6. Alternating Current

6.1. Alternating Current Source

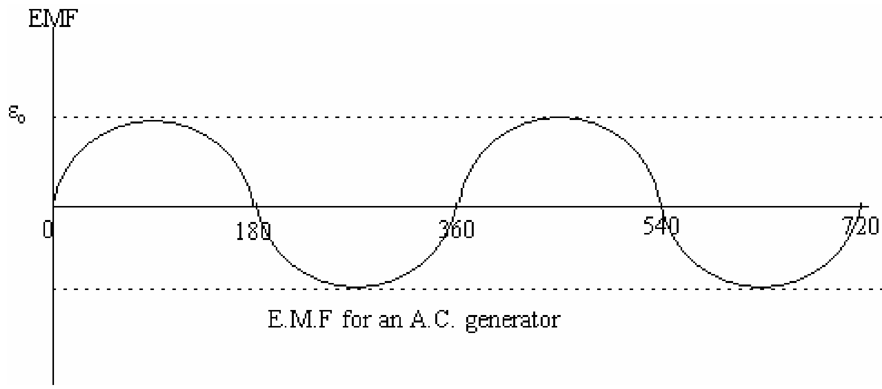
6.1.1. A coil rotating in a magnetic field, producing induced e.m.f. is an example of an alternating current source. An alternating current provides current which alternately forward and backward in the circuit. The e.m.f. of the souce is changing between positive and negative values.

A coil rotating in a magnetic field has an alternating e.m.f. of the form
$\varepsilon(t) = \varepsilon_o \sin \omega t$
where ε_o is the maximum value or the peak value of the e.m.f. and
ω is the rate of change of the e.m.f.
$\omega = 2\pi f$, (unit for ω is rads^{-1})
f = frequency of the source , which is the number of cycle per second (unit for f is Hertz (Hz))
f= 1/T , T is the period, the time for one complete cycle (unit for T is second)
This is also called the "wave form" of the alternating voltage source.

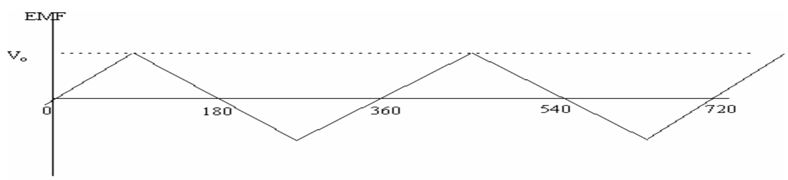
The household main is supplied by an alternating voltage source of the form
$V(t) = 339 \sin 100\pi t$ volts.
As the voltage is changing with time, voltmeters designed for alternating current do not give the instantaneous value of the voltage, but an "average value" called the root.mean.square value (r.m.s.). The r.m.s. value of the voltage relates with the peak value as $V_{rms} = \frac{V_o}{\sqrt{2}}$.
Thus the household main has V_{rms} = 240 Volt, and a frequency f = 50 Hz.

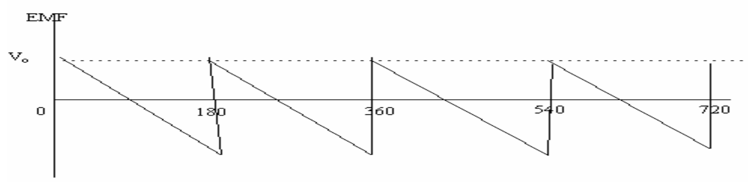
Similarly the current also changes with time due to the voltage. Therefore ammeters for alternating current are designed to read the "average value" which is the root.mean.square (r.m.s.). The r.m.s value of the voltage is $I_{rms} = \frac{I_o}{\sqrt{2}}$

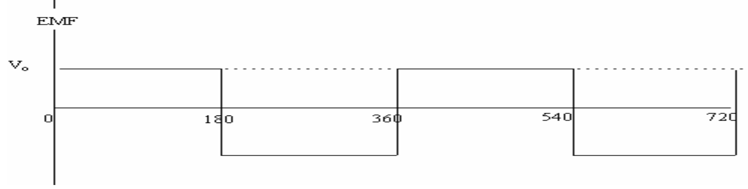
Note 1: A plot of the e.m.f. against time produces a graph which changes sinusoidally with time, thus the "wave form". This is not a wave equation, there is no wavelength associated with this equation.



Note 2: Other alternating voltage source do exist, eg. Triangular, sawtooth, square form. These forms are more difficult to describe mathematically.





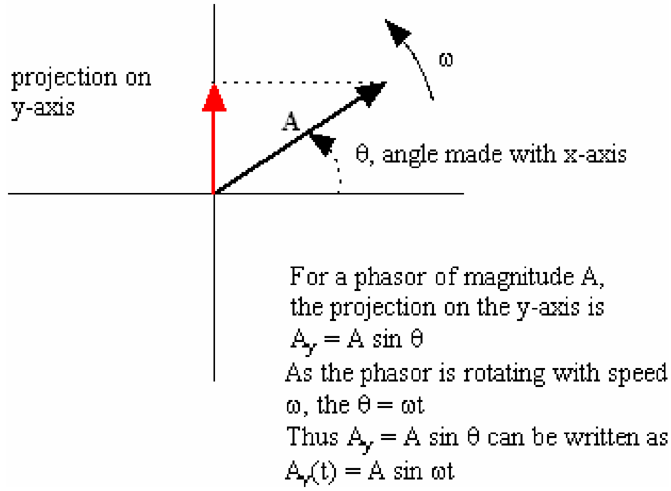


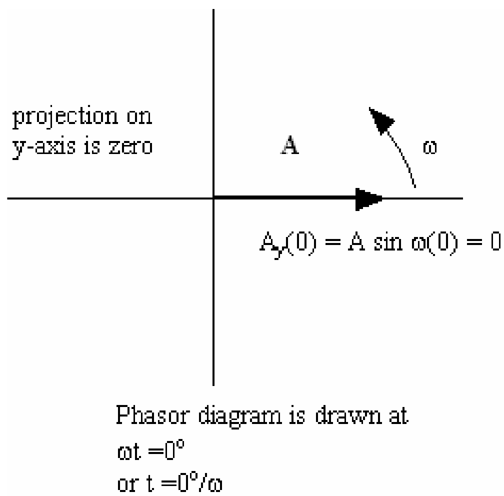
6.2. Phasor

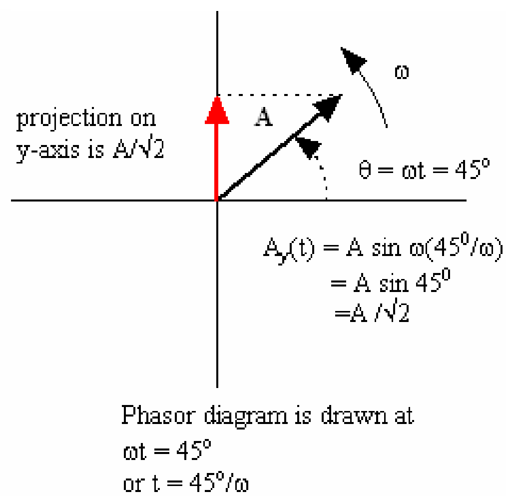
A sinusoidal function is easily recognized when a plot against time is obtained. Its value at any given time can be obtained by looking at the time axis and its corresponding value from the vertical axis.

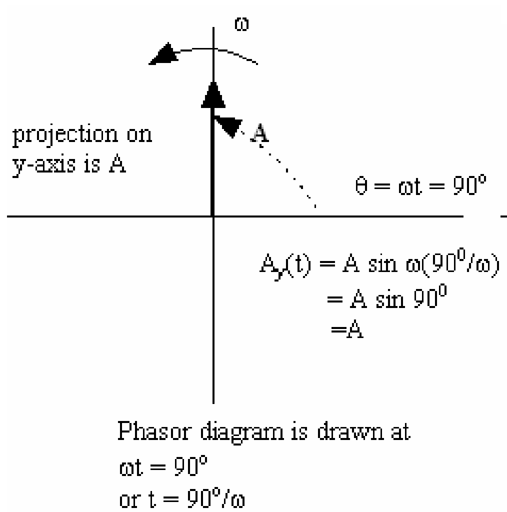
A phasor diagram is a representation of the sinusoidal value at a particular time. To obtain a sinusoidal graph a series of phasor diagrams have to be drawn. However a phasor diagram is important as it can show the phase different between sinusoidal function readily.

