The symmedian point

Now it's time to introduce the main object of our investigations, the symmedian point:

The **symmedian point** of a triangle is defined as the isogonal conjugate of the centroid of the triangle (with respect to this triangle). In other words: If S is the centroid of a triangle ABC, and L is the isogonal conjugate of this point S with respect to triangle ABC, then this point L is called the **symmedian point** of triangle ABC.

So the point L is the isogonal conjugate of the point S with respect to triangle ABC, i. e. the point of intersection of the reflections of the lines AS, BS, CS in the angle bisectors of the angles CAB, ABC, BCA. The lines AS, BS, CS are the three medians of triangle ABC (since S is the centroid of triangle ABC); thus, the point L is the point of intersection of the reflections of the medians of triangle ABC in the corresponding angle bisectors of triangle ABC. In other words, the lines AL, BL, CL are the reflections of the medians of triangle ABC in the corresponding angle bisectors of triangle ABC. These lines AL, BL, CL are called the **symmedians** of triangle ABC.

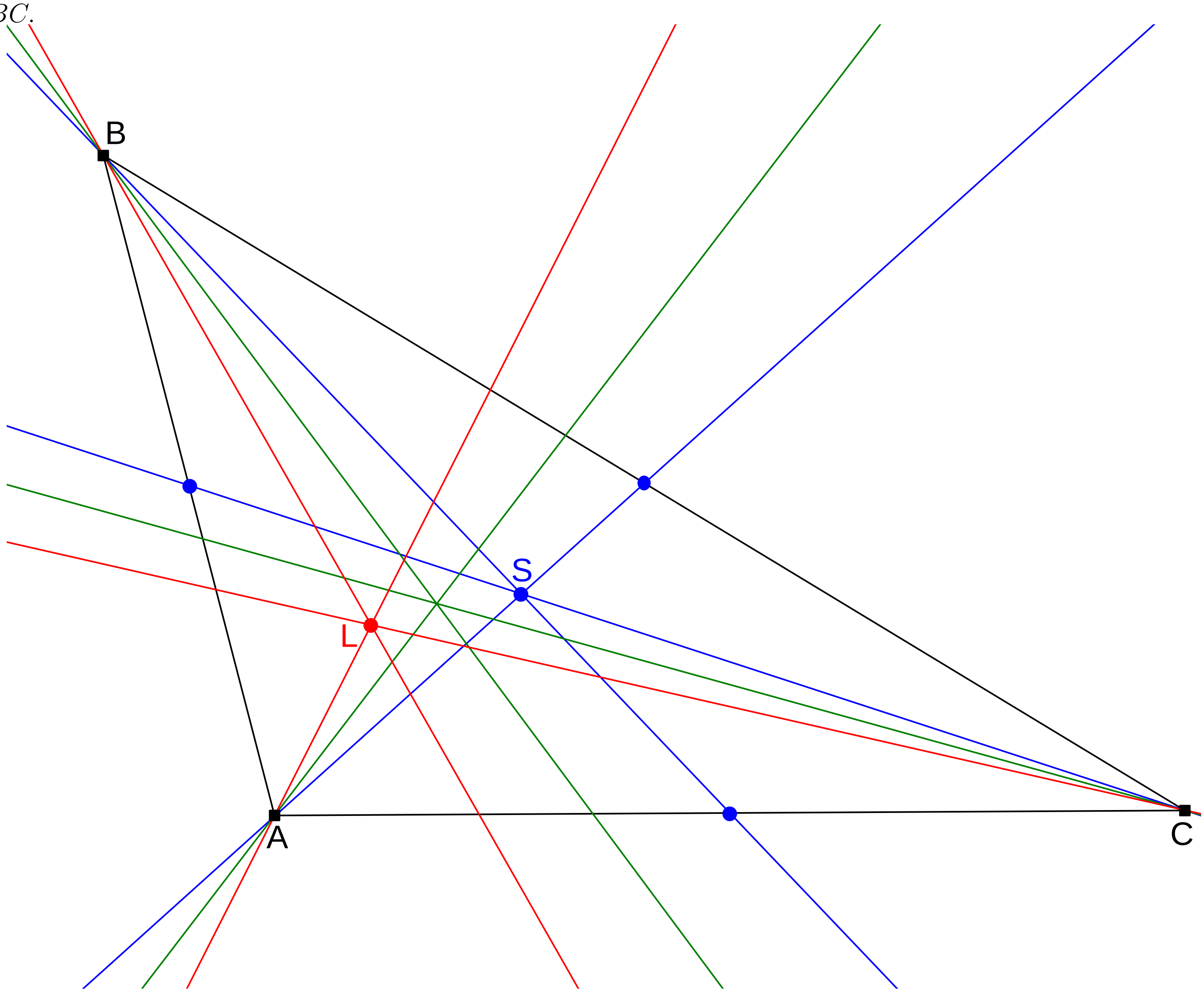

Fig. 5

Since the point L is the isogonal conjugate of the point S with respect to triangle ABC, we can apply Theorem 2 to the points $P = S$ and $Q = L$, and obtain:

Theorem 3. Let S be the centroid and L the symmedian point of a triangle ABC. Then:

a) We have $\measuredangle BAL = -\measuredangle CAS$, $\measuredangle CAL = -\measuredangle BAS$, $\measuredangle CBL = -\measuredangle ABS$, $\measuredangle ABL = -\measuredangle CBS$, $\measuredangle ACL = -\measuredangle BCS$ and $\measuredangle BCL = -\measuredangle ACS$. (See Fig. 5.)

b) Let X, Y, Z be the points of intersection of the medians AS, BS, CS of triangle ABC with the circumcircle of triangle ABC (different from A, B, C). Let X', Y', Z' be the points of intersection of the symmedians AL, BL, CL with the circumcircle of triangle ABC (different from A, B, C). Then, $XX' \parallel BC$, $YY' \parallel CA$ and $ZZ' \parallel AB$. (See Fig. 6.)

c) The perpendicular bisectors of the segments BC, CA, AB are simultaneously the perpendicular bisectors of the segments XX', YY', ZZ'.

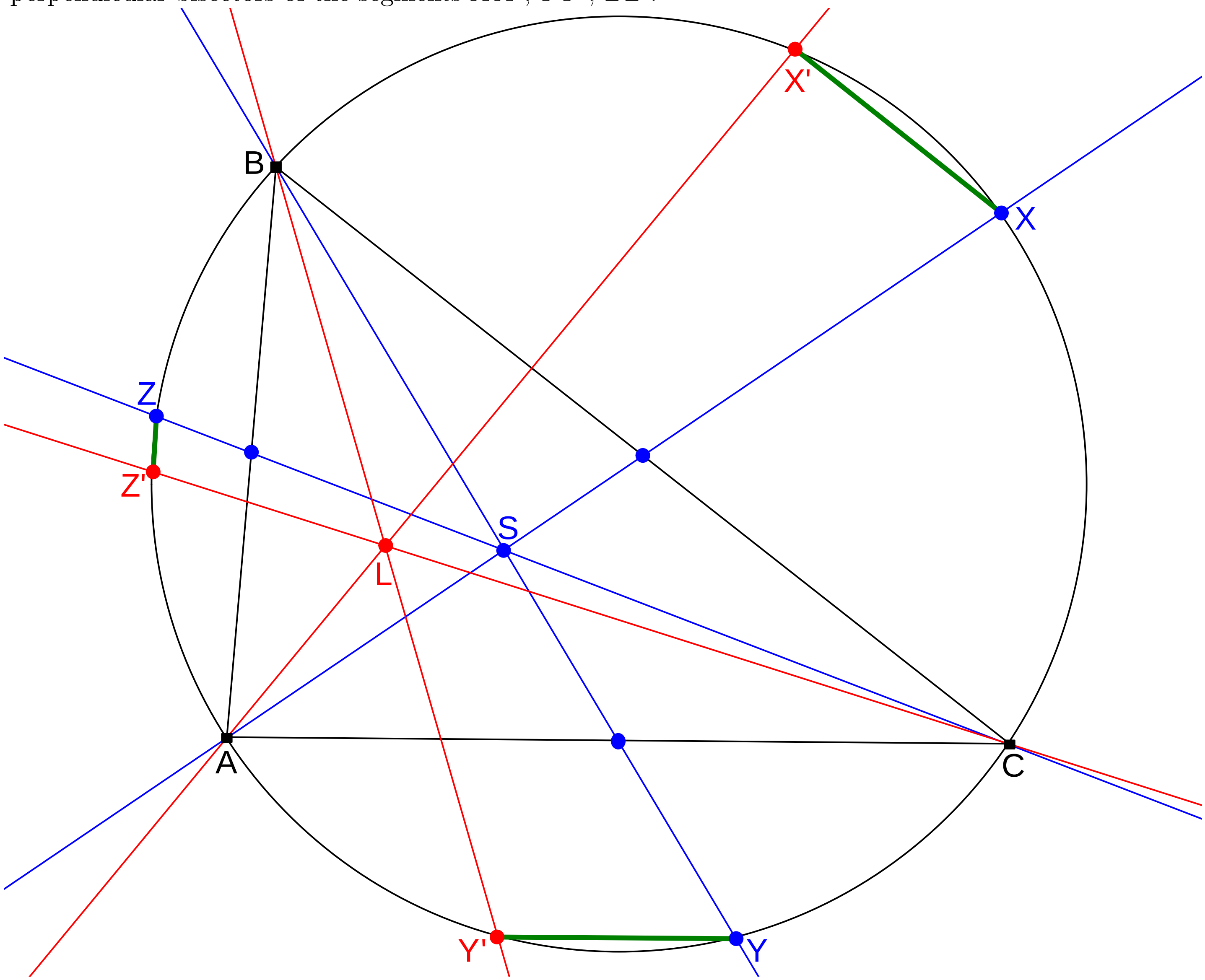

Fig. 6

Another basic property of the symmedian point will be given without proof, since it was shown in [1], Chapter 7, §4 (iii) and in [3], §24:

Theorem 4. Let the tangents to the circumcircle of triangle ABC at the points B and C meet at a point D; let the tangents to the circumcircle of triangle ABC at the points C and A meet at a point E; let the tangents to the circumcircle of triangle ABC at the points A and B meet at a point F. Then, the lines AD, BE, CF are the symmedians of triangle ABC and pass through its symmedian point L. (See Fig. 7.)

The triangle DEF is called the **tangential triangle** of triangle ABC.

