1. Moment Of Inertia

1.1. Rotational Kinematics –Description Of Motion Of A Rotating Body

1.1.1. Linear Kinematics

Consider the case of linear kinematics; it concerns the description of an object traveling in a straight line. The description of the motion consist of the knowledge of the body's position, velocity and acceleration at a particular time (x, v, a, t).

Velocity is rate of change of displacement (v = ds/dt), while acceleration is the rate of change of velocity (a = dv/dt).
(Note: please refer to the textbook about the differences between instantaneous and average values).

The mathematical manipulations of the definitions give rise to the linear equations (applicable for constant acceleration).

Linear motion
$v = u + at$
$s = ut + \frac{1}{2}at^2$
$s = \frac{(u+v)}{2}t$
$v^2 = u^2 + 2as$

1.1.2. Circular Motion

Consider now the motion of a particle in a circular path on the x-y plane, described by the radius (distance of particle from center of circle (0.0)), and the angle between the radius with the positive x- axis.

The angle is measured from the positive x-axis in a counter clockwise direction. As the radius is constant in length, the angle then determines the position of the particle.

Angular displacement is the change in the value of the angle between new position and an earlier position. A positive angular displacement is obtained if the particle moves in a counterclockwise direction. If the particle moves in a clockwise direction, a negative displacement is obtained.

Angular speed is the time rate of change of angular displacement. A positive angular speed indicates a counterclockwise motion, while a negative angular speed indicates a clockwise motion.

Angular acceleration is the time rate of change of angular speed. A positive angular acceleration indicates the angular speed is increasing during a counterclockwise motion, while a negative angular acceleration indicates the angular speed is decreasing during a counterclockwise motion. (A negative angular acceleration also may indicate that the magnitude of angular speed is increasing during a clockwise motion).

The equations of motion of a particle moving in a circle can be readily obtained from the equations of motion of a particle in a straight line by substituting the linear displacement with angular displacement, linear velocity with angular velocity and linear acceleration with angular acceleration.

(Note: When the particle travels along the circular path, the displacement is nearly linear, and given as ds = r dθ, differentiating this gives the tangential velocity and its relationship with angular velocity,

$$\frac{ds}{dt} = \frac{d(r\theta)}{dt} = r\frac{d\theta}{dt}$$

now $\frac{ds}{dt} = v$ and $\frac{d\theta}{dt} = \omega$;
therefore $v = r\omega$ gives the relationship between velocity and angular velocity. Differentiating velocity again gives tangential acceleration and its relationship with angular acceleration.)

Circular Motion
$\omega = \omega_o + \alpha t$
$\theta = \omega_o t + \frac{1}{2}\alpha t^2$
$\theta = \frac{(\omega_o + \omega)}{2} t$
$\omega^2 = \omega_o^2 + 2\alpha\theta$

1.1.3. Rigid body

In both linear motion, there is not much concern if the moving object is a particle or a body.
A particle a single point body, having mass but does not occupy space. A body however has mass and occupies space, it has a shape. In linear motion if the parts of the body travels at different velocity, then the shape of the body changes. The description of the whole body becomes rather difficult to describe. Thus we restrict the body such that all parts move at the same speed / velocity to maintain the same shape and size throughout its motion. Thus the body is called a rigid body. For a rigid body the distance between two points on the body remains constant.

If a non rigid body is made to rotate, its shape and size may change, imagine a piece of pizza dough (or roti canai) being rotated, its size will increase. Thus we restrict our study of rotational motion to rigid body only.

1.1.4. Rotation of rigid body

When a rigid body is made to rotate about an axis, the path of each particle in the body describes a circle centered on the axis of rotation. The size of each circular path depends on the distance between the particle and the axis of rotation. Near particles make smaller circle, while far away particles make larger circles. Each particle can be said to move in a circular motion. If lines are drawn connecting the particles to the center of the circle (which is also the axis of rotation), the lines will be observed to rotate about the axis of rotation. Each line rotates at the same angular speed, as no lines moves faster than another (i.e. The particle's distance from one another does not change – condition for a rigid body)
Therefore the rotation of a rigid body can be described as one of circular motion. The angular speed of the body is the angular speed of all radiuses drawn from each particle to the axis of rotation.

(Note: The tangential velocity of each particle is different as it depends on the distance from the axis of rotation ($v = r\,\omega$).)

1.2. Moments Of Inertia

1.2.1. Torque

Torque is the cross product of the torque arm and the applied force, $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. The magnitude of the torque is given by $\tau = rF\sin\theta$, (where θ is the angle between r and F) and its direction determine by the right hand screw rule. If the force turns the body counter clockwise, torque is positive, if the body turns clockwise, torque is negative. (Full vector treatment can also be used).

Now consider a point particle restricted to move in a circular path. A force acts on a particle while the particle still moves in the circular path. Then $F\sin\theta$ contributes to the tangential acceleration, while $F\cos\theta$ contributes to the centrifugal force.