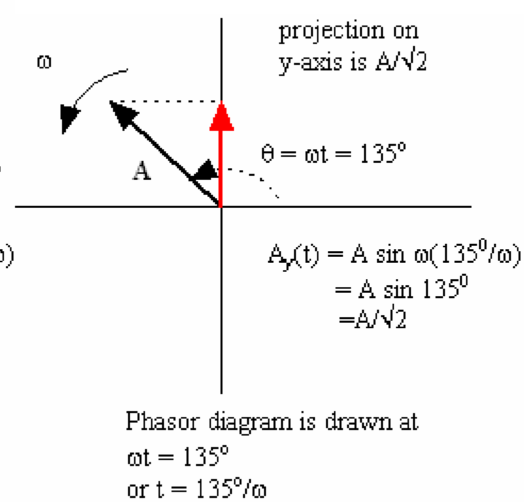
A phasor in the phasor diagram is represented by an arrow with its end fixed on the origin. The arrow is rotating counterclockwise with a constant angular speed ω. The instantaneous value is the projection on the y-axis.

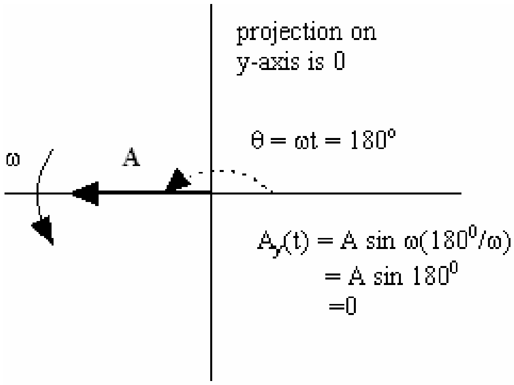

projection on
y-axis
ω
A
θ, angle made with x-axis
For a phasor of magnitude A,
the projection on the y-axis is
$A_y = A \sin \theta$
As the phasor is rotating with speed
ω, the $\theta = \omega t$
Thus $A_y = A \sin \theta$ can be written as
$A_y(t) = A \sin \omega t$


projection on
y-axis is zero
A
ω
$A_y(0) = A \sin \omega(0) = 0$
Phasor diagram is drawn at
$\omega t = 0^o$
or $t = 0^o/\omega$

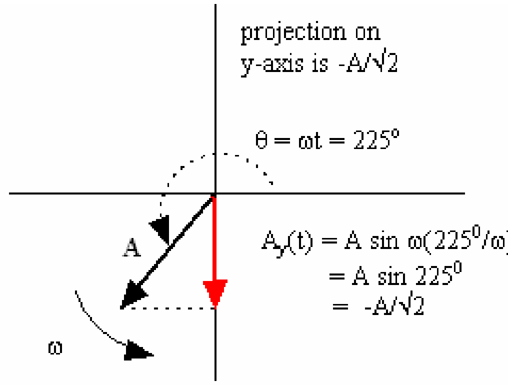

projection on
y-axis is $A/\sqrt{2}$
A
ω
$\theta = \omega t = 45^o$
$A_y(t) = A \sin \omega(45^0/\omega)$
$= A \sin 45^0$
$= A/\sqrt{2}$
Phasor diagram is drawn at
$\omega t = 45^o$
or $t = 45^o/\omega$


ω
projection on
y-axis is A
A
$\theta = \omega t = 90^o$
$A_y(t) = A \sin \omega(90^0/\omega)$
$= A \sin 90^0$
$= A$
Phasor diagram is drawn at
$\omega t = 90^o$
or $t = 90^o/\omega$

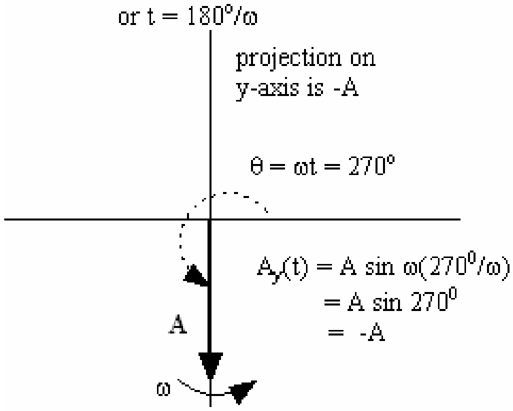

projection on
y-axis is $A/\sqrt{2}$
ω
A
$\theta = \omega t = 135^o$
$A_y(t) = A \sin \omega(135^0/\omega)$
$= A \sin 135^0$
$= A/\sqrt{2}$
Phasor diagram is drawn at
$\omega t = 135^o$
or $t = 135^o/\omega$



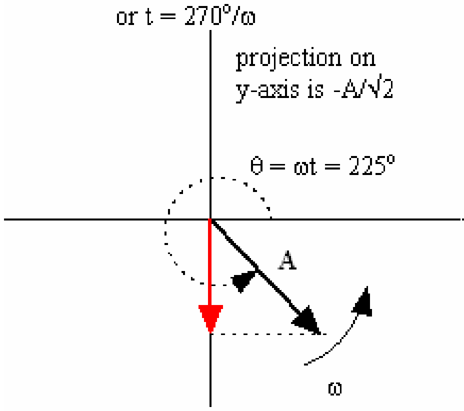
Phasor diagram is drawn at
$\omega t = 180^{o}$
or $t = 180^{o}/\omega$

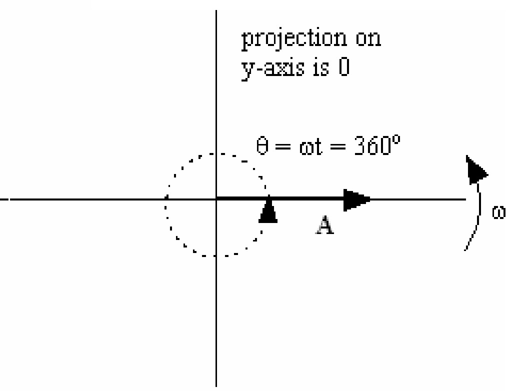


Phasor diagram is drawn at
$\omega t = 225^{o}$
or $t = 225^{o}/\omega$



Phasor diagram is drawn at
$\omega t = 270^{o}$
or $t = 270^{o}/\omega$



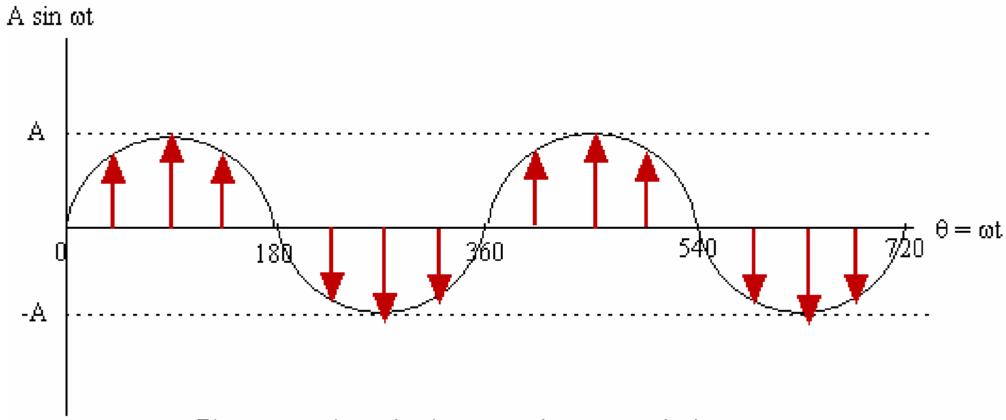


Phasor diagram is drawn at
$\omega t = 315^{o}$
or $t = 315^{o}/\omega$

$A_y(t) = A \sin \omega(315^0/\omega)$
$= A \sin 315^0$
$= -A/\sqrt{2}$

Phasor diagram is drawn at
$\omega t = 360^{o}$
or $t = 360^{o}/\omega$

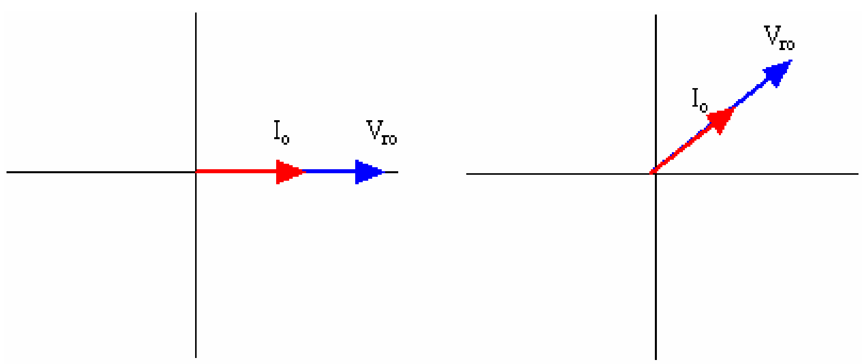
$A_y(t) = A \sin \omega(360^0/\omega)$
$= A \sin 360^0$
$= 0$



Phasor y- axis projections superimpose on A sin ωt

6.3. Current and Voltage for a Resistor

The current through a resistor is in phase with the voltage across the resistor. If the current is given as $I(t) = I_o \sin(\omega t + \phi)$, the the voltage is $V_r(t) = V_{ro} \sin(\omega t + \phi)$.