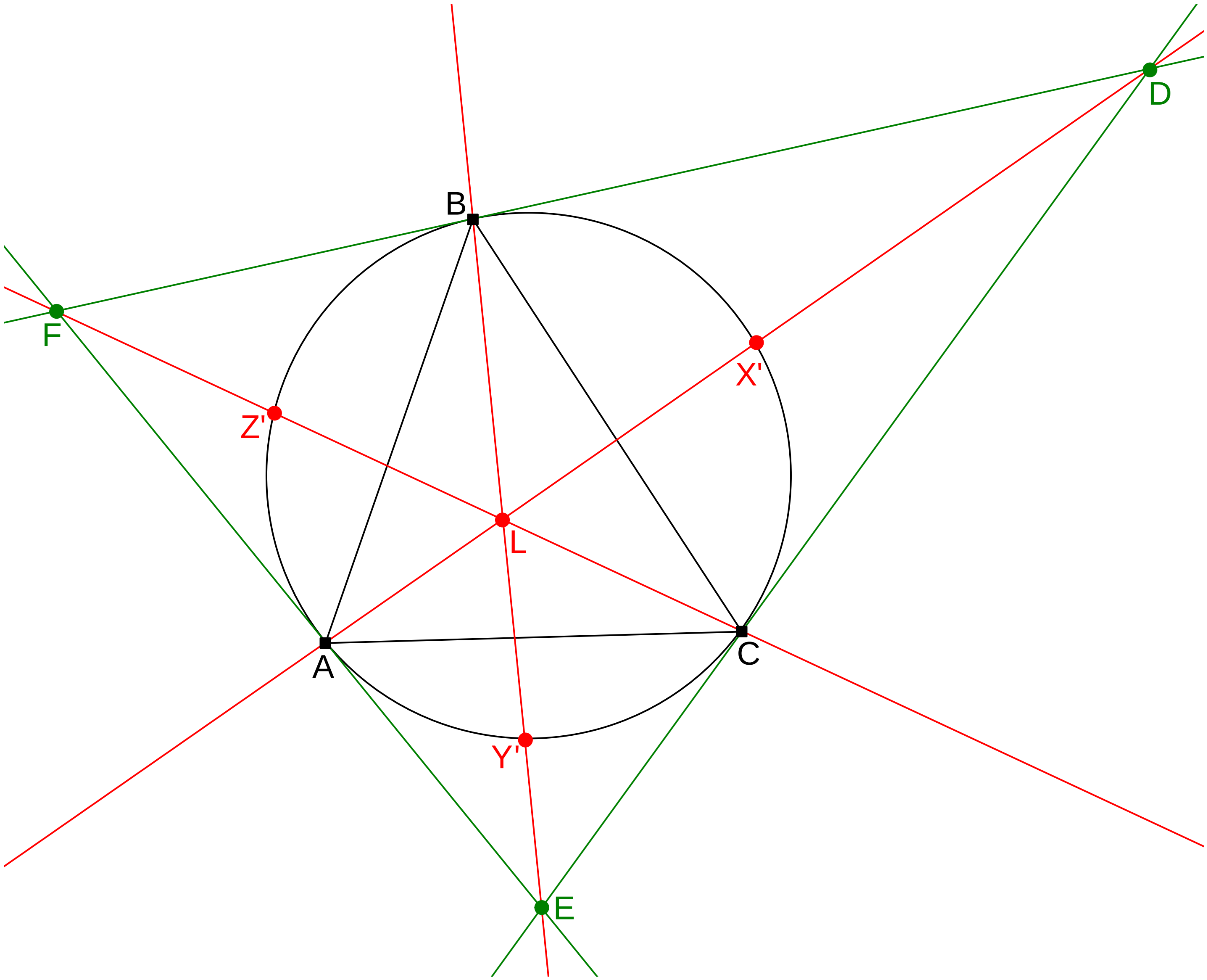

Fig. 7

The last property of the symmedian point which we will use relates it to the midpoints of the altitudes of the triangle (Fig. 8):

Theorem 5. Let G be the foot of the altitude of triangle ABC issuing from the vertex A, and let G' be the midpoint of the segment AG. Furthermore, let A' be the midpoint of the side BC of triangle ABC. Then, the line $A'G'$ passes through the symmedian point L of triangle ABC.

For the proof of this fact, we refer to [1], Chapter 7, §4 (vii).

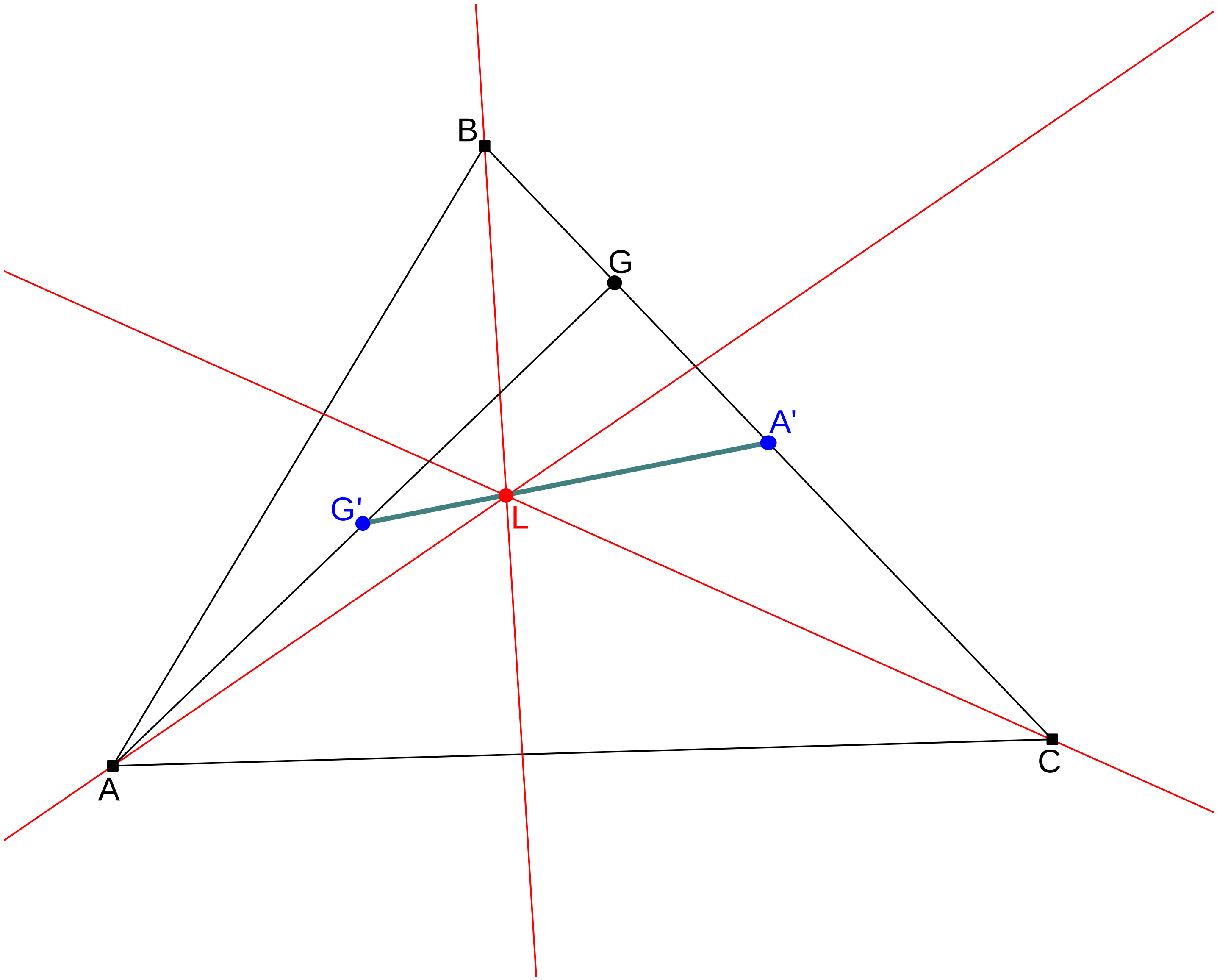

Fig. 8

3. The midpoints of two symmedians

Now we are prepared for stating and proving the three properties of the symmedian point. The first one is as follows:

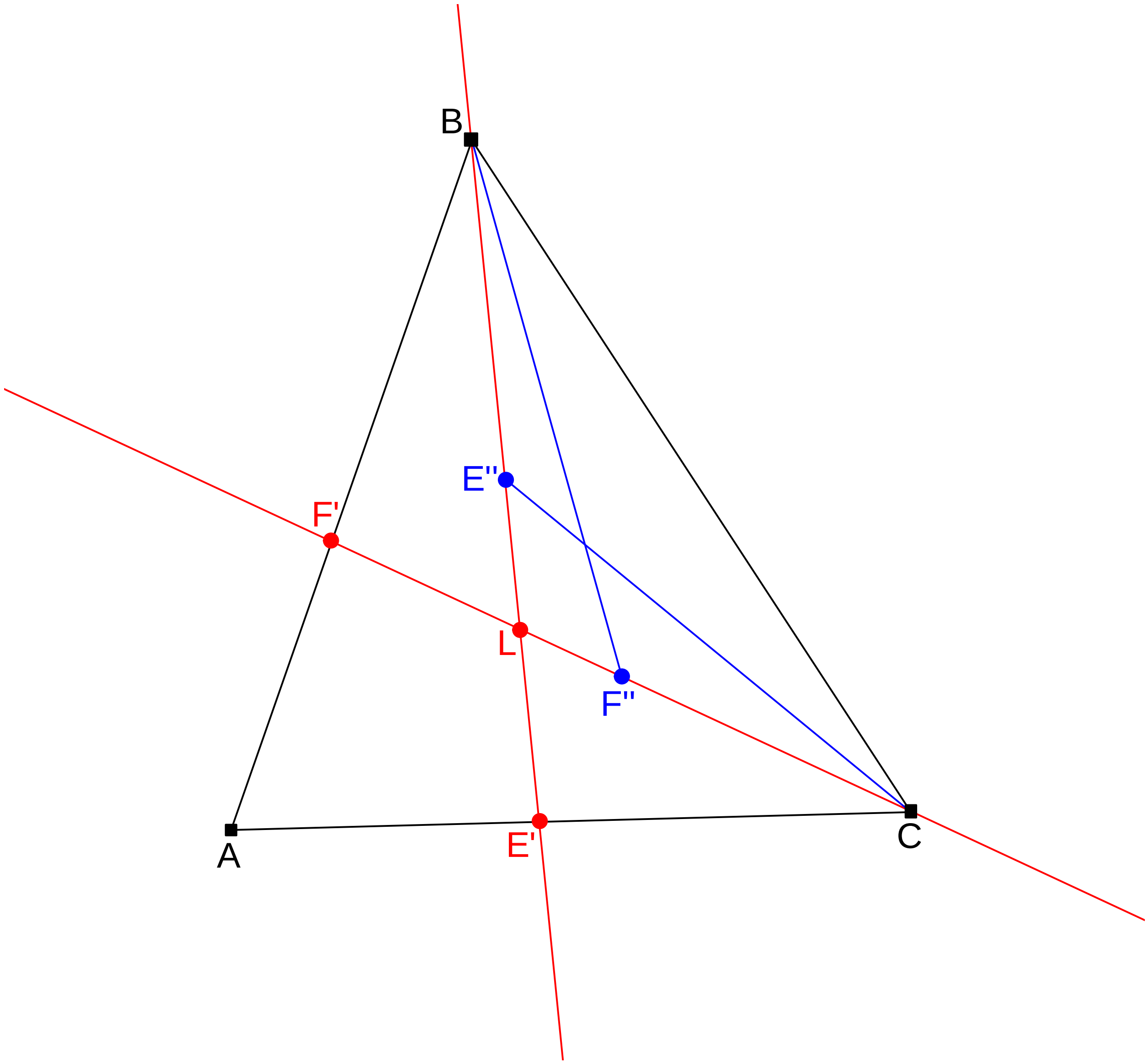

Fig. 9

Theorem 6. Let L be the symmedian point of a triangle ABC. The symmedians BL and CL of triangle ABC intersect the sides CA and AB at the points E' and F', respectively. Denote by E'' and F'' the midpoints of the segments BE' and CF'. Then:

a) We have $\measuredangle BCE'' = -\measuredangle CBF''$. (See Fig. 9.)

b) The lines BF'' and CE'' are symmetric to each other with respect to the perpendicular bisector of the segment BC. (See Fig. 10.)

