Delocking of flux-flow states in dc-driven magnetically coupled Josephson junctions

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We report analytical and numerical results for delocking of flux-flow states in two magnetically coupled annular Josephson junctions (JJ's). JJ stacks with different damping parameters are considered. An analytical model of the delocking is developed for dense fluxon chains. Analytically predicted and numerically found delocking thresholds are in good agreement.

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I. INTRODUCTION

Recently, stacked Josephson junctions (SJJ) have received much attention due to several new effects found in such systems. In particular, SJJs were proposed as coherently operating magnetically coupled oscillators with enhanced power and frequency yield [1]. Spectral characteristics of the two-fold SJJ have been investigated numerically [2]. It has been found that the spread of parameters affects mainly the locking margins, rather than the spectrum of the output voltage.

In this work, our objective is to investigate in detail the margins of the delocking region between flux-flow states in two annular magnetically coupled JJ's. We assume that both junctions are biased by the same dc current, while their dissipation constants are different. This situation is relevant for a typical experiment. In fact, the subgap resistances could differ by as much as one order of magnitude whereas the ratio of critical currents does not exceed 2. It is necessary to mention that a similar locking/delocking problem was considered, both theoretically and experimentally, in Ref. [3]. However, there are some essential differences from our approach: the dissipation constants were identical and only one junction was biased. In another recent work [4], synchronization and desynchronization of two JJs operating in flux-flow regimes in relatively short linear coupled junctions was considered from the viewpoint of the Fiske-step theory. For the single fluxon case, the analytical and numerical analysis of delocking was accomplished in [5] for the case of different Swihart velocities of JJ's. Due to the small value of coupling parameter $|\Delta| = 0.05$ in Ref. [5], the authors gave incorrect interpretation of the delocked state. In the delocked state fluxons in the stack move with velocities $\bar{c}_-$ and $\bar{c}_+$ that characterize the whole system rather than with velocities $\bar{c}_1$ and $\bar{c}_2$ that characterize uncoupled JJ's. Having small Δ, $\bar{c}_1$ is very close to $\bar{c}_-$ and $\bar{c}_2$ is very close to $\bar{c}_+$.

The paper is organized as follows. In section II, the analytical model is formulated and an equation predicting delocking points is derived. Results of numerical simulations and their comparison with the model are presented in section III. Concluding remarks are collected in section IV.

II. THEORETICAL MODEL

The theoretical model describing the dynamics of N-fold stacks of magnetically coupled long Josephson junctions (LJJ) was developed by Sakai *et al.* [6]. For two weakly coupled junctions, in the standard notation, it takes the form

$$\phi^A_{xx} - \phi^A_{tt} - \sin\phi^A = -\gamma + \alpha^A\phi^A_t - \Delta\phi^B_{xx}; \quad (1)$$
$$\phi^B_{xx} - \phi^B_{tt} - \sin\phi^B = -\gamma + \alpha^B\phi^B_t - \Delta\phi^A_{xx}, \quad (2)$$

where $\Delta > 0$ is the coupling constant, γ is the dc bias current density, and $\alpha^{A,B}$ are dissipation constants in the two junctions. We will develop an analytical approximation for the case of large fluxon density. A similar analytical approach was already considered in [3]. Here, we assume the same bias currents but different loss parameters of the JJ's. Moreover, we extend the approach of [3] to the relativistic case. The most important effect, the decoupling of the fluxon chains in the junctions, mainly takes place in the relativistic region. The subsequent comparison with numerical simulations will show that the analytical model provides a satisfactory agreement only in the general relativistic form.

We will consider the case of two fluxon chains locked in out-of-phase mode. The limiting velocity in this mode is equal to $\bar{c}_- = \bar{c}_0/\sqrt{1+\Delta}$. Neglecting the perturbation term in the r.h.s. of Eqs. (1) and (2), one can use the well-known exact flux-flow solution to the unperturbed sine-Gordon equation. In the high-density limit, when the mean magnetic field $H \equiv \overline{\phi^A_x} = \overline{\phi^B_x}$ is a large parameter, the solution can be conveniently represented by the expansion

$$\phi^{A,B} = (x - ut - \xi_{A,B})H - \frac{\sin\left[(x-ut-\xi_{A,B})H\right]}{H^2\left(1-u^2\right)}, \quad (3)$$

where u $(0 < u < 1)$ is the fluxon chain velocity assumed to be the same in both JJ's, and $\xi_{A,B}$ are arbitrary constants. We have to mention here, that the fluxon velocity in (3) is normalized to the modified Swihart velocity $\bar{c}_- = \bar{c}_0/\sqrt{1+\Delta}$.