The current phasor and voltage phasor are in phase. Both phasors rotate at the same angular speed ω. If $I(t) = I_o \sin \omega t$ then $V_r(t) = V_{ro} \sin \omega t$
In the above diagram both are on the x-axis, $\theta = 0$

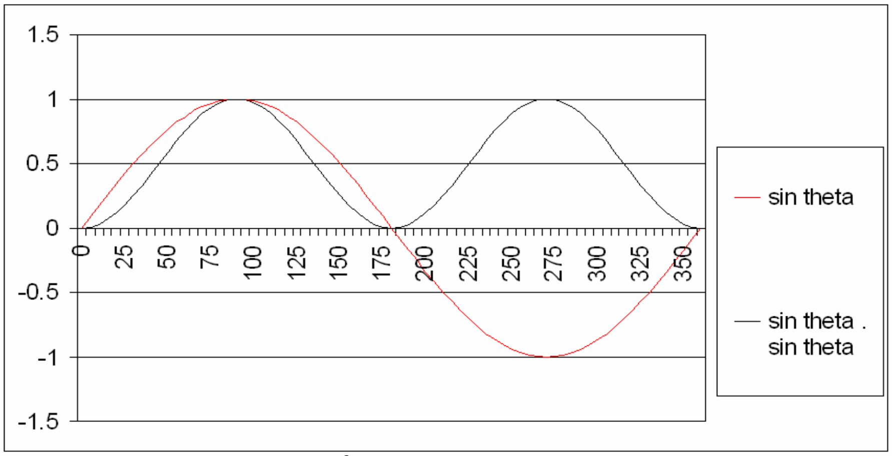
Both phasors at a particular instance t. θ , the angle they make with the x- axis is $\omega t = \theta$

Using Ohm's Law, we have $V = IR$, then $V_r(t) = I_oR \sin(\omega t + \phi)$, where $V_{ro} = I_oR$.

The instantaneous power $P(t) = (V(t)\, I(t)$
$= I_oR \sin(\omega t + \phi)\, I_o \sin(\omega t + \phi)$
$= I_o^2R \sin(\omega t + \phi) \sin(\omega t + \phi)$

The graph $\sin(\omega t + \phi) \sin(\omega t + \phi)$ is shown below. ϕ is called the initial phase and set to zero in the graph below.

mohdnoormohdali



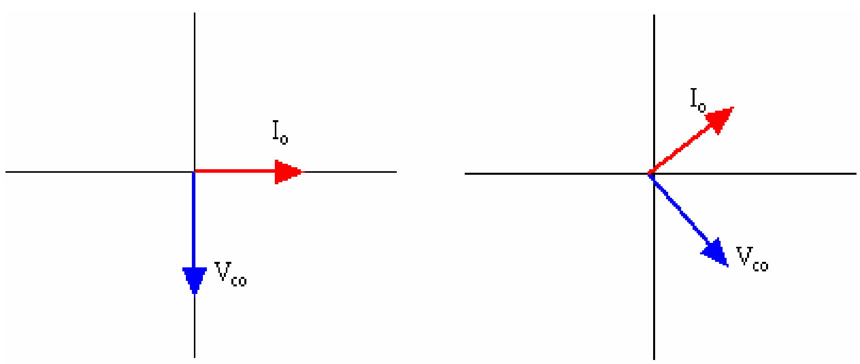
The average power is $\langle P \rangle = I_o^2 R/2$

$= I_{rms}^2 R = V_{rms} I_{rms}$

where $V_{rms} = \dfrac{V_0}{\sqrt{2}}$ and $I_{rms} == \dfrac{I_0}{\sqrt{2}}$

6.4. Currrent and Voltage for a Capacitor

The current through a capacitor is leading the voltage across the capacitor by $\pi/2$ radian. If the current is given as $I(t) = I_o \sin(\omega t + \phi)$, the the voltage is $V_c(t) = V_{co} \sin(\omega t + \phi - \pi/2)$



The current phasor and voltage phasor are not in phase. The current phasor leads the voltage pasor by a phase different of $\pi/2$ radian.
If $I(t) = I_o \sin \omega t$ then
$V_c(t) = V_{co} \sin(\omega t - \pi/2)$

Both phasors at a particular instance t.

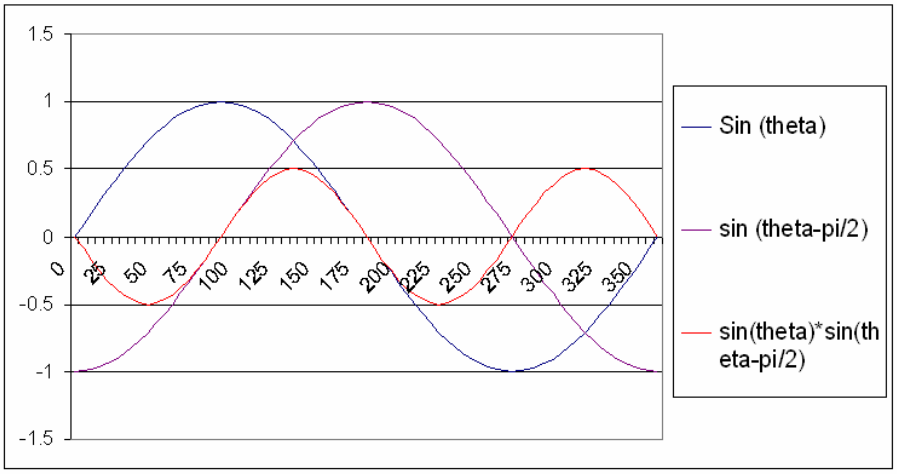
Using Ohm's Law, we have V = IR, then $V_c(t) = I_o X_c \sin(\omega t + \phi - \pi/2)$, where $V_{co} = I_o X_c$.

X_c is called the capacitive reactance, where $X_c = \dfrac{1}{\omega C} = \dfrac{1}{2\pi f C}$. The unit of reactance is ohm.

The instantaneous power $P(t) = (V(t)\ I(t)$

$$= I_o X_c \sin(\omega t + \phi - \pi/2)\ I_o \sin(\omega t + \phi)$$
$$= I_o^2 X_c \sin(\omega t + \phi - \pi/2) \sin(\omega t + \phi)$$

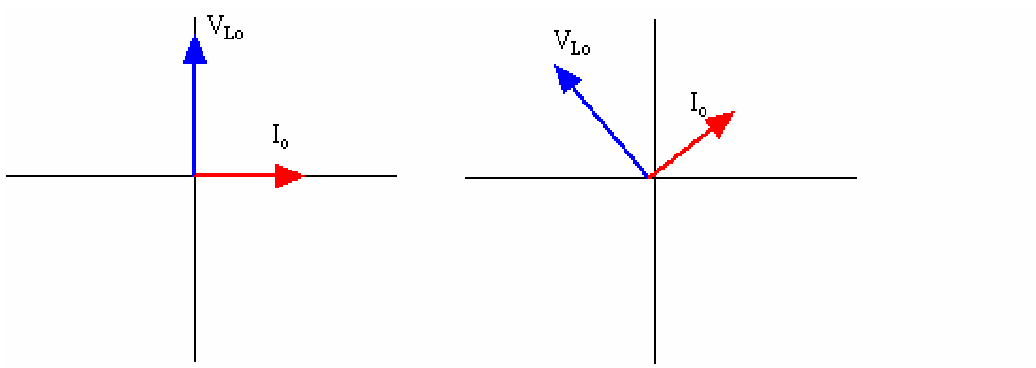
The graph $\sin(\omega t + \phi - \pi/2) \sin(\omega t + \phi)$ is shown below. ϕ is set to zero in the graph below.



The average power is <P> = 0. Notice in the graph above that there is positive power and negative power. Thus energy is stored in the capacitor and then returned to the circuit.