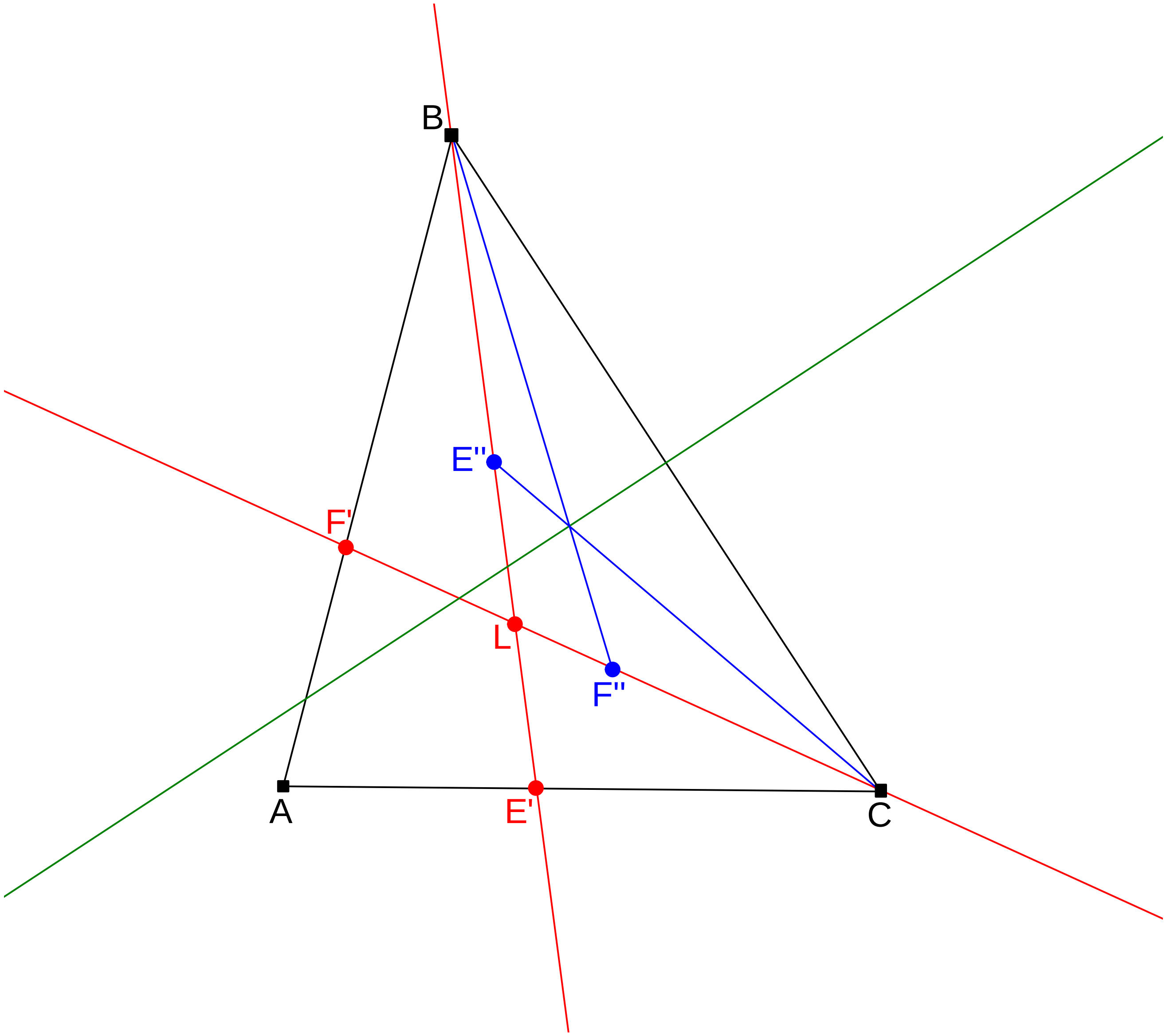

Fig. 10

Proof of Theorem 6. (See Fig. 11.) We will make use of the tangential triangle DEF defined in Theorem 4.

In the above, we have constructed the midpoint A' of the side BC of triangle ABC. Now let B' and C' be the midpoints of its sides CA and AB. The line $B'C'$ intersects the line DE at a point R.

First we will show that $AR \parallel CF$.

We will use directed segments. After Theorem 4, the lines AD, BE, CF concur at one point, namely at the point L. Hence, by the Ceva theorem, applied to the triangle DEF, we have

$$\frac{EA}{AF} \cdot \frac{FB}{BD} \cdot \frac{DC}{CE} = 1.$$

Let the parallel to the line DE through the point F intersect the lines BC and CA at the points F_a and F_b, respectively. Then, since $F_aF_b \parallel DE$, Thales yields $\dfrac{CE}{FF_b} = \dfrac{EA}{AF}$

and $\dfrac{F_aF}{DC} = \dfrac{FB}{BD}$; hence,

$$\frac{F_aF}{FF_b} = \frac{CE}{FF_b} \cdot \frac{F_aF}{DC} \cdot \frac{DC}{CE} = \frac{EA}{AF} \cdot \frac{FB}{BD} \cdot \frac{DC}{CE} = 1.$$

Thus, the point F is the midpoint of the segment F_aF_b.

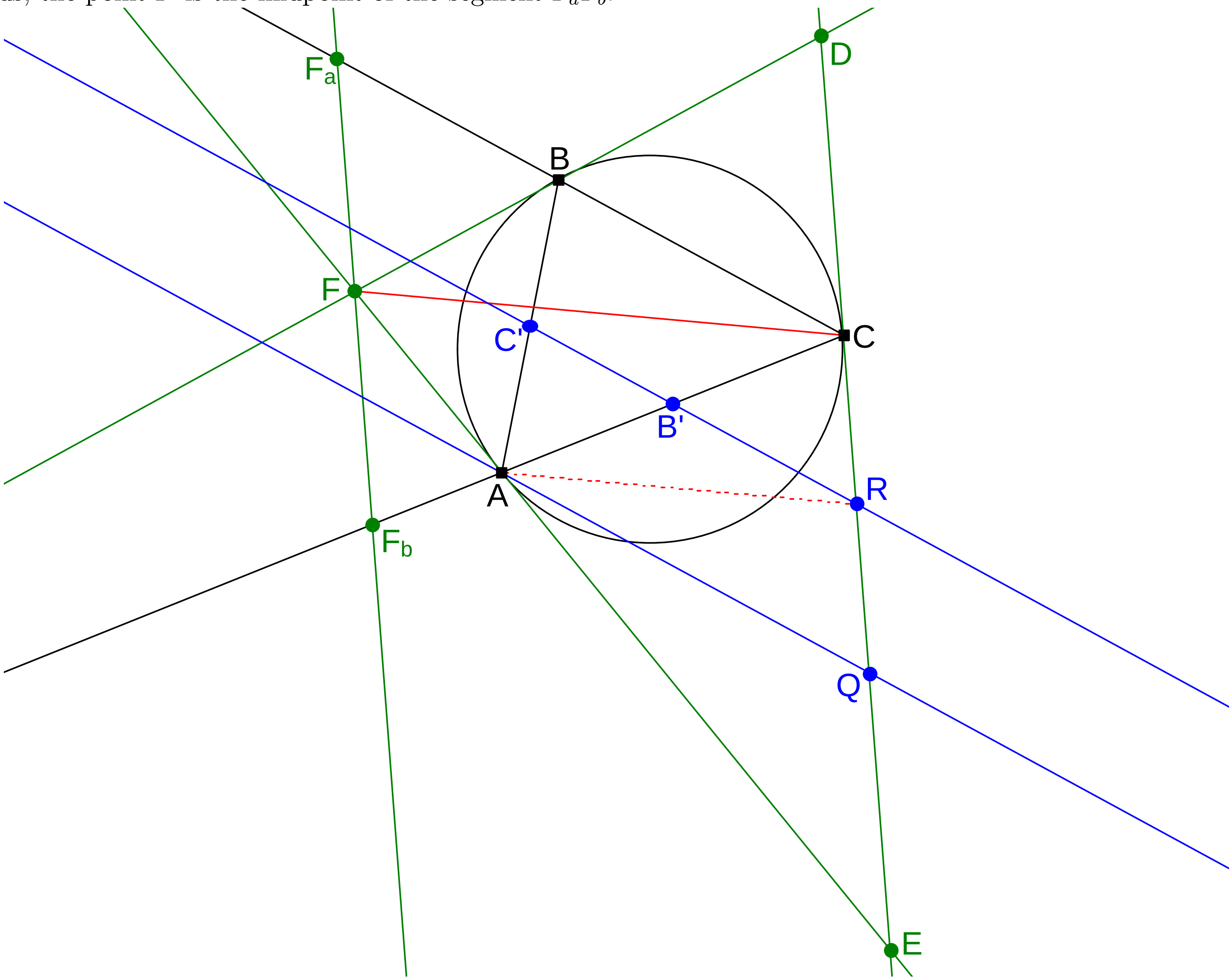

Fig. 11

Now, let the parallel to the line BC through the point A meet the line DE at Q. Since B' and C' are the midpoints of the sides CA and AB of triangle ABC, we have $B'C' \parallel BC$. Comparing this with $AQ \parallel BC$, we get $B'C' \parallel AQ$, and thus, Thales yields $\dfrac{CR}{RQ} = \dfrac{CB'}{B'A}$. But since B' is the midpoint of the segment CA, we have $\dfrac{CB'}{B'A} = 1$; thus $\dfrac{CR}{RQ} = 1$, and it follows that R is the midpoint of the segment QC.

We have $F_aF_b \parallel QC$ (this is just a different way to say $F_aF_b \parallel DE$), we have $F_bC \parallel CA$ (trivial, since the lines F_bC and CA coincide), and we have $CF_a \parallel AQ$ (this is an equivalent way of stating $BC \parallel AQ$). Hence, the corresponding sides of triangles F_aF_bC and QCA are parallel; thus, these triangles are homothetic. In other words, there exists a homothety mapping the points F_a, F_b, C to the points Q, C, A. Then, of course, this homothety must map the midpoint F of the segment F_aF_b to the midpoint R of the segment QC. Hence, this homothety maps the line CF to the line AR. Since a homothety maps any line to a parallel line, we thus have $AR \parallel CF$.

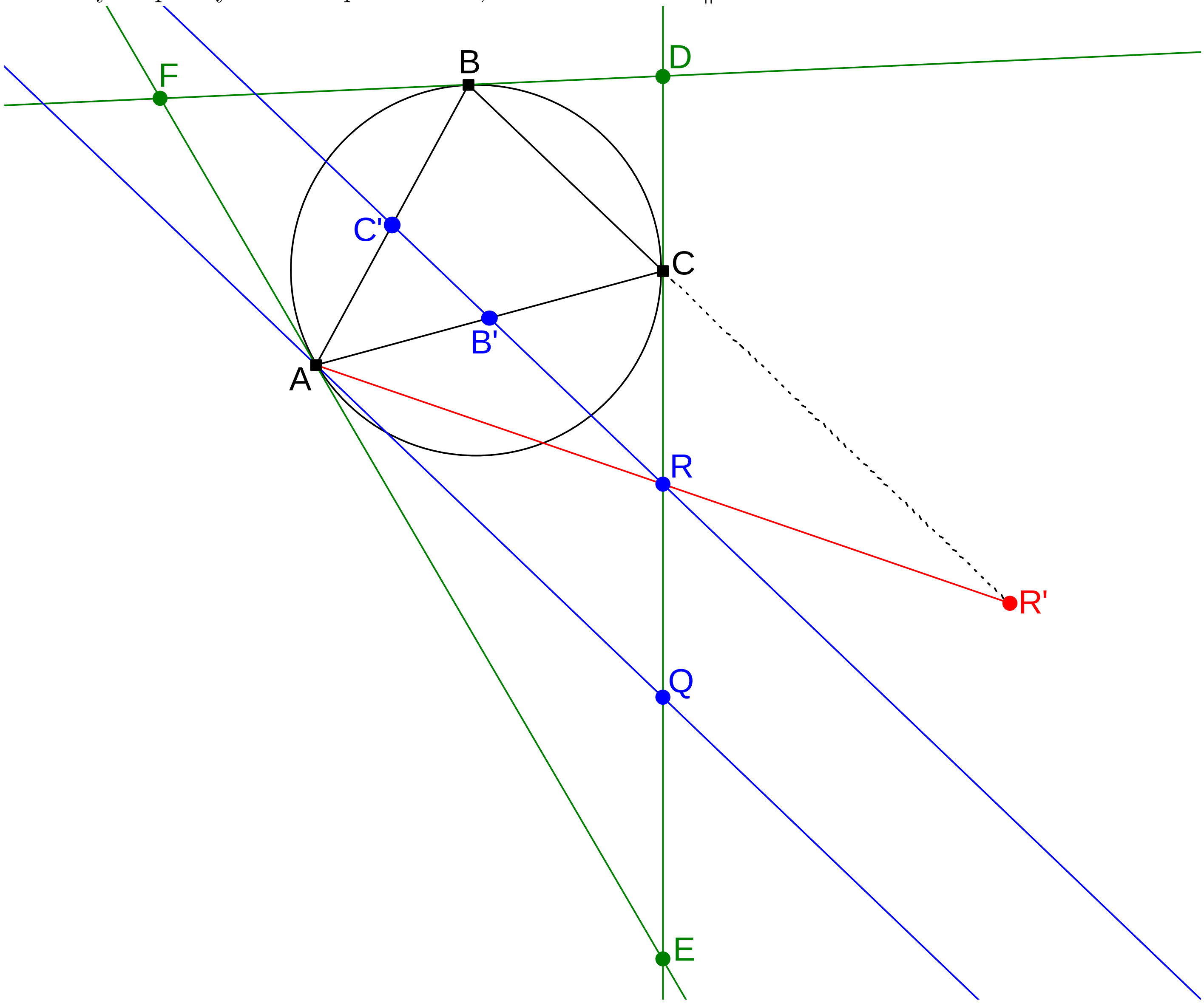

Fig. 12

(See Fig. 12.) Since R is the midpoint of the segment QC, the point C is the reflection of the point Q in the point R. Let R' be the reflection of the point A in the point R. Since reflection in a point maps any line to a parallel line, we thus have $R'C \parallel AQ$. Together with $AQ \parallel BC$, this becomes $R'C \parallel BC$. Thus, the point R' must lie on the line BC.