Following the pattern of Ref. [3], we consider each fluxon chain as an effective particle with the single degree of freedom ξ_A or ξ_B. Our immediate objective is to derive equations of motion for these coordinates. To this end, we notice that the momentum of each "particle" is

$$P_{A,B} = -\int_0^L \phi_x^{A,B}\phi_t^{A,B}\,dx = H^2uL, \tag{4}$$

L being the length of the junction (L and H are related according to the periodic boundary condition, $HL = 2\pi N$, where N is the number of the flux quanta trapped in the annular junction, i.e. H is quantized). The Newton's equation of motion reads that the time derivative of the momentum is the net force applied to the particle. In our model, we have three forces: friction F_α, driving F_γ, and coupling F_Δ forces. First of all, it is straightforward to find the friction and driving forces:

$$F_\alpha^{A,B} = -\alpha^{A,B}P_{A,B} = -\alpha^{A,B}H^2uL, \tag{5}$$

$$F_\gamma = \gamma HL. \tag{6}$$

Next, the simplest way to calculate the coupling force F_Δ is using the coupling Hamiltonian H_Δ which generates the coupling terms in Eqs. (1) and (2):

$$H_\Delta = -\Delta\int_0^L \phi_x^A\phi_x^B\,dx. \tag{7}$$

Substituting the approximation (3) into (7), it is easy to obtain

$$H_\Delta = -\frac{1}{2}L\Delta\frac{\cos\left[(\xi_A-\xi_B)H\right]}{H^2\left(1-u^2\right)^2}. \tag{8}$$

Finally, the coupling force F_Δ acting on each "particle" can be obtained from (8) as

$$F_\Delta^{A,B} = -\frac{\partial H_\Delta}{\partial \xi_{A,B}} = \mp\frac{1}{2}L\Delta\frac{\sin\left[(\xi_A-\xi_B)H\right]}{H\left(1-u^2\right)^2}. \tag{9}$$

To get a stationary solution corresponding to a constant velocity u, we insert the expressions (5), (6), and (9) into the Newton's equations of motion for the two "particles"

$$\frac{dP_{A,B}}{dt} = F_\gamma^{A,B} - F_\alpha^{A,B} + F_\Delta^{A,B}.$$

In the stationary case $dP_{A,B}/dt = 0$ and the two Newton's equations amount to

$$\gamma = \frac{1}{2}\left(\alpha^A+\alpha^B\right)Hu, \tag{10}$$

$$\Delta\frac{\sin\left[(\xi_A-\xi_B)H\right]}{H\left(1-u^2\right)^2} = \left(\alpha^A-\alpha^B\right)H^2u. \tag{11}$$

From a formal point of view, a solution to these equations exists provided that it produces $|\sin\left[(\xi_A-\xi_B)H\right]| \le 1$. Physically, the point $|\sin\left[(\xi_A-\xi_B)H\right]| = 1$ implies cease of existence of a coupled state of the two fluxon chains, i.e., delocking. Eliminating the velocity u from Eqs. (10) and (11), we finally arrive at an inequality which determines the value of the bias current density at which the system is locked:

$$\frac{\alpha^A-\alpha^B}{\alpha^A+\alpha^B}\gamma - \frac{\Delta}{2H^2\left[1-\frac{4\gamma^2}{H^2(\alpha^A+\alpha^B)^2}\right]^2} \le 0. \tag{12}$$

Taking the equal sign in (12) one gets the expression for the critical value of γ when the system switches from the locked to unlocked state (delocking) corresponding to points A and C in Fig. 1 (see below). It is very easy to solve the fifth-degree algebraic Eq. (12) numerically. In the next section, we compare the prediction for the critical value of γ following from this equation with results of direct numerical simulations of Eqs. (1) and (2). It will be shown that Eq. (12) has two physically relevant solutions, which implies that the two fluxon chains remain locked at either very small or very large values of bias current, and are delocked at intermediate current range.