6.5. Current and Voltage for an Inductor

The voltage across an inductor is leading the current through the inductor by $\pi/2$ radian. If the current is given as $I(t) = I_o \sin(\omega t + \phi)$, the the voltage is $V_L(t) = V_{Lo} \sin(\omega t + \phi + \pi/2)$



The current phasor and voltage phasor are not in phase. The voltage phasor leads the current phasor by a phase different of $\pi/2$ radian.
If $I(t) = I_o \sin \omega t$ then
$V_L(t) = V_{Lo} \sin(\omega t + \pi/2)$

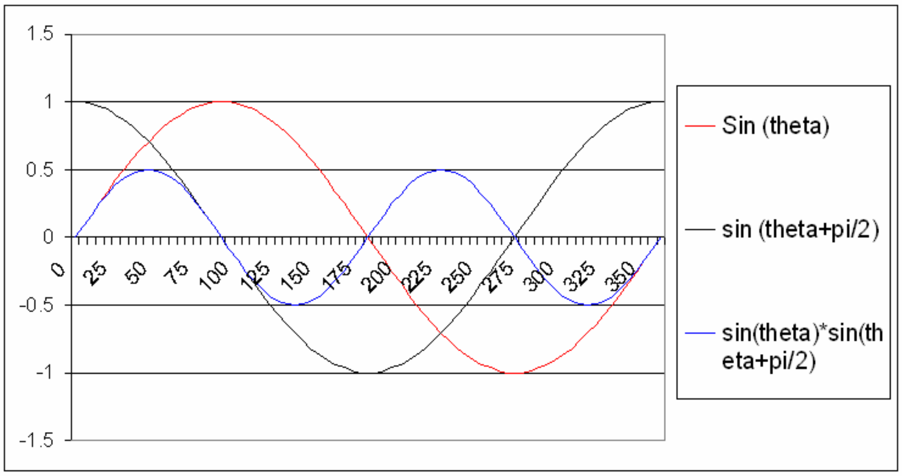
Both phasors at a particular instance t.

mohdnoormohdali

Using Ohm's Law, we have V = IR, then $V_L(t) = I_oX_L \sin(\omega t + \phi + \pi/2)$, where $V_{Lo} = I_oX_L$.
X_c is called the inductive reactance, where $X_L = \omega L = 2\pi fL$. The unit of reactance is ohm.
The instantaneous power $P(t) = (V(t)\ I(t)$

$$= I_oX_L \sin(\omega t + \phi + \pi/2)\ I_o \sin(\omega t + \phi)$$
$$= I_o^2X_L \sin(\omega t + \phi + \pi/2) \sin(\omega t + \phi)$$

The graph $\sin(\omega t + \phi + \pi/2) \sin(\omega t + \phi)$ is shown below. ϕ is set to zero in the graph below.



The average power is <P> = 0. Notice in the graph above that there is positive power and negative power. Thus energy is stored in the capacitor and then returned to the circuit.

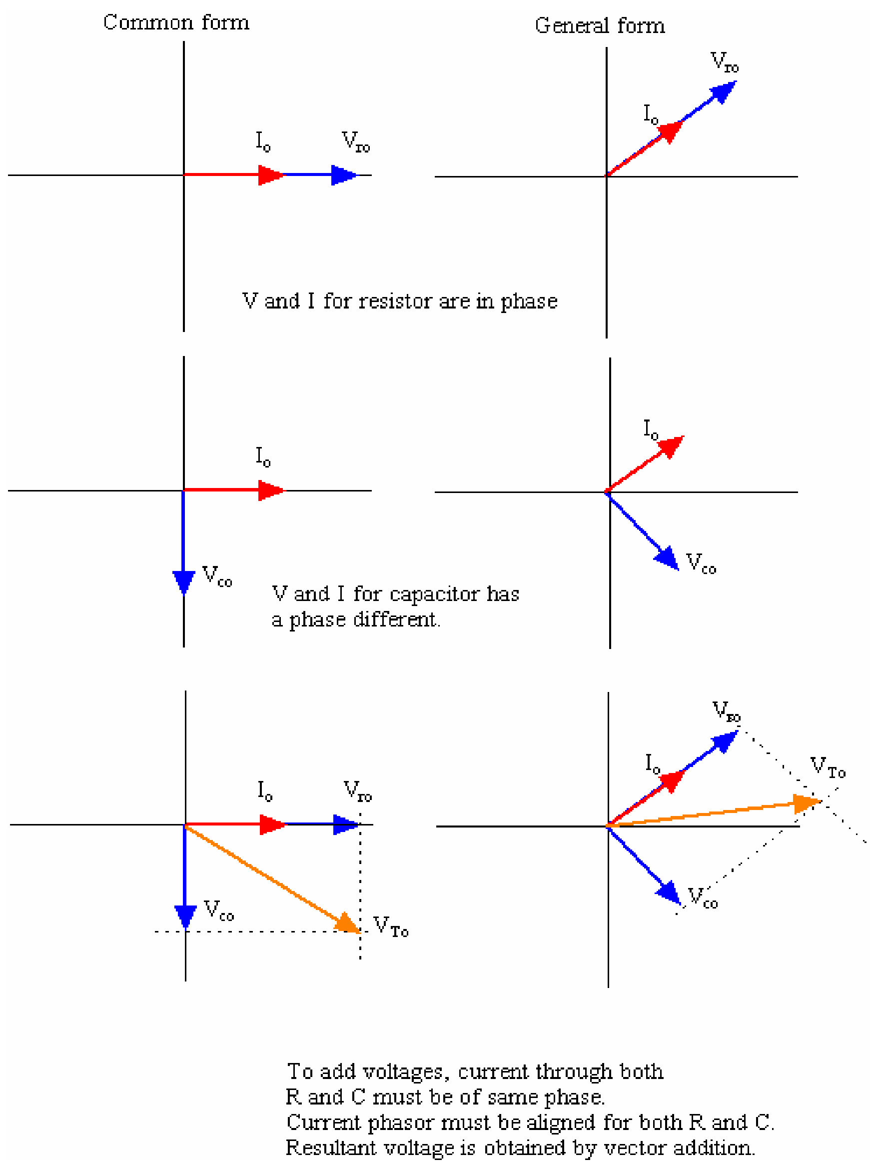
6.6. RC series Circuit

In a series circuit the current through the circuit is the same at any point. Therefore the current through the resistor is the same as the current through the capacitor. The voltage across both components is the sum of the voltage across the resistor and the voltage across the capacitor.

If the current $I(t) = I_o \sin(\omega t)$, then the voltage across the resistor is
$V_r(t) = V_{ro} \sin(\omega t)$ while the the voltage across the capacitor is
$V_c(t) = V_{co} \sin(\omega t - \pi/2)$

The sum of the voltages is then $V_T(t) = V_{ro} \sin(\omega t) + V_{co} \sin(\omega t - \pi/2)$
As the sum is quite tedious to solve mathematically, we will use phasors to add the voltages.

Note: The initial phase angle ϕ is not important in the calculations. ϕ determines the value of the function when t = 0. The initial phase can be dropped to simplify the equations.



The voltage phasors of the resistor and the capacitor are exactly at right angle to each other. The magnitude of the resultant phasor V_{To} can be obtain from Phytagorean Theorem;

$$V_{To}^2 = V_{ro}^2 + V_{co}^2$$

$$V_{To} = \sqrt{\left[V_{ro}^2 + V_{co}^2\right]}$$

And the angle between V_{To} and I_o is

$\phi = \tan^{-1}\dfrac{-V_{co}}{V_{ro}}$, where ϕ is then a negative value.

Therefore $V_T(t) = V_{To} \sin(\omega t + \phi)$ where ϕ is negative , and V_T is lagging behing the current.

Note: ϕ is the initial phase angle for V_T , the initial phase angle for V_c and V_r has ben set to zero.

By convention to show explicitly that V_T is lagging the above equation is written as $V_T(t) = V_{To} \sin(\omega t - |\phi|)$ and simplified to $V_T(t) = V_{To} \sin(\omega t - \phi)$, where ϕ is now the magnitude (positive value).

$V_{To} = \sqrt{\left[V_{ro}^2 + V_{co}^2\right]}$ can be written in other form.