Since R' is the reflection of the point A in the point R, the point R is the midpoint of the segment $R'A$.

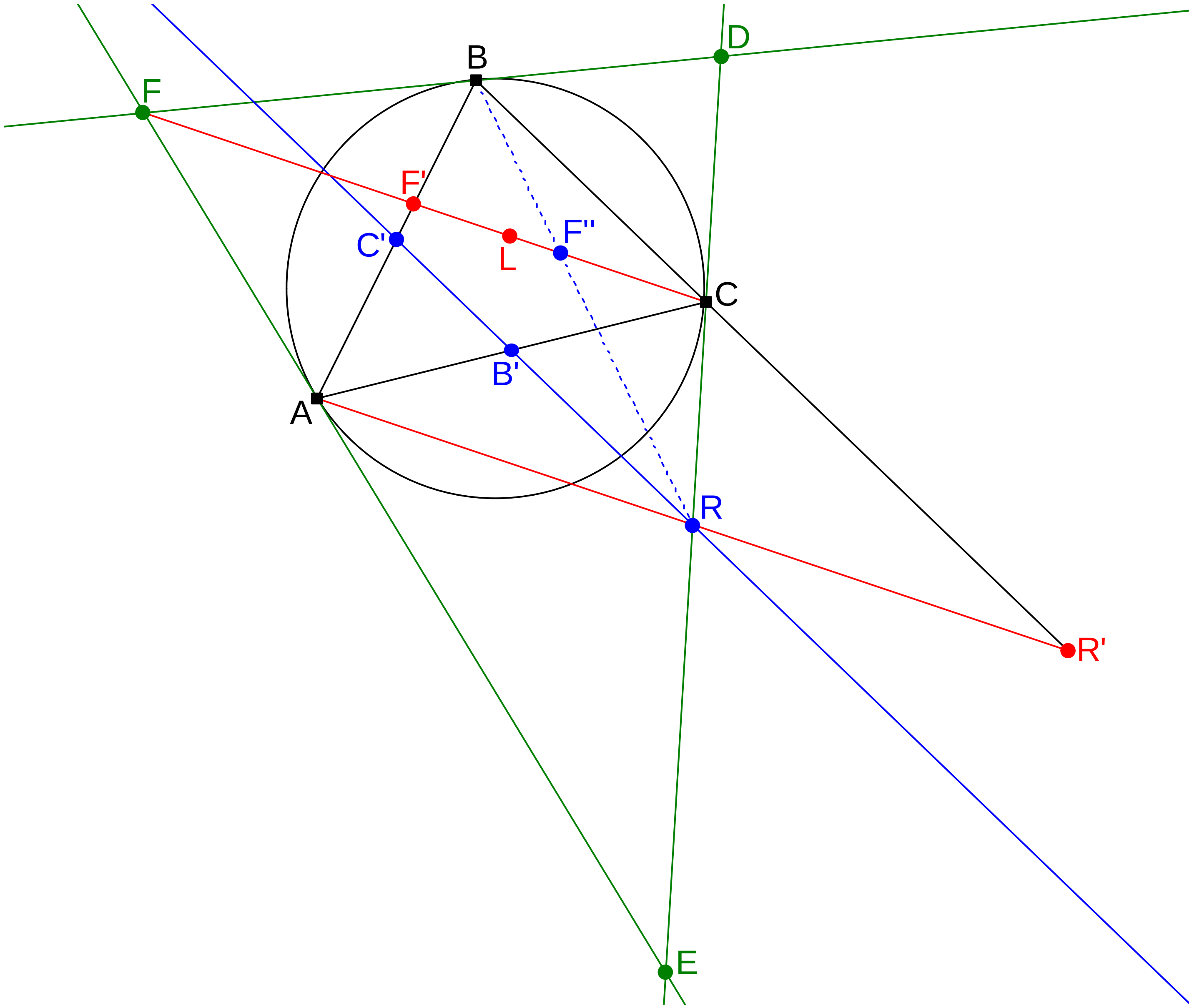

Fig. 13

(See Fig. 13.) After Theorem 4, the line CF passes through the symmedian point L of triangle ABC. In other words, the line CF coincides with the line CL. Hence, the point F', defined as the point of intersection of the lines CL and AB, is the point of intersection of the lines CF and AB. Consequently, from $AR \parallel CF$ we infer by Thales that $\frac{BC}{BR'} = \frac{BF'}{BA}$. The homothety with center B and factor $\frac{BC}{BR'} = \frac{BF'}{BA}$ maps the points R' and A to the points C and F'; hence, this homothety must also map the midpoint R of the segment $R'A$ in the midpoint F'' of the segment CF'. Hence, the points R and F'' lie on one line with the center of our homothety, i. e. with the point B. In other words, the line BF'' coincides with the line RB.

(See Fig. 14.) So we have shown that the line BF'' coincides with the line RB, where R is the point of intersection of the lines $B'C'$ and DE. Similarly, the line CE'' coincides with the line TC, where T is the point of intersection of the lines $B'C'$ and FD.

Hence, in order to prove Theorem 6 **b)**, it is enough to show that the lines RB

and TC are symmetric to each other with respect to the perpendicular bisector of the segment BC.

What is trivial is that the reflection with respect to the perpendicular bisector of the segment BC maps the point B to the point C and the point C to the point B. Furthermore, it maps the circumcircle of triangle ABC to itself (since the center of this circumcircle lies on the perpendicular bisector of the segment BC, i. e. on the axis of reflection). Hence, this reflection maps the tangent to the circumcircle of triangle ABC at the point B to the tangent to the circumcircle of triangle ABC at the point C. In other words, this reflection maps the line FD to the line DE. On the other hand, this reflection maps the line $B'C'$ to itself (since the line $B'C'$ is parallel to the line BC, and thus perpendicular to the perpendicular bisector of the segment BC, i. e. to the axis of reflection). Hence, our reflection maps the point of intersection T of the lines $B'C'$ and FD to the point of intersection R of the lines $B'C'$ and DE. Also, as we know, this reflection maps the point C to the point B. Thus, this reflection maps the line TC to the line RB. In other words, the lines RB and TC are symmetric to each other with respect to the perpendicular bisector of the segment BC. And this proves Theorem 6 **b)**.

Now, establishing Theorem 6 **a)** is a piece of cake: The reflection with respect to a line leaves directed angles invariant in their absolute value, but changes their sign. Since the reflection in the perpendicular bisector of the segment BC maps the line CE'' to the line BF'' (according to Theorem 6 **b)**), while it leaves the line BC invariant, we thus have $\measuredangle(BC;\ BF'') = -\measuredangle(BC;\ CE'')$. In other words, $\measuredangle CBF'' = -\measuredangle BCE''$. Hence, $\measuredangle BCE'' = -\measuredangle CBF''$. Thus, Theorem 6 **a)** is proven as well.

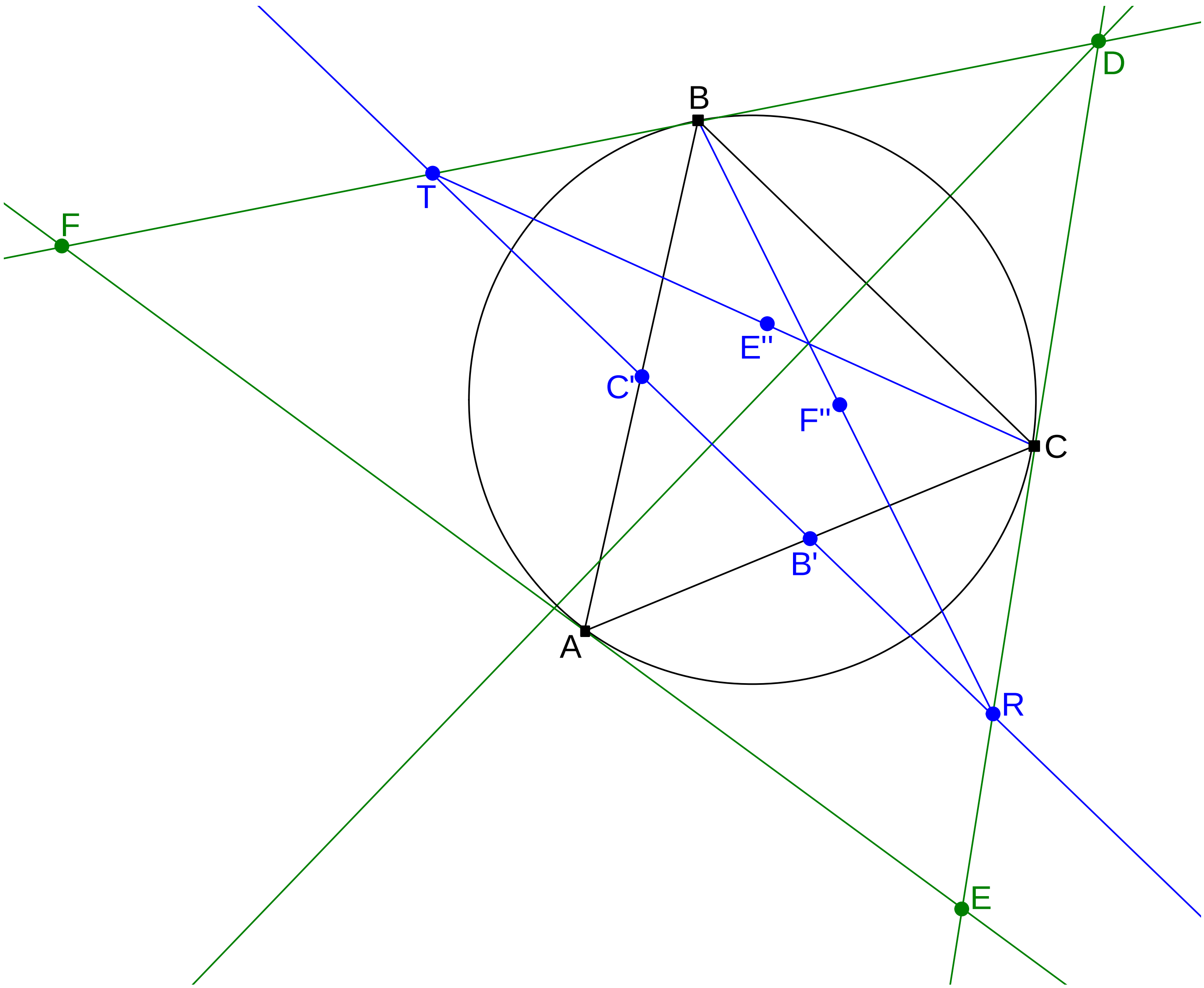

Fig. 14

The proof of Theorem 6 is thus complete. During this proof, we came up with two auxiliary results which could incidentally turn out useful, so let's compile them to a theorem:

Theorem 7. In the configuration of Theorem 6, let B' and C' be the midpoints of the sides CA and AB of triangle ABC, and let R be the point of intersection of the lines $B'C'$ and DE. Then:

a) We have $AR \parallel CF$.

b) The points R, F'' and B lie on one line. (See Fig. 15.)