III. NUMERICAL RESULTS

Details about the numerical methods used in simulations will be published elsewhere [2]. To simulate the delocking in the present model, we took $L = 5$, $\alpha^A = 0.1$, and the periodic boundary conditions.

$$\begin{cases} \phi^{A,B}\Big|_{x=0} = \phi^{A,B}\Big|_{x=L} - 2\pi N^{A,B} \;; \\ \phi_x^{A,B}\Big|_{x=0} = \phi_x^{A,B}\Big|_{x=L} \;. \end{cases} \tag{13}$$

The number of fluxons N is preserved during simulation due to (13). N defines the fluxon density in each JJ and is equivalent to the applied external magnetic field in the case of linear geometry. We always choose equal fluxon densities $N_A = N_B = N$ in the junctions. The periodic boundary conditions allow us to neglect Fiske resonances in the system, thus, concentrating our attention on the flux-flow regimes.

A typical numerically obtained I-V characteristic is shown in Fig. 1. The current is proportional to γ, and the average voltage is proportional to the fluxon velocity. One can see that the junctions are locked at zero driving current. This can be easily understood. In a static state, the fluxons in the two junctions repel each other (coupling Force F_Δ) and form a sort of triangular array, which corresponds to the locked state. Applying a small bias current, one makes the fluxon array move as a whole. With increase of the bias current and the corresponding velocity, the difference of the friction forces acting upon the two chains also increases. Beyond the decoupling

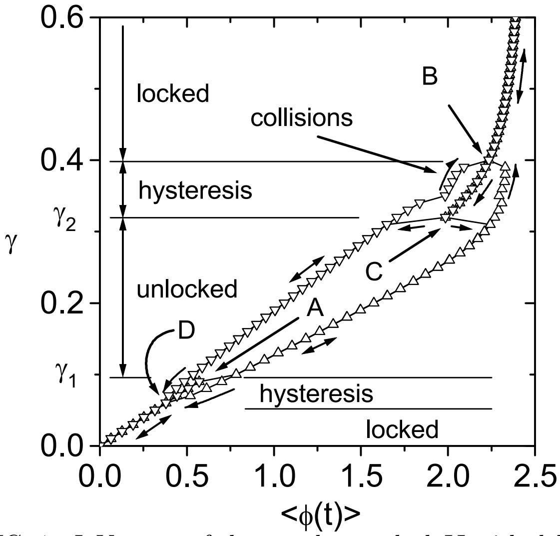

FIG. 1. I–V curve of the annular stacked JJ with delocking region. The parameters of the system are: $\Delta = 0.5$, $\alpha^A/\alpha^B = 1.5$, $N = 3$, $L = 5$. A and C are delocking points, B and D are relocking points.

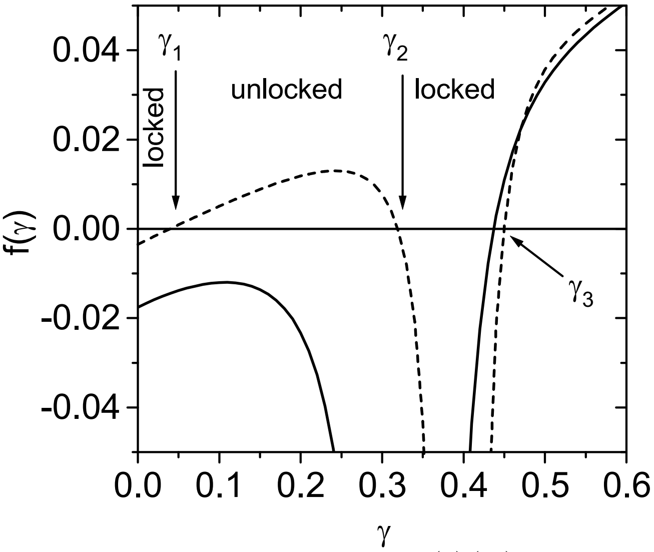

FIG. 2. Typical behavior of function $f(\gamma)$ (12). Continuous curve – strong coupling with $\Delta = 0.5$ (1 root), no delocking region; dashed curve – weak coupling with $\Delta = 0.1$ (3 roots), delocking region is present. Other parameters: $\alpha^A = 0.12$, $\alpha^B = 0.1$, $N = 3$

point the coupling force F_Δ between the chains can no longer compensate this difference.