Substituting with $V_{ro} = I_oR$ and $V_{co} = I_oX_c$ will give

$$V_{To} = \sqrt{\left[(I_oR)^2 + (I_oX_c)^2\right]}$$

$$V_{To} = I_o\sqrt{\left[(R)^2 + (X_c)^2\right]}$$

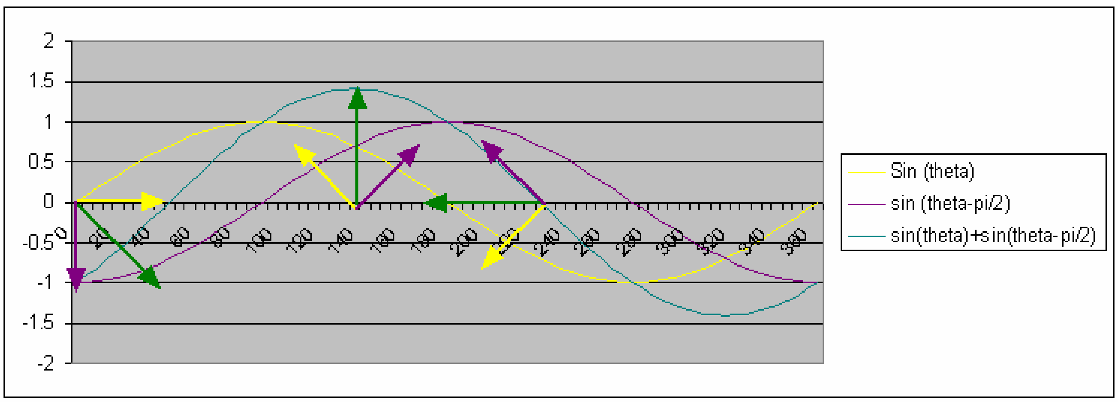
$$V_{To} = I_oZ$$

where;

$$Z = \sqrt{\left[(R)^2 + (X_c)^2\right]}$$

Z is called the impedance of the circuit. The equation for Z above leads to an impedance diagram which also behave as phasors.

Addition of phasors and the graph of the resultant phasor.



The instantaneous power of the circuit is $P(t) = V_T(t)\, I(t)$

Substituting for $V_T(t)$ and $I(t)$

$P(t) = V_{To} \sin(\omega t - \phi)\, I_o \sin(\omega t)$

$P(t) = V_{To}\, I_o \sin(\omega t - \phi) \sin(\omega t)$

$P(t) = V_{To}\, I_o\, [(\sin \omega t \cos \phi - \cos \omega t \sin \phi) \sin \omega t\,]$

$P(t) = V_{To}\, I_o\, [(\sin \omega t \sin \omega t \cos \phi - \cos \omega t \sin \omega t \sin \phi)\,]$

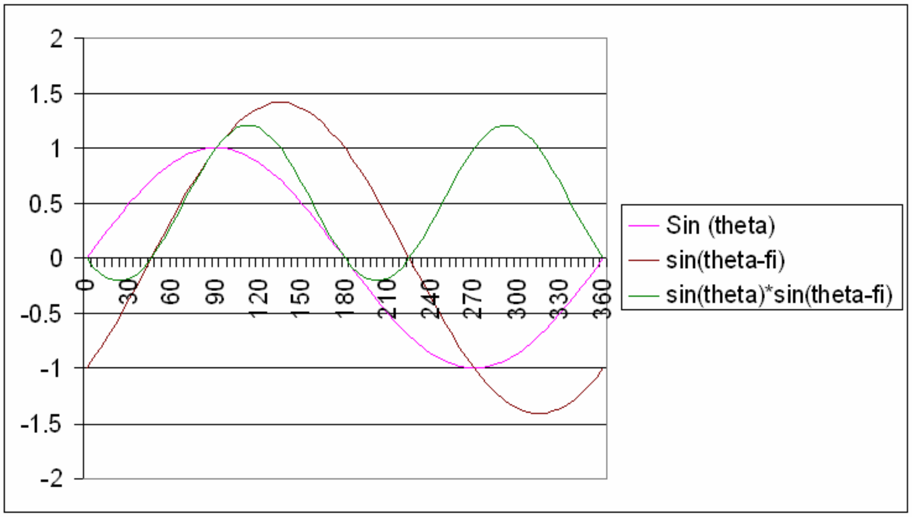
If the mean power is to be obtained (looking at the power over on complete cycle) $\sin \omega t \sin \omega t \cos \phi$ reduces to ½ $\cos \phi$ while $\cos \omega t \sin \omega t \sin \phi$ reduced to zero.

mohdnoormohdali

Therefore mean power is $\langle P \rangle = \frac{1}{2} V_{To} I_o \cos \phi$

$\langle P \rangle = V_{Trms} I_{rms} \cos \phi$ which looks similar to other power equation except for the factor of $\cos \phi$. Thus $\cos \phi$ is called the power factor as it determines the magnitude of the mean power.

The instantaneous power function is shown below. The leading factor $V_{To} I_o$ is ignored (set to arbitrary value)



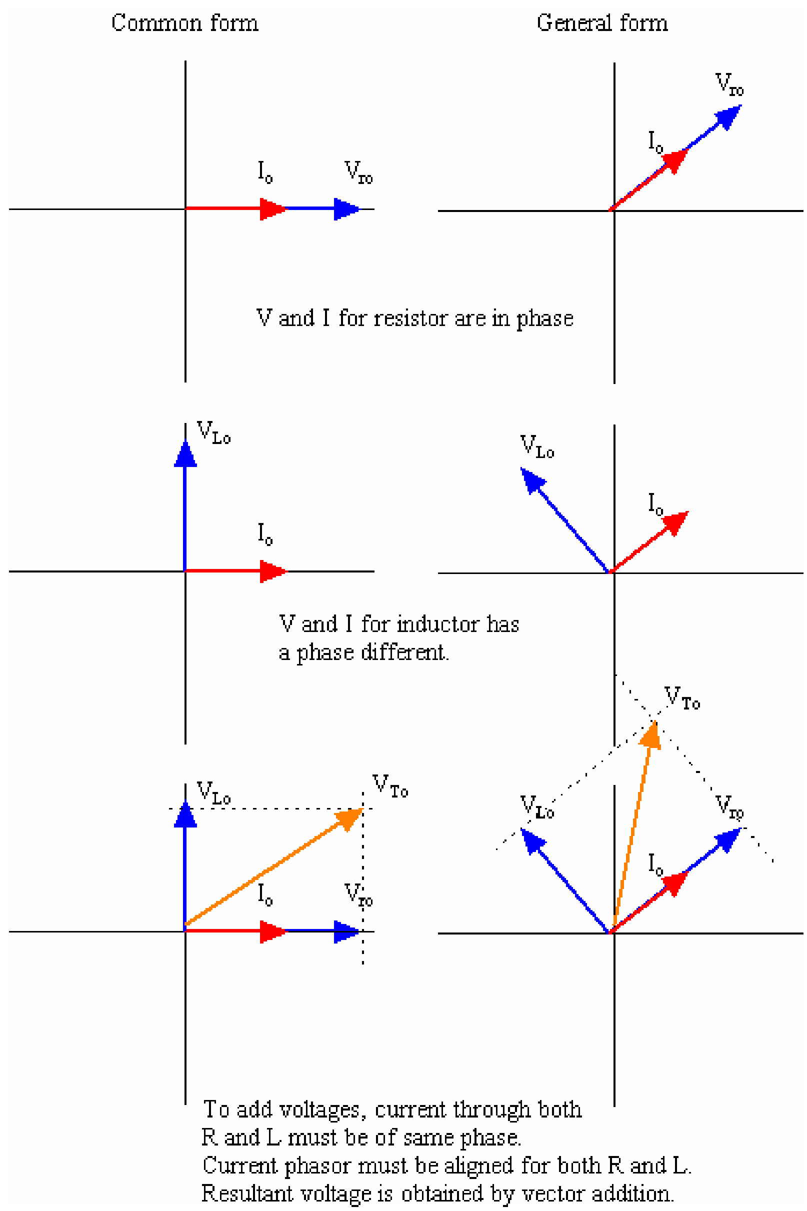
Notice that there is negative power which is the energy returned by the capacitor to the circuit.

6.7. RL series circuit

As in RC series circuit, the current through the RL series circuit is the same at any point. Therefore the current through the resistor is the same as the current through the inductor.
The voltage across both components is the sum of the voltage across the resistor and the voltage across the inductor.

If the current $I(t) = I_o \sin(\omega t)$, then the voltage across the resistor is
$V_r(t) = V_{ro} \sin(\omega t)$ while the the voltage across the inductor is
$V_L(t) = V_{Lo} \sin(\omega t + \pi/2)$

The sum of the voltages is then $V_T(t) = V_{ro} \sin(\omega t) + V_{Lo} \sin(\omega t + \pi/2)$
As before, we will use phasors to add the voltages.



The voltage phasors of the resistor and the inductor are exactly at right angle to each other. The magnitude of the resultant phasor V_{To} can be obtain from Phytagorean Theorem;

$$V_{To}^2 = V_{ro}^2 + V_{Lo}^2$$

$$V_{To} = \sqrt{\left[V_{ro}^2 + V_{Lo}^2\right]}$$

And the angle between V_{To} and I_o is

$\phi = \tan^{-1}\dfrac{V_{Lo}}{V_{ro}}$, where ϕ is then a positive value.