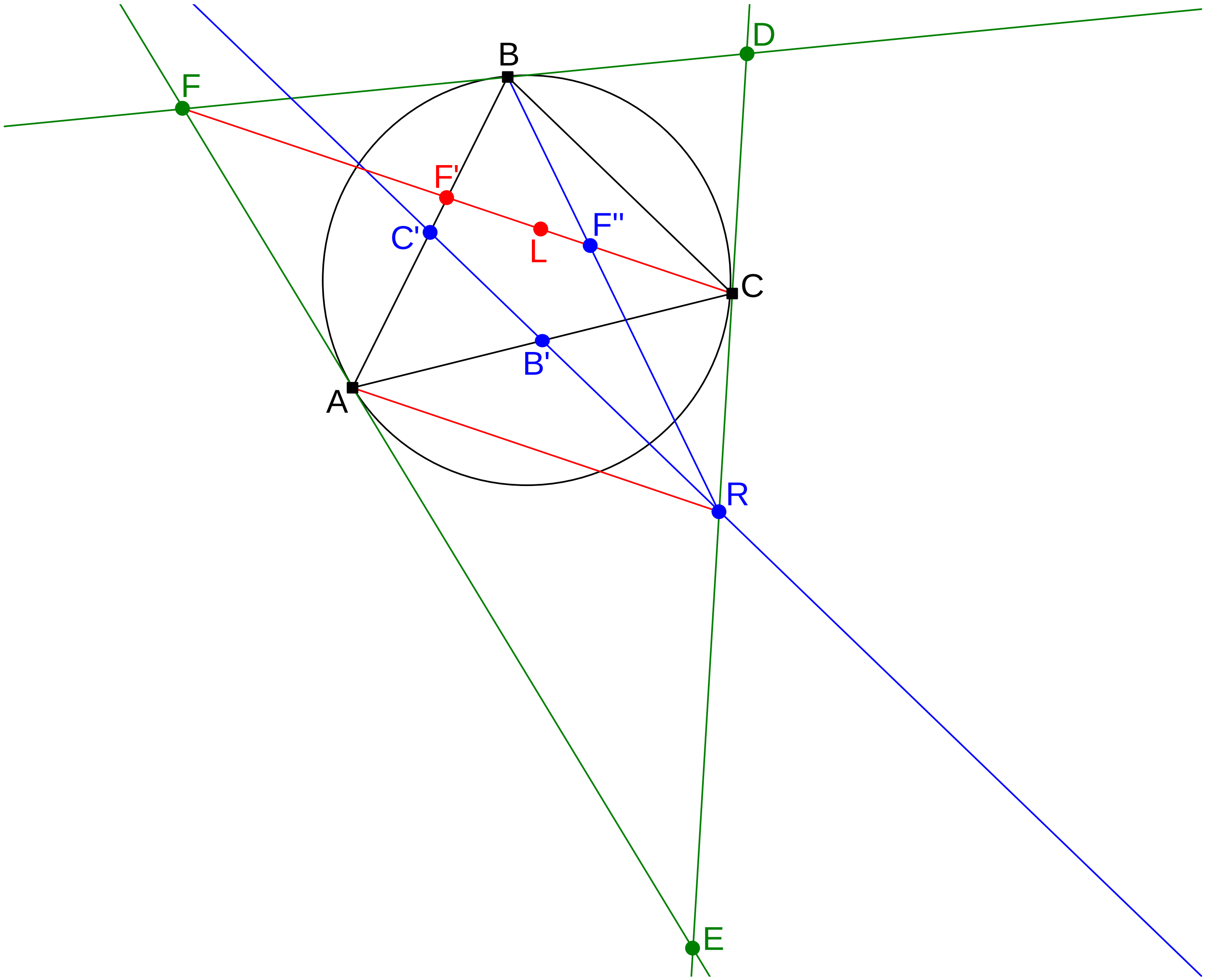

Fig. 15

Note that Theorem 6 **a)** forms a part of the problem G5 from the IMO Shortlist 2000. The two proposed solutions of this problem can be found in [2], p. 49-51, and both of them require calculation. (The original statement of this problem G5 doesn't use the notion of the symmedian point; instead of mentioning the symmedians BL and CL, it speaks of the lines BE and CF, what is of course the same thing, according to Theorem 4).

4. The point J on SL such that $\frac{SJ}{JL} = \frac{2}{3}$

Our second fact about the symmedian point originates from a locus problem by Antreas P. Hatzipolakis. Here is the most elementary formulation of this fact (Fig. 16):

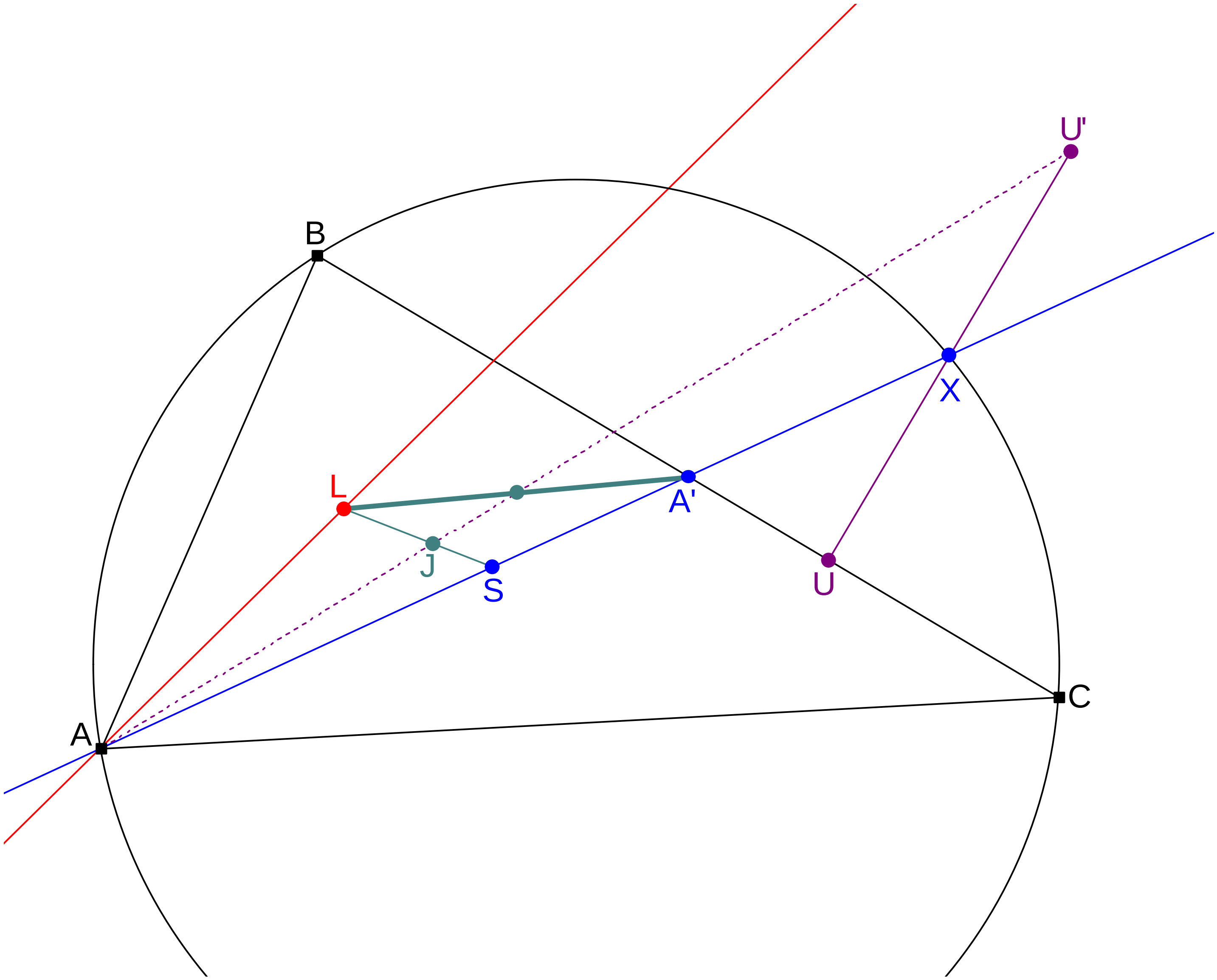

Fig. 16

Theorem 8. Let ABC be a triangle, and let A' be the midpoint of its side BC. Let S be the centroid and L the symmedian point of triangle ABC. Let J be the point on the line SL which divides the segment SL in the ratio $\frac{SJ}{JL} = \frac{2}{3}$.

Let X be the point of intersection of the median AS of triangle ABC with the circumcircle of triangle ABC (different from A). Denote by U the orthogonal projection of the point X on the line BC, and denote by U' the reflection of this point U in the point X.

Then, the line AU' passes through the point J and bisects the segment LA'.

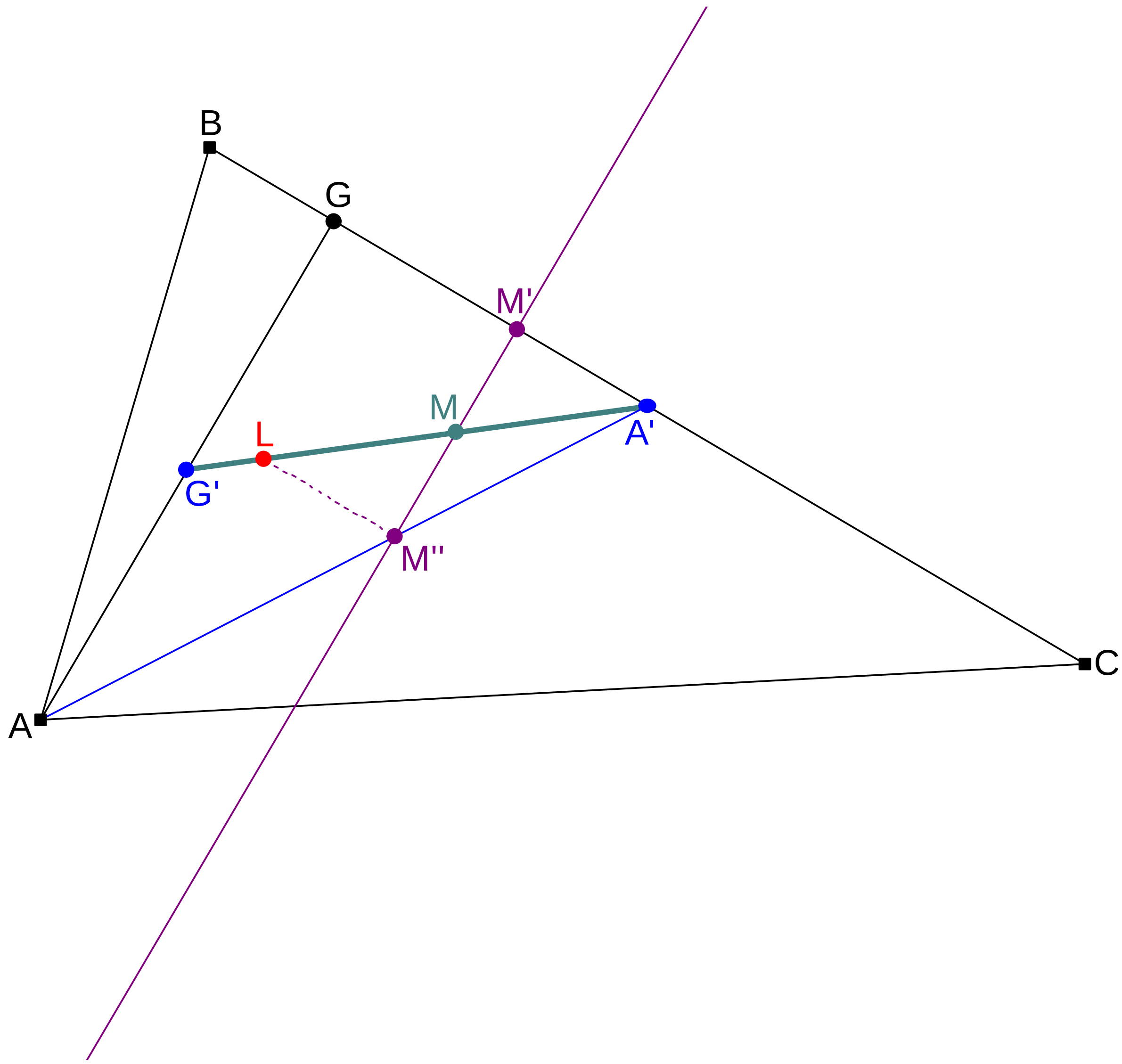

Fig. 17

Proof of Theorem 8. Since S is the centroid of triangle ABC, the line AS is the median of triangle ABC issuing from the vertex A, and thus passes through the midpoint A' of its side BC. Also, it passes through X (remember the definition of X). Hence, the four points A, S, A' and X lie on one line.