The decoupling point is marked by A in Fig. 1. After decoupling, the chains are sliding relative to each other. However, with further increase of the bias current, the motion of the decoupled chains becomes relativistic, i.e. their velocities approach the limiting (Swihart) velocity $\bar{c}_-$. The velocities are close again, the friction force difference gets small, and, finally, the locked state is recovered (point B in Fig. 1). This explanation of the delocking and relocking is a qualitative interpretation of the mathematical formalism developed in the previous section.

The simulated I-V curve in Fig. 1 demonstrates a small-scale hysteresis around the delocking and relocking points. The analytical model presented above does not account for this hysteresis. The delocking points obtained from (12) correspond to the points A and C (delocking) rather than B and D (relocking) in Fig. 1.

In Fig. 1 one can also see an interesting feature on I-V curve close to the hysteresis region between points B and C marked as "collisions". This is a mode which is different from viscous sliding of two fluxon chains relative to each other. The fluxon chains become less dense in the region near the resonance and single fluxon-fluxon collisions start to be important. The fluxon moving with higher velocity in one JJ collides with the slower fluxon from the other JJ. A part of the kinetic energy is transferred from the fast fluxon to the slow one which results in a change of velocities of fluxon chains as a whole. This change of velocities can be seen as the above mentioned feature in the I-V curve. This collisions can be clearly seen if one simulates a long ($L = 20$) stack with one fluxon in each JJ and different damping parameters. Depending on parameters, the collision region may appear in the I-V curve. Other interesting modes may be observed as well for different values of parameters.

The delocking points A and C can be regarded as zeros of the function $f(\gamma)$, defined as the l.h.s. of (12). Fig. 2 shows a typical behavior of the function $f(\gamma)$ for weak ($\Delta = 0.1$) and strong ($\Delta = 0.5$) coupling. One can see that the function $f(\gamma)$ has either 1 or 3 roots. The range of γ, where $f(\gamma) < 0$, corresponds to the locked state, while the region between the first two roots (for $\Delta = 0.1$) corresponds to the delocked chains. Interestingly, the function $f(\gamma)$ has the third root which corresponds to the delocking at very high γ. We will not consider this root because the approximation of the dense fluxon chain does not work very close to the top of the flux-flow step. Indeed, due to the relativistic contraction, the fluxon size reduces and the chain becomes sparse. The numerical simulations show poor agreement between the top of the step obtained numerically and the value given by the third root of $f(\gamma)$. The locking range is much wider than it is predicted by the third root of $f(\gamma)$. If the coupling Δ is large enough, the function $f(\gamma)$ has only one root, which corresponds to the completely locked fluxon chains at all values of γ.

We have performed a series of simulations for different fluxon densities (with $N = 3, 5$, and 10 trapped fluxons in $L = 5$ long annular junction), different coupling strengths ($\Delta = 0.1, 0.2, 0.3, 0.5$), and various ratios of the damping coefficients α^A/α^B in the two junctions (from 1 up to 1.5). The results of simulation for $N = 3$ are shown in Fig. 3. The analytical curves for the same parameter set are shown in this figure as well. Excellent agreement between the analytical curves and the simulations for both delocking points is found. The simulated and analytical data for the case $N = 5$ are shown in

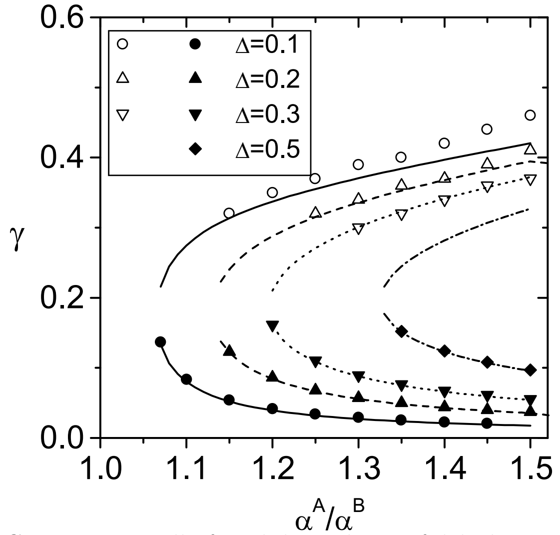

FIG. 3. Numerically found dependence of delocking points on the ratio of the damping parameters α^A/α^B for different coupling strengths (symbols), and comparison with analytical prediction (curves). Each junction contains $N = 3$ fluxons.