Therefore $V_T(t) = V_{To} \sin(\omega t + \phi)$ where V_T is leading the current.

$V_{To} = \sqrt{\left[V_{ro}^2 + V_{Lo}^2\right]}$ can be written in other form.

Substituting with $V_{ro} = I_o R$ and $V_{Lo} = I_o X_L$ will give

$$V_{To} = \sqrt{\left[(I_o R)^2 + (I_o X_L)^2\right]}$$

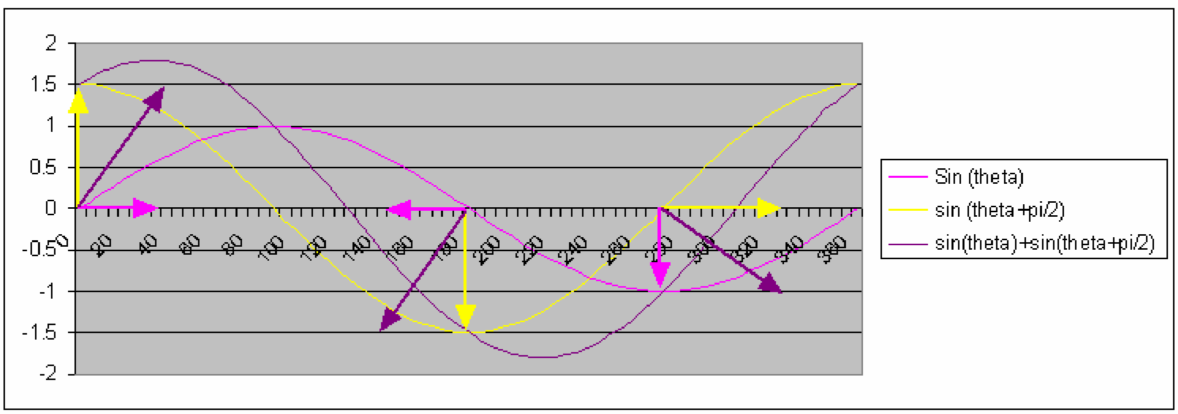
$$V_{To} = I_o\sqrt{\left[(R)^2 + (X_L)^2\right]}$$

$$V_{To} = I_o Z$$

where;

$$Z = \sqrt{\left[(R)^2 + (X_L)^2\right]}$$

Z is called the impedance of the circuit. The equation for Z above leads to an impedance diagram which also behave as phasors.



The instantaneous power of the circuit is $P(t) = V_T(t)\, I(t)$

Substituting for $V_T(t)$ and $I(t)$

$P(t) = V_{To} \sin(\omega t + \phi)\, I_o \sin(\omega t)$

$P(t) = V_{To}\, I_o \sin(\omega t + \phi) \sin(\omega t)$

$P(t) = V_{To}\, I_o\, [(\sin \omega t \cos \phi + \cos \omega t \sin \phi) \sin \omega t]$

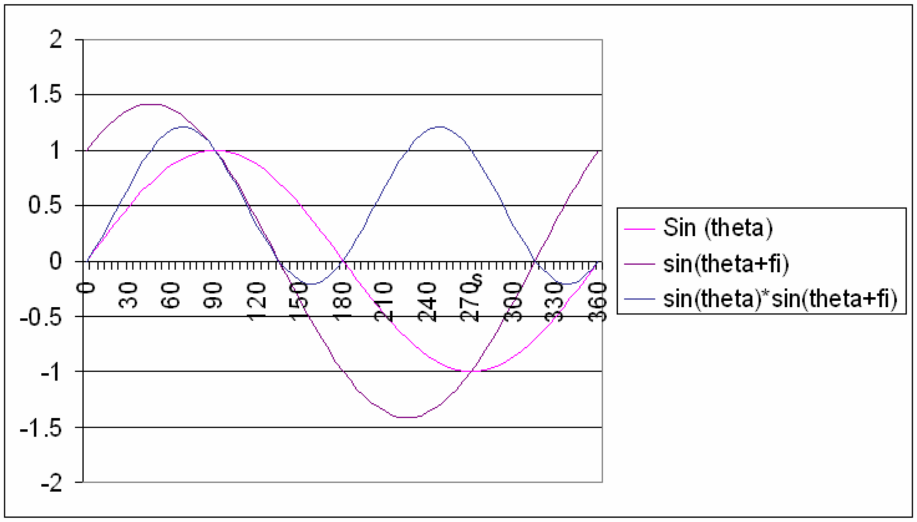
$P(t) = V_{To}\, I_o\, [(\sin \omega t \sin \omega t \cos \phi + \cos \omega t \sin \omega t \sin \phi)]$

If the mean power is to be obtained (looking at the power over on complete cycle) $\sin \omega t \sin \omega t \cos \phi$ reduces to ½ $\cos \phi$ while $\cos \omega t \sin \omega t \sin \phi$ reduced to zero.

Therefore mean power is $\langle P \rangle = \frac{1}{2} V_{To} I_o \cos \phi$

$\langle P \rangle = V_{Trms} I_{rms} \cos \phi$ which looks similar to other power equation except for the factor of $\cos \phi$. Thus $\cos \phi$ is called the power factor as it determines the magnitude of the mean power.

The instantaneous power function is shown below. The leading factor $V_{To}\, I_o$ is ignored (set to an arbitrary value)



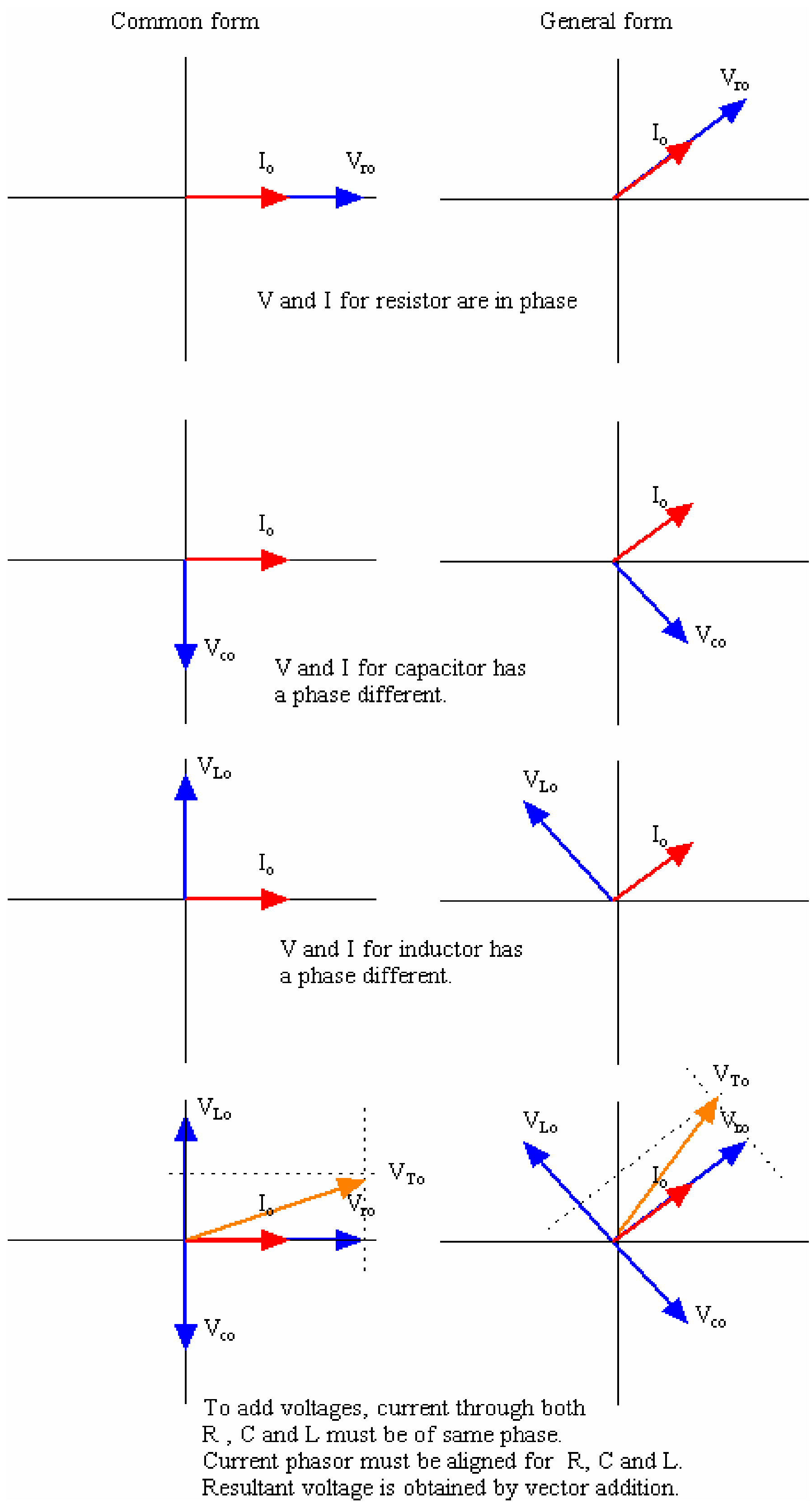
Notice that there is negative power which is the energy returned by the inductor to the circuit.