(See Fig. 17.) Let M be the midpoint of the segment LA'. Then, the point L is the reflection of the point A' in the point M.

We will use the auxiliary points constructed in Theorem 5. This means: Let G be the foot of the altitude of triangle ABC issuing from the vertex A, and let G' be the midpoint of the segment AG.

According to Theorem 5, the line $A'G'$ passes through the symmedian point L; in other words, the points L, A' and G' lie on one line. Of course, the midpoint M of the segment LA' must also lie on this line.

Let M' be the foot of the perpendicular from the point M to the line BC, and let M'' be the point where this perpendicular meets the line AA'.

The line AG is, as an altitude of triangle ABC, perpendicular to its side BC. The line $M''M'$ is also perpendicular to BC. Hence, $AG \parallel M''M'$, and thus Thales yields

$\frac{M''M}{MM'} = \frac{AG'}{G'G}$. But since G' is the midpoint of the segment AG, we have $\frac{AG'}{G'G} = 1$; thus, $\frac{M''M}{MM'} = 1$, so that M is the midpoint of the segment $M''M'$. In other words, the point M'' is the reflection of the point M' in the point M. On the other hand, the point L is the reflection of the point A' in the point M. Since reflection in a point maps any line to a parallel line, we thus have $LM'' \parallel A'M'$. In other words: $LM'' \parallel BC$.

(See Fig. 18.) Now, let X' be the point of intersection of the symmedian AL of triangle ABC with the circumcircle of triangle ABC (different from A). After Theorem 3 **b)**, we then have $XX' \parallel BC$, and after Theorem 3 **c)**, the perpendicular bisector of the segment BC is simultaneously the perpendicular bisector of the segment XX'.

Now, let N' be the orthogonal projection of the point X' on the line BC.

The lines AG, XU and $X'N'$ are all perpendicular to the line BC; thus, they are parallel to each other: $AG \parallel XU \parallel X'N'$.

As we know, the points L, A' and G' lie on one line. Let this line intersect the line XU at a point N. Then, since $AG \parallel XU$, Thales yields $\frac{XN}{NU} = \frac{AG'}{G'G}$. But as we know, $\frac{AG'}{G'G} = 1$. Hence, $\frac{XN}{NU} = 1$, so that the point N is the midpoint of the segment XU.

We have $XX' \parallel UN'$ (this is just another way to say $XX' \parallel BC$) and $XU \parallel X'N'$. Hence, the quadrilateral $XX'N'U$ is a parallelogram. Since $X'N' \perp BC$, we have $\measuredangle X'N'U = 90^\circ$; thus, this parallelogram has a right angle, and thus is a rectangle. In a rectangle, the perpendicular bisectors of opposite sides coincide; hence, in the rectangle $XX'N'U$, the perpendicular bisector of the segment XX' coincides with the perpendicular bisector of the segment UN'. On the other hand, as we know, the perpendicular bisector of the segment XX' coincides with the perpendicular bisector of the segment BC. Thus, the perpendicular bisector of the segment UN' coincides with the perpendicular bisector of the segment BC. Now, both segments UN' and BC lie on one line; hence, this coincidence actually yields that the midpoint of the segment UN' coincides with the midpoint of the segment BC. In other words, the midpoint A' of the segment BC is simultaneously the midpoint of the segment UN'.

Since A' and N are the midpoints of the sides UN' and XU of triangle $N'XU$, we have $A'N \parallel N'X$. But the line $A'N$ is simply the line LA'. Hence, $LA' \parallel N'X$.

Since the quadrilateral $XX'N'U$ is a parallelogram, we have $UX = N'X'$, where we use directed segments and the two parallel lines XU and $X'N'$ are oriented in the same direction. On the other hand, $UX = XU'$, since U' is the reflection of the point U in the point X. Hence, $XU' = N'X'$. Together with $XU' \parallel N'X'$ (this is just an equivalent version of $XU \parallel X'N'$), this yields that the quadrilateral $XU'X'N'$ is a parallelogram, and thus $N'X \parallel U'X'$. Together with $LA' \parallel N'X$, this leads to $LA' \parallel U'X'$.

From $LM'' \parallel BC$ and $XX' \parallel BC$, we infer $LM'' \parallel XX'$.

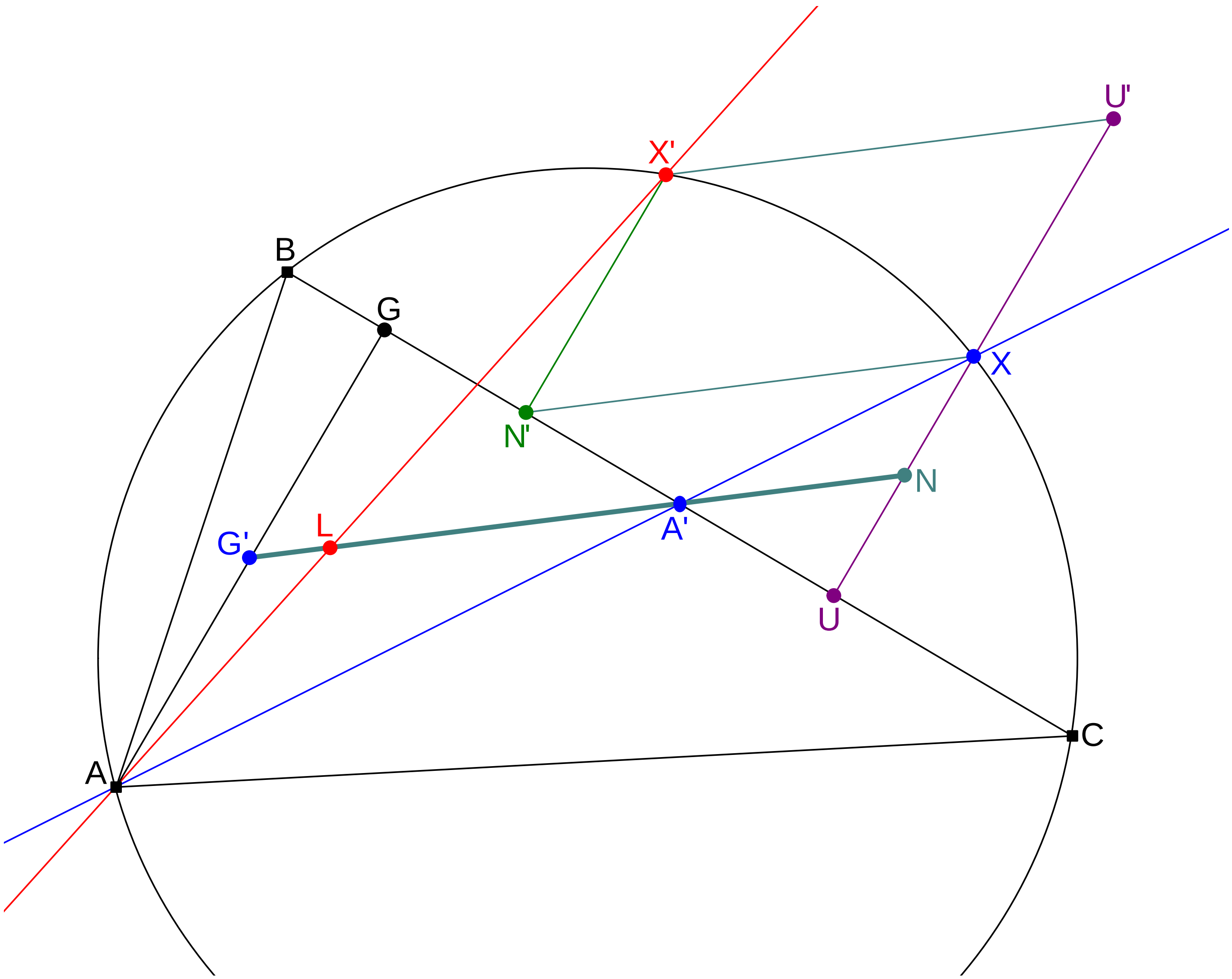

Fig. 18

(See Fig. 19.) Since $M''M \perp BC$ and $XU' \perp BC$ (the latter is just a different way to write $XU \perp BC$), we have $M''M \parallel XU'$. Furthermore, $ML \parallel U'X'$ (this is equivalent to $LA' \parallel U'X'$) and $LM'' \parallel X'X$ (this is equivalent to $LM'' \parallel XX'$). Hence, the corresponding sides of triangles $LM''M$ and $X'XU'$ are parallel. Thus, these triangles are homothetic; hence, the lines LX', $M''X$, MU' concur at one point (namely, at the center of homothety). In other words, the point of intersection of the lines LX' and $M''X$ lies on the line MU'. But the point of intersection of the lines LX' and $M''X$ is simply the point A, and hence, we obtain that the point A lies on the line MU'. To say it differently, the line AU' passes through the point M, hence through the midpoint of the segment LA'. In other words, the line AU' bisects the segment LA'.

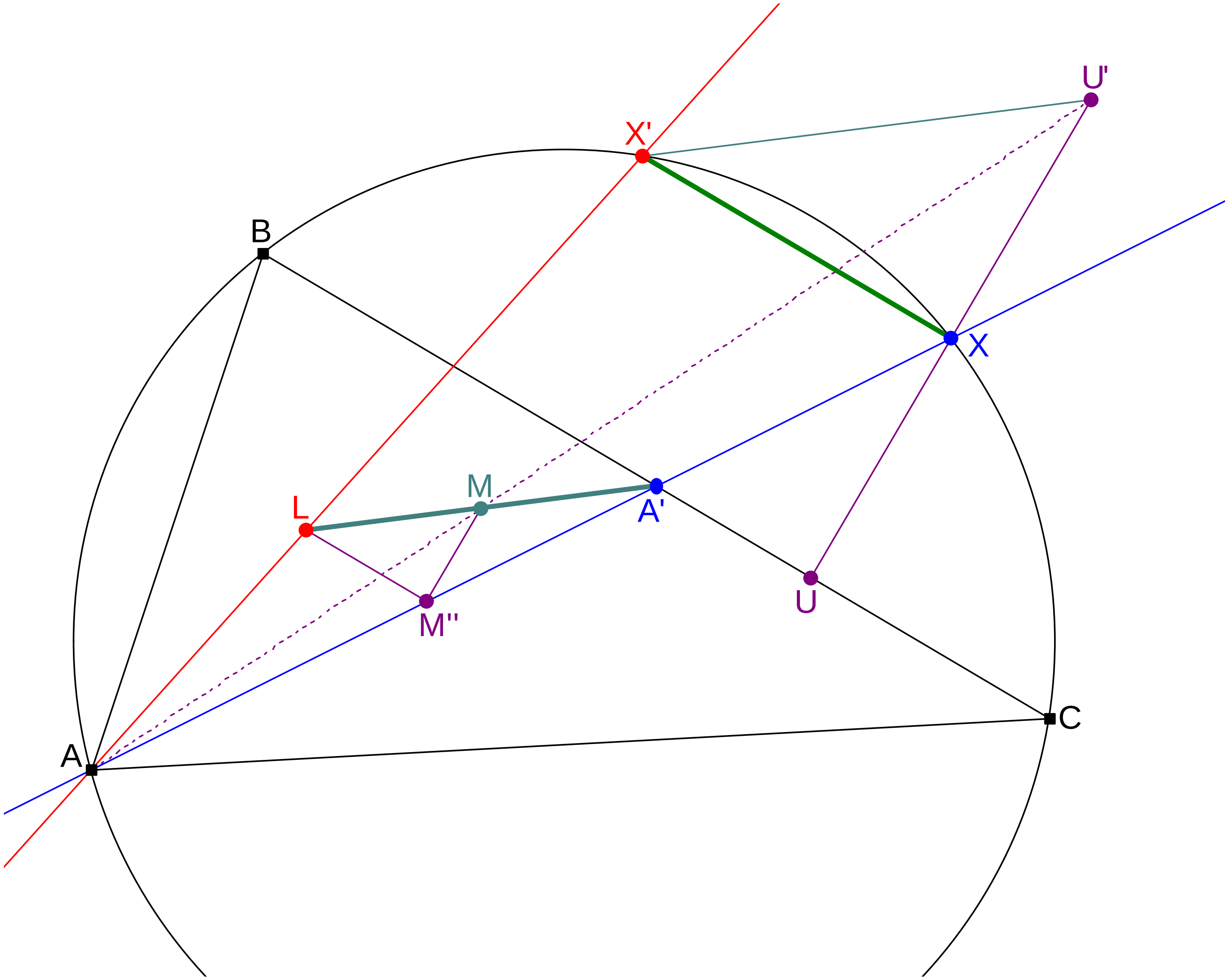

Fig. 19

This shows a part of Theorem 8; the rest is now an easy corollary:

(See Fig. 20.) It is well-known that the centroid S of triangle ABC divides the median AA' in the ratio $\frac{AS}{SA'} = 2$. Thus, $\frac{SA'}{AS} = \frac{1}{2}$, so that $\frac{AA'}{AS} = \frac{AS + SA'}{AS} = 1 + \frac{SA'}{AS} = 1 + \frac{1}{2} = \frac{3}{2}$, and thus $\frac{A'A}{AS} = -\frac{AA'}{AS} = -\frac{3}{2}$. According to its definition, the point J lies on the line SL and satisfies $\frac{SJ}{JL} = \frac{2}{3}$; finally, $\frac{LM}{MA'} = 1$, since M is the midpoint of the segment LA'. Hence,

$$\frac{A'A}{AS} \cdot \frac{SJ}{JL} \cdot \frac{LM}{MA'} = \left(-\frac{3}{2}\right) \cdot \frac{2}{3} \cdot 1 = -1.$$

By the Menelaos theorem, applied to the triangle $LA'S$ and the points A, J, M on its sidelines $A'S$, SL, LA', this yields that the points A, J, M lie on one line. In other words, the line AM passes through the point J. But the line AM is the same as the line AU' (since the line AU' passes through M); hence, we can conclude that the line AU' passes through the point J. Thus, Theorem 8 is completely proven.