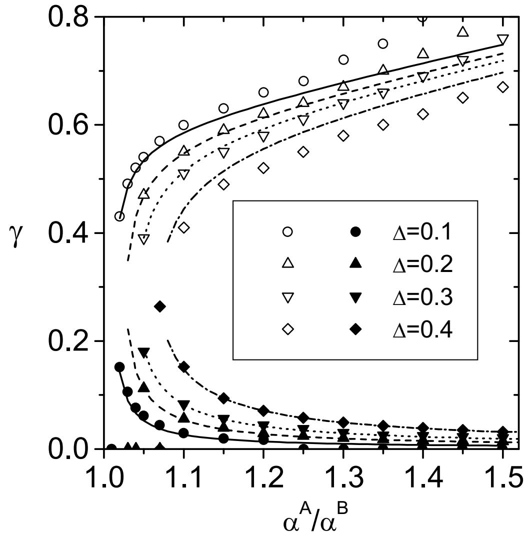

FIG. 4. The same as in Fig. 3 but for $N = 5$.

Fig. 4. We have also performed simulations for $N = 10$, which qualitatively showed the same behavior.

One can note that for α^A/α^B close to 1.5 and large γ a small deviation of the analytical model from the simulation results takes place. This results from the fact that the fluxon chains are not very dense in this region due to relativistic contraction. Nevertheless, simulations give a wider locked region than the analytical model.

IV. CONCLUSION

We have demonstrated that Eq. (12) derived by means of the perturbation theory yields a very good approximation for the two delocking points on the I-V curves of two magnetically coupled annular Josephson junctions. Using this equation, one can find a critical value of the coupling parameter $\Delta^\star$ so that the junctions never delock if $\Delta > \Delta^\star$. In this case, the function $f(\gamma)$ defined above (as the l.h.s of Eq. (12)) has only one root (see Fig. 2). If $\Delta < \Delta^\star$, a delocking region exists, and the function $f(\gamma)$ (12) has 3 roots (see Fig. 2). At $\Delta = \Delta^\star$ the two smallest roots coincide, so that, at some value of $\gamma = \gamma^\star$, $f(\gamma^\star) = 0$ and $f'(\gamma^\star) = 0$ simultaneously. Solving these equations, one obtains the following relation:

$$\Delta^\star\sqrt{1+\Delta^\star} = \frac{16}{125}\sqrt{5}H^3|\alpha^A - \alpha^B| \approx 0.286H^3|\alpha^A - \alpha^B| \tag{14}$$

Using (14), one can easily evaluate the minimum value of the coupling parameter $\Delta^\star$ required to avoid delocking in the system for given values of α^A, α^B and magnetic field (fluxon density) H. The deviation discussed at the end of section III does not affect this result because the system locks better than predicted by the analytical model.

V. ACKNOWLEDGMENTS

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[1] A. V. Ustinov, H. Kohlstedt, and C. Heiden, Appl. Phys. Lett. **65** 1457 (1994)

[2] A. Wallraff, E. Goldobin, and A. V. Ustinov, "Numerical Analysis of the Coherent Radiation Emission by Two Stacked Josephson Flux-flow Oscillators.", to be published in J. Appl. Phys. **80** (11), 1 December 1996.

[3] P. Barbara, J. Mygind, N. F. Pedersen, G. Costabile, A. D'Anna, R. D. Parmentier, and B. A. Malomed, in: "Nonlinear Superconducting Devices and High-T_c Materials", ed. by R. D. Parmentier and N. F. Pedersen (World Scientific: Singapore, 1995).

[4] N. Grønbech-Jensen, J. A. Blackburn, and M. R. Samuelsen, Phys. Rev. B **53**, 12364 (1996).

[5] N. Grønbech-Jensen, O. H. Olsen, and M. R. Samuelsen, Phys. Lett. A **179**, 27 (1993).

[6] S. Sakai, P. Bodin, N. F. Pedersen, J. Appl. Phys. **73**, 2411 (1993).