6.8. RLC series circuit.

If the current $I(t) = I_o \sin(\omega t)$, then
the voltage across the resistor is $V_r(t) = V_{ro} \sin(\omega t)$,
the the voltage across the capacitor is $V_c(t) = V_{co} \sin(\omega t - \pi/2)$, and
the the voltage across the inductor is $V_L(t) = V_{Lo} \sin(\omega t + \pi/2)$.

The sum of the voltages is then $V_T(t) = V_{ro} \sin(\omega t) + V_{co} \sin(\omega t - \pi/2) + V_{Lo} \sin(\omega t + \pi/2)$

As before, we will use phasors to add the voltages.


Common form
General form
I$_o$
V$_{ro}$
V and I for resistor are in phase
V$_{co}$
V and I for capacitor has
a phase different.
V$_{Lo}$
V and I for inductor has
a phase different.
V$_{To}$
To add voltages, current through both
R , C and L must be of same phase.
Current phasor must be aligned for R, C and L.
Resultant voltage is obtained by vector addition.

The magnitude of the resultant phasor V_{To} can be obtain from Phytagorean Theorem;

$$V_{To}^2 = V_{ro}^2 + (V_{Lo} - V_{co})^2$$

$$V_{To} = \sqrt{\left[V_{ro}^2 + (V_{Lo} - V_{co})^2\right]}$$

And the angle between V_{To} and I_o is

$\phi = \tan^{-1}\dfrac{V_{Lo} - V_{co}}{V_{ro}}$, where ϕ can be a positive or negative value.

Therefore V_T could be leading or lagging behind the current.

$V_{To} = \sqrt{\left[V_{ro}^2 + (V_{Lo} - V_{co})^2\right]}$ can be written in other form.

Substituting with $V_{ro} = I_oR$, $V_{Lo} = I_oX_L$ and $V_{co} = I_oX_c$ will give

$$V_{To} = \sqrt{\left[(I_oR)^2 + (I_oX_L - I_oX_c)^2\right]}$$

$$V_{To} = I_o\sqrt{\left[(R)^2 + (X_L - X_c)^2\right]}$$

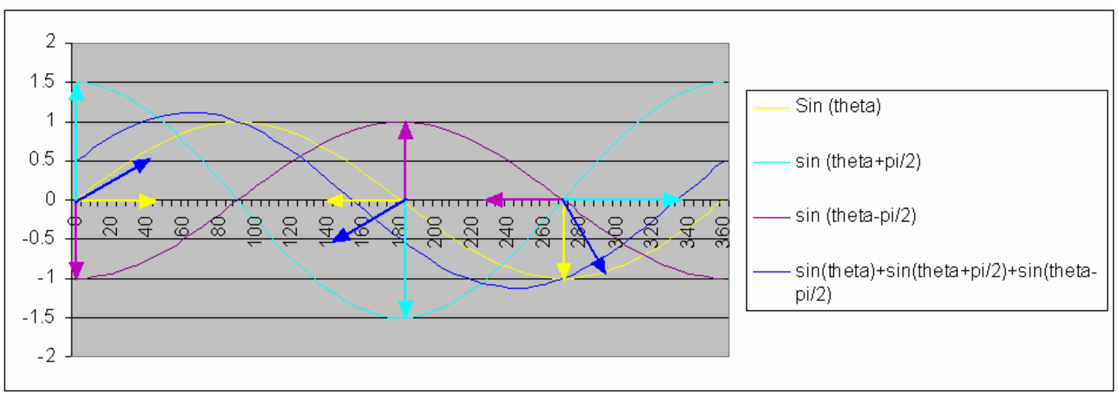
$$V_{To} = I_oZ$$

where;

$$Z = \sqrt{\left[(R)^2 + (X_L - X_c)^2\right]}$$

Z is called the impedance of the PLC series circuit.

Below are the functions in graphical form.



As in the previous cases the instantaneous power of the circuit is P(t) = V_T(t) I(t)

Substituting for V_T(t) and I(t)

P(t) = V_{To} sin (ωt + ϕ) I_o sin (ωt)

P(t) = V_{To} I_o sin (ωt + ϕ) sin (ωt)

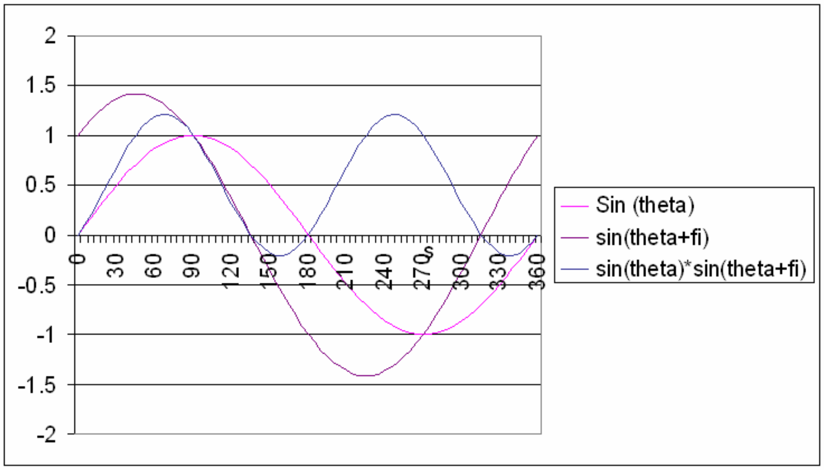
P(t) = V_{To} I_o [(sin ωt cos ϕ + cos ωt sin ϕ) sin ωt]

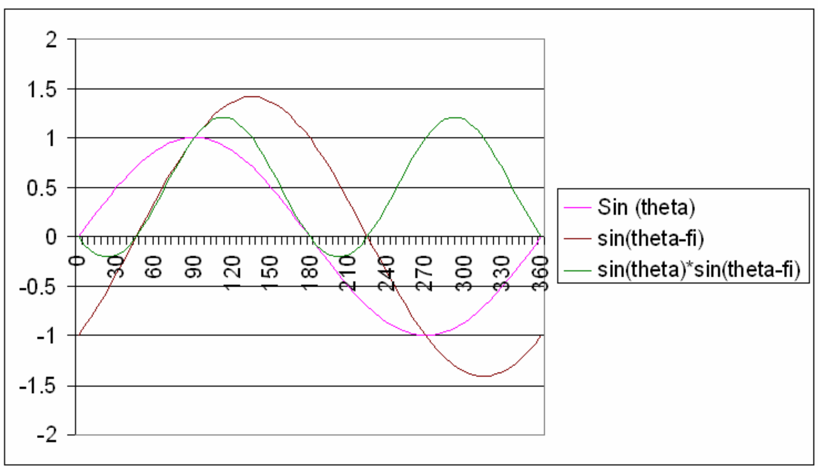
P(t) = V_{To} I_o [(sin ωt sin ωt cos ϕ + cos ωt sin ωt sin ϕ)]

The mean power is $<P> = ½ V_{To} I_o \cos \phi$

$<P> = V_{Trms}I_{rms} \cos \phi$

The instantaneous power function is shown below. The leading factor $V_{To} I_o$ is ignored (set to an arbitrary value)





6.9. Power

6.9.1. Instantaneous power

Instantaneous power is the product of the instantaneous voltage and instantaneous current.

$P(t) =(V(t) I(t)$

It has little practical value as the power of the circuit is not a constant and is changing with time.

6.9.2. Average (Mean) Power / True Power

Average or Mean Power, $<P> = V_{Trms}I_{rms} \cos \phi$, is essentially the power dissipated across the resistance.