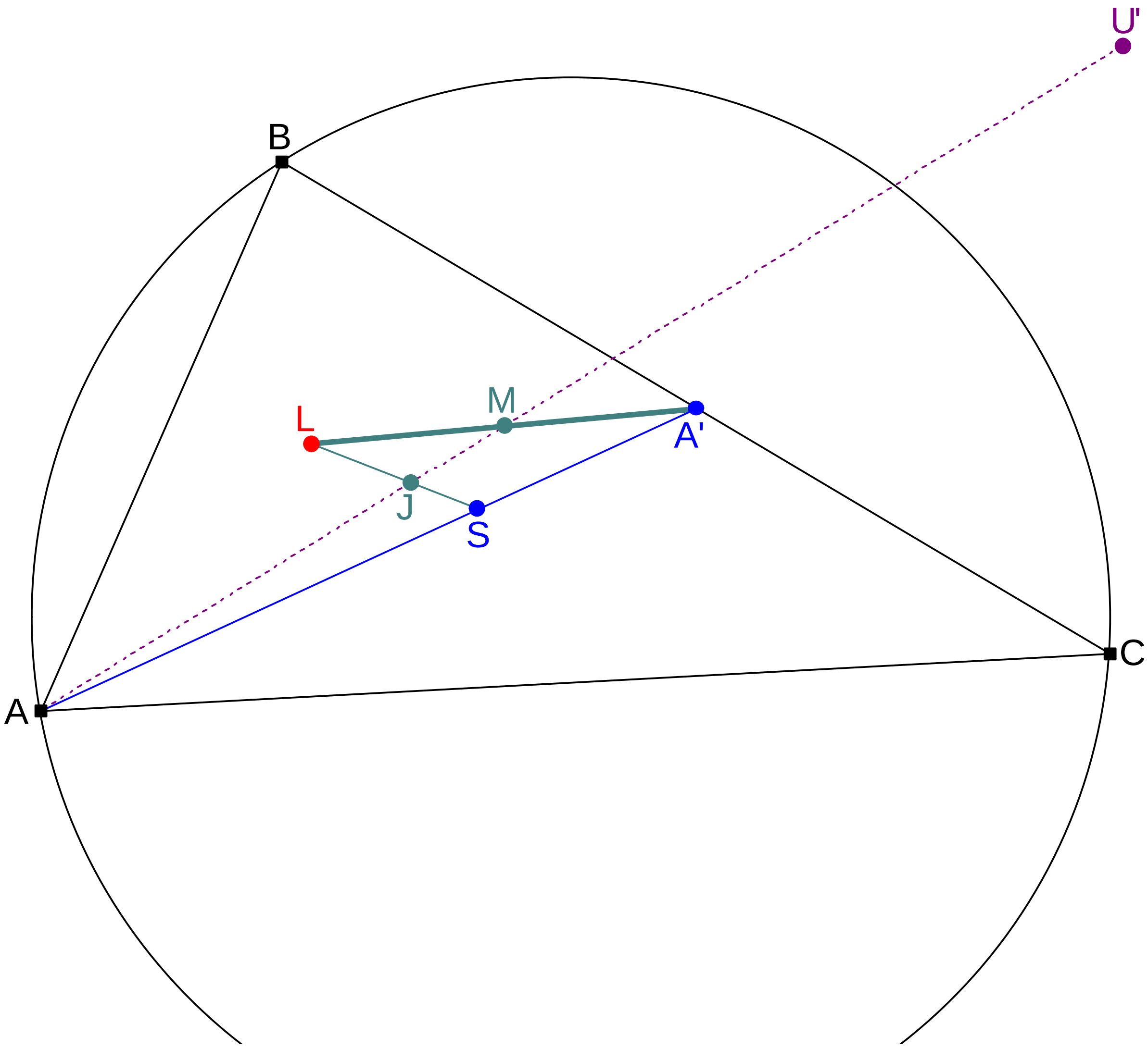

Fig. 20

Theorem 8 gives an assertion about the line AU'; we can obtain two analogous assertions by cyclic permutation of the vertices A, B, C. Combining these three assertions, we get the following symmetric variant of Theorem 8:

Theorem 9. Let ABC be a triangle, and let A', B', C' be the midpoints of its sides BC, CA, AB. Let S be the centroid and L the symmedian point of triangle ABC. Let J be the point on the line SL which divides the segment SL in the ratio $\frac{SJ}{JL} = \frac{2}{3}$.

Let X, Y, Z be the points of intersection of the medians AS, BS, CS of triangle ABC with the circumcircle of triangle ABC (different from A, B, C). Let U, V, W be the orthogonal projections of the points X, Y, Z on the lines BC, CA, AB, and let U', V', W' be the reflections of these points U, V, W in the points X, Y, Z.

Then, the lines AU', BV', CW' pass through the point J and bisect the segments LA', LB', LC', respectively. (See Fig. 21.)

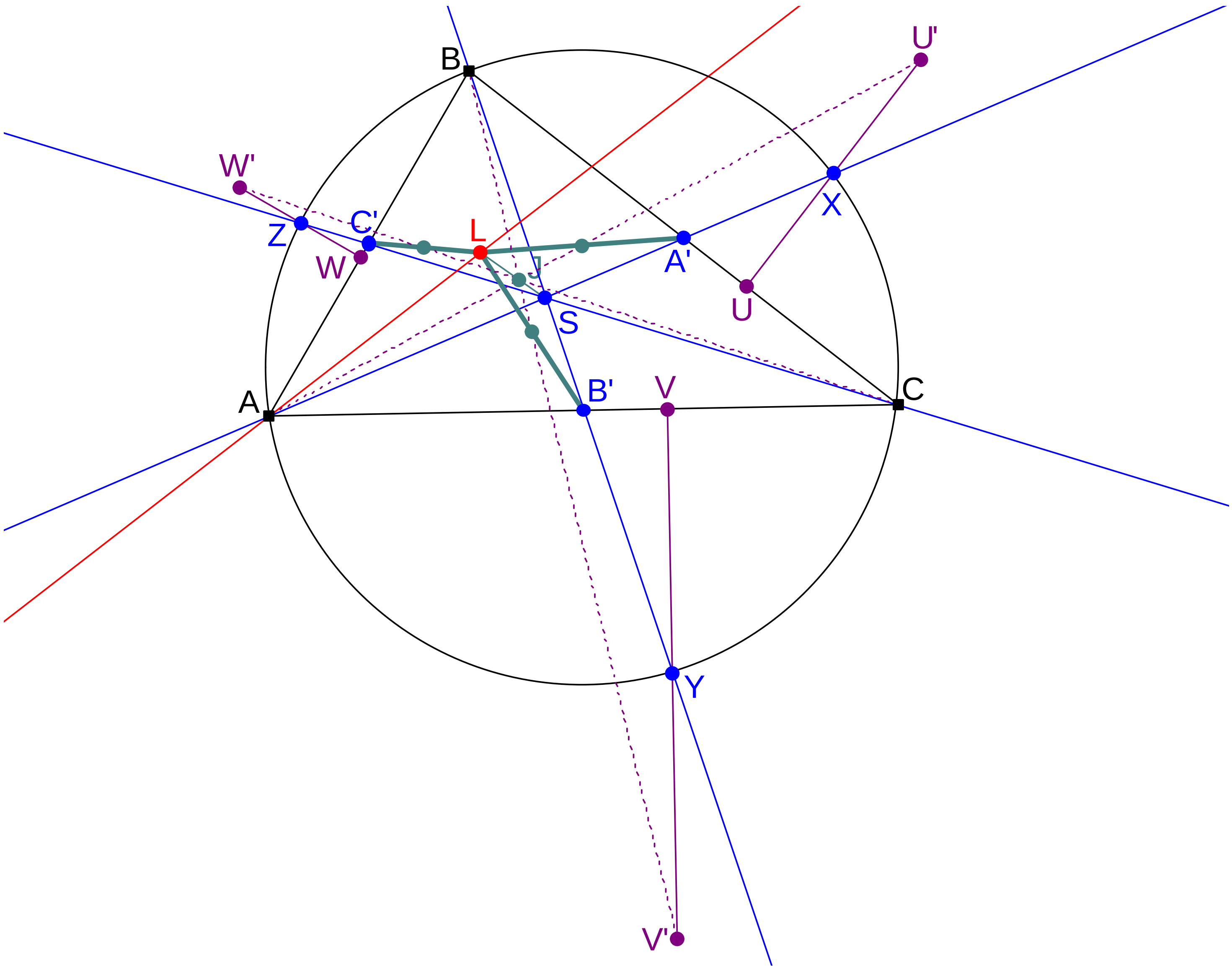

Fig. 21

5. A remarkable cross-ratio

Finally, as a side-product of our above observations, we will establish our third property of the symmedian point (a rather classical one compared with the former two).

(See Fig. 22.) According to Theorem 3 **c)**, the perpendicular bisector of the segment BC is simultaneously the perpendicular bisector of the segment XX'. The point A', being the midpoint of the segment BC, lies on the perpendicular bisector of the segment BC; hence, it must lie on the perpendicular bisector of the segment XX'. Thus, $A'X = A'X'$. Therefore, the triangle $XA'X'$ is isosceles, what yields $\measuredangle A'X'X = \measuredangle X'XA'$. In other words, $\measuredangle(A'X';\ XX') = \measuredangle(XX';\ AX)$. But $XX' \parallel BC$ implies $\measuredangle(A'X';\ XX') = \measuredangle(A'X';\ BC)$ and $\measuredangle(XX';\ AX) = \measuredangle(BC;\ AX)$; thus, $\measuredangle(A'X';\ BC) = \measuredangle(BC;\ AX)$. This rewrites as $\measuredangle X'A'B = \measuredangle BA'A$.