<P> = $V_{Trms} I_{rms} \cos\phi$, substituting $\cos\phi = R/Z$ and $V_{Trms} = I_{rms} Z$
reduces to <P> = $I_{rms}^2 R$

6.9.3. Reactive Power

Reactive power is the power stored and returned by the reactive components (inductor / capacitor) $P = I_{rms}^2 X$
$X = X_L$ or X_c for RC or RL series circuit, or $X = X_L - X_c$ for RLC circuit.
Reactive power is not dissipated but returned to the circuit in each cycle.

6.9.4. Apparent Power

Apparent power is the power that appears to the source, it is the power that has to be provided by the cource to the circuit.
Apparent power $P = I_{rms}^2 Z$
In another form $P = I_{rms}^2 \sqrt{\left[(R)^2 + (X_L - X_c)^2\right]}$

Comparing the power, $P_{apparent} > P_{mean}$ while $P_{reactive}$ is trapped in the circuit.

6.10. Frequency dependency

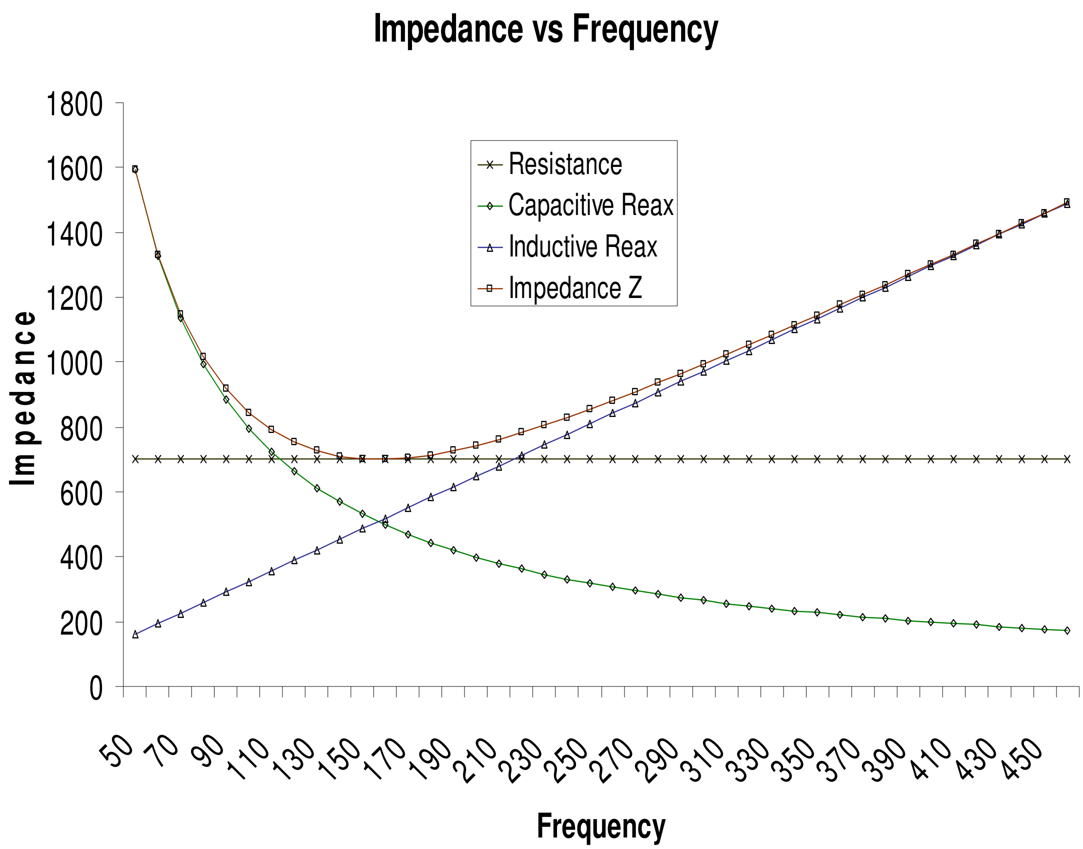
Resistance, R, is not frequency dependent. It remains a constant value throughout any change source frequency.

Capacitive reactance, X_c and inductive reactance, X_L, however are frequency depencence.

$X_c = 1/\omega C = 1/2\pi fC$, for low frequency X_c is high, while at higher frequency X_c becomes lower approaching zero at very high frequency.

$X_L = \omega L = 2\pi fL$, for low frequency X_L is low, while at higher frequency X_L becomes higher approaching infinity at very high frequency.

Resistance and reactive impedance are shown in the graph below.



6.11. Resonance

Resonance in series RLC circuit occurs when the impedance is at a minimum value. The inductive and reactive impedance are equal in magnitude but 180^{o} out of phase thus cancelling each other. The total impedance is contributed by the resistance in the circuit only. At resonance the current in the circuit is at its most possible maximum value.

$$V_o = I_o Z,$$

$$I_o = \frac{V_o}{Z},$$

$$Z = \sqrt{\left[(R)^2 + (X_L - X_c)^2\right]}$$

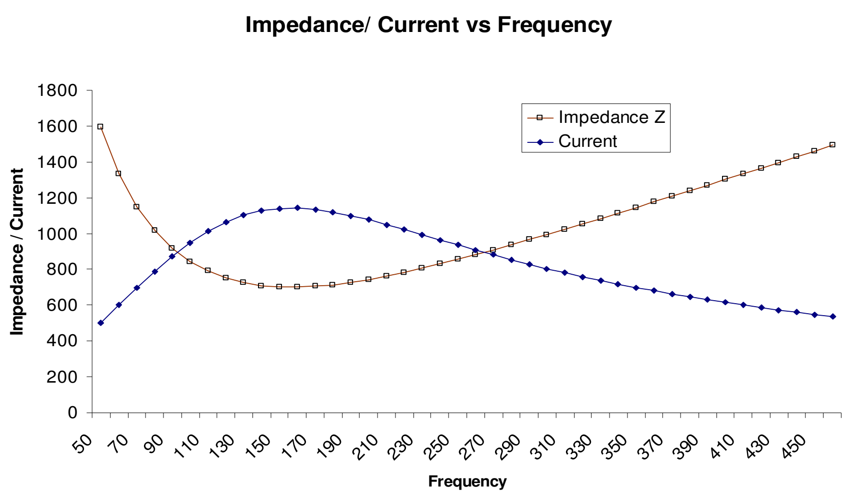
Z is at a minimum when $X_L - X_c = 0$.

Thus

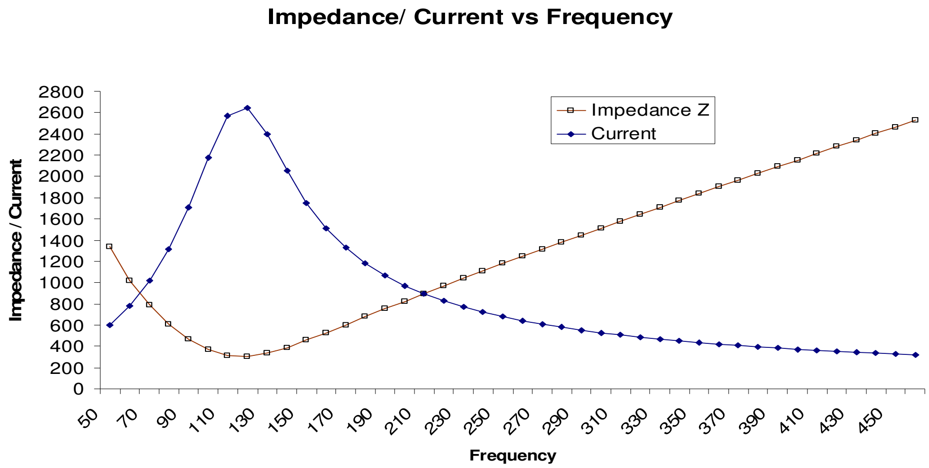
$$X_L - X_c = 0$$
$$\omega L - \frac{1}{\omega C} = 0$$
$$\omega L = \frac{1}{\omega C}$$
$$\omega^2 = \frac{1}{LC}$$
$$\omega = \sqrt{\frac{1}{LC}}$$
$$2\pi f_o = \sqrt{\frac{1}{LC}}$$
$$f_o = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

f_o is called the resonance frequency of the circuit.

The current against frequency graph can be drawn to show that it maximize at resonance.



Lowest point of impedance curve corresponds to same frequency for the highest point on the current curve.



The shape of the current curve becomes sharper if resistance is lower. The "spread" of the hump can be designed to obtain a balance between maximum current and the ability to tune in to the resonance frequency.

A measure of this ability is the Q factor, where $Q = \omega_o / \Delta\omega$.

$\Delta\omega$ is the bandwidthy (the range of frequencies where current is more than $I_o/2$.

Therefore a sharper peak has a jigher Q value than a shallower peak.