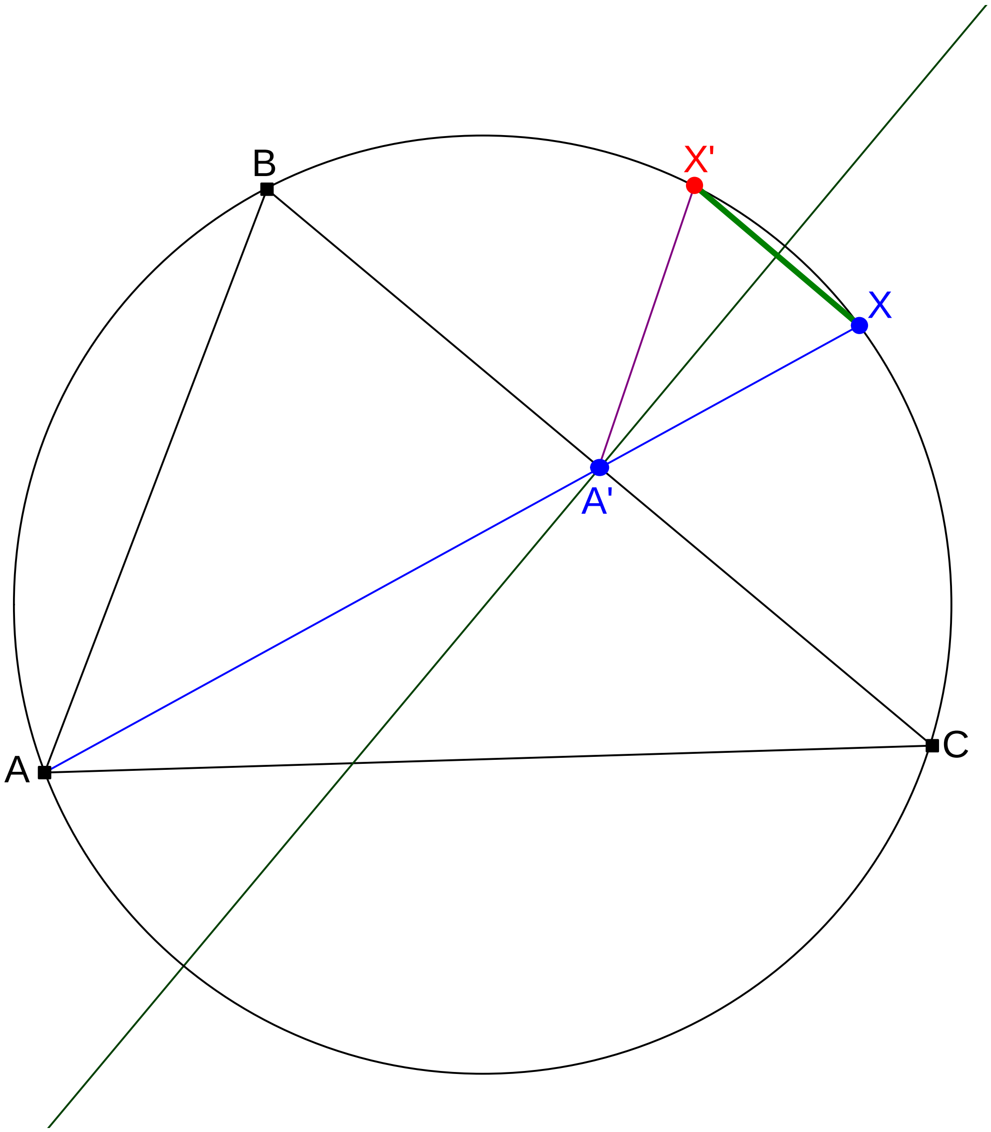

Fig. 22

(See Fig. 23.) Now let A_1 be the reflection of the point A in the line BC. On the other hand, G is the foot of the perpendicular from A to BC. Thus, G is the midpoint of the segment AA_1. Hence, $\frac{AA_1}{GA_1} = 2$, and thus $\frac{AA_1}{A_1G} = -\frac{AA_1}{GA_1} = -2$.

Since A_1 is the reflection of the point A in the line BC, we have $\measuredangle A_1A'B = \measuredangle BA'A$. Comparison with $\measuredangle X'A'B = \measuredangle BA'A$ yields $\measuredangle A_1A'B = \measuredangle X'A'B$; thus, the points A', A_1 and X' lie on one line. If we denote by D' the point where the symmedian AL of triangle ABC meets the side BC, then the points A', G and D' lie on one line. Finally, the points A', G' and L lie on one line, and the points A', A and A lie on one line (trivial). But the points A, G, G' and A_1 lie on one line, and the points A, D', L and X' lie on one line. Hence, by the invariance of the cross-ratio under central projection,

$$\frac{AL}{LD'} : \frac{AX'}{X'D'} = \frac{AG'}{G'G} : \frac{AA_1}{A_1G}.$$

But since G' is the midpoint of the segment AG, we have $\frac{AG'}{G'G} = 1$. Furthermore,

$\frac{AA_1}{A_1G} = -2$. Hence, we obtain

$$\frac{AL}{LD'} : \frac{AX'}{X'D'} = 1 : (-2) = -\frac{1}{2}.$$

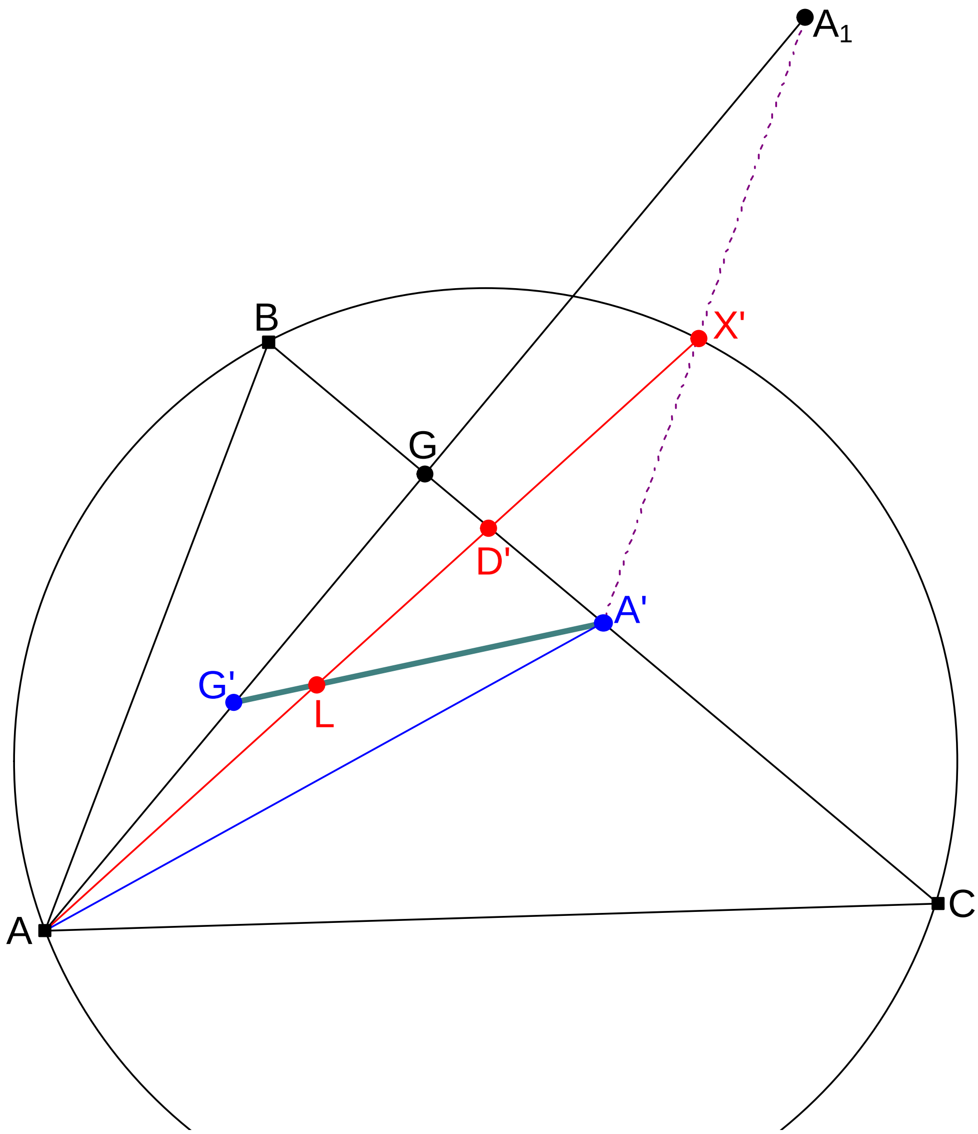

Fig. 23

We formulate this as a theorem:

Theorem 10. Let L be the symmedian point of a triangle ABC. Let the symmedian AL of triangle ABC meet the side BC at a point D' and the circumcircle of triangle ABC at a point X' (different from A). Then,

$$\frac{AL}{LD'} : \frac{AX'}{X'D'} = -\frac{1}{2}.$$

(See Fig. 24.)

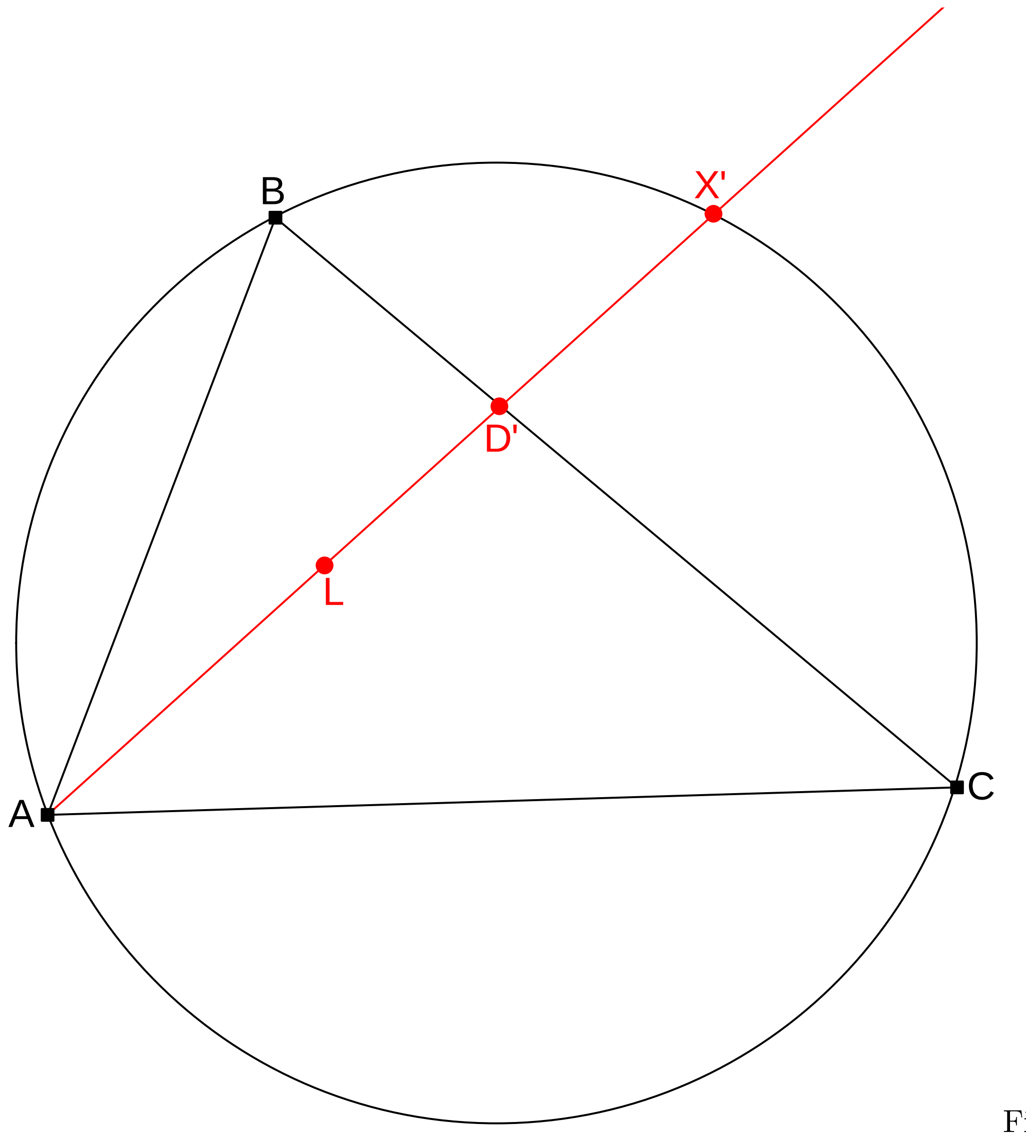

Fig. 24

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(you have to register at MathLinks in order to be able to download the files, but registration is free and painless; a mirror can be found at `http://www.ajorza.org` , but this server is currently down).

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