$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$
$\boldsymbol{\tau} = \mathbf{r} \times m\mathbf{a}$
$\boldsymbol{\tau} = \mathbf{r} \times m(r\boldsymbol{\alpha})$
$\tau = m\,r^2\alpha$
(Note: The angular acceleration, $\boldsymbol{\alpha}$ and the tangential acceleration, **a** are in the same direction. **r** and $\boldsymbol{\alpha}$ are perpendicular to each other**,** thus $\boldsymbol{\tau}$ is perpendicular to both **r** and $\boldsymbol{\alpha}$)

Now $\tau = m\,r^2\alpha$ can be written as $\tau = I\alpha$, where $I = m\,r^2$ is called the moment of inertia or the rotational inertia. This form is similar to $F = ma$, Newton's Second Law for linear motion. Thus $\tau = I\alpha$ is Newton's Second Law for rotational motion.

Expanding to a multi point body,

$\Sigma\boldsymbol{\tau} = \Sigma\,(\mathbf{r} \times \mathbf{F})$
$\Sigma\boldsymbol{\tau} = \Sigma\,(\mathbf{r} \times m\mathbf{a})$
$\Sigma\boldsymbol{\tau} = \Sigma\,(\mathbf{r} \times m(r\boldsymbol{\alpha}))$
$\Sigma\tau = \Sigma m\, r^2\alpha$, where α is common for all points on a rotating body.
Thus $I = \Sigma m\, r^2$ is the moment of inertia or the rotational inertia for the body.
This is the moment of inertia or the rotational inertia defined from torque.

1.2.2. Definition Of Moment Of Inertia

The total kinetic energy of a purely rotating body is the sum of the kinetic energy of all particles in the body.

$$KE_{rotational} = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$$

replacing v with $v = r\omega$ will give the equation

$$KE_{rotational} = \sum_{i=1}^{N} \frac{1}{2} m_i (r\omega)_i^2 = \sum \frac{1}{2} m_i r_i^2 \omega_i^2 = \frac{1}{2} \sum m_i r_i^2 \omega^2$$ as ω is common for all particles.

The equation further reduces to $KE_{rotational} = \frac{1}{2} I\omega^2$, where $I = \sum m_i r_i^2$ is callled the moment of inertia (or rotational inertia) of the rotating body about that particular axis.

This is moment of inertia or the rotational inertia defined from kinetic energy.

1.2.3. Moment of inertia for a body of continuous mass

For a body of continuous form (many particles) the summation can be replaced by an integration. $I = \int mr^2$.

The moment of inertia is difficult to calculate, especially the integral form.

To overcome this difficulty we shall use the parallel axis theorem, the perpendicular axis theorem, the techniques for calculating moment of inertia of a composite body and the moment of inertia for some regular bodies to obtain the moment of inertia of a body about a given axis of rotation.
The moments of inertia for some regular bodies. In each case the axis of rotation passes through the object's center of mass.

Object	**MI**
Thin rod	$\frac{ML^2}{12}$
Ring	MR^2

Disc	$\frac{MR^2}{2}$
Cylinder	$\frac{MR^2}{2}$
Cylinder	$\frac{MR^2}{4}+\frac{ML^2}{12}$
Rectangular plate	$\frac{M\left(a^2+b^2\right)}{12}$
Rectangular plate	$\frac{Ma^2}{12}$
Sphere (hollow)	
Sphere (solid)	

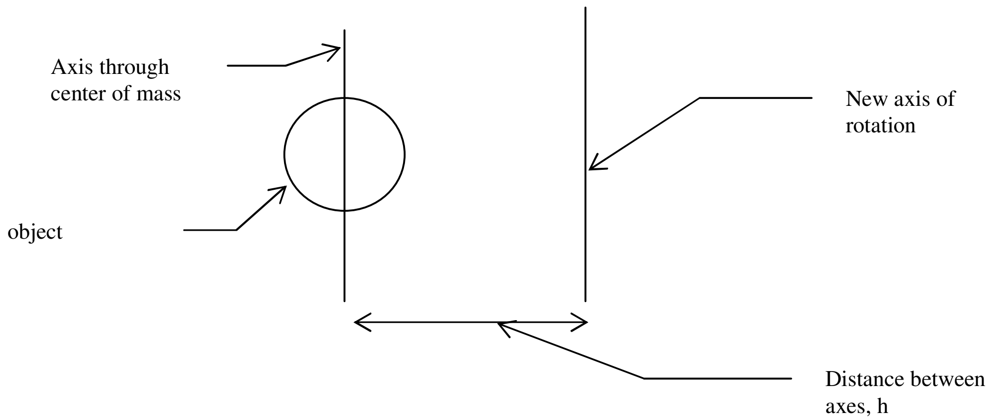
Example: Moment of inertia of a disk.

A uniform density disk has mass M and radius R. The moment of inertia about an axis through the center of mass is determined as:
(Area) Density, $\rho = M / \pi R^2$, mass of a mass element dm = $\rho\, r\, d\theta\, dr$

Therefore,
$$I=\int mr^2=\int r^2\rho r d\theta .dr$$
$$I=\iint r^3\rho .dr.d\theta$$
$$I=\int \rho .\frac{r^4}{4}\Big]_0^R .d\theta$$
$$I=\rho\frac{R^4}{4}\int d\theta$$
$$I=\rho\frac{R^4}{4}\theta\Big]_0^{2\pi}$$
$$I=\rho\frac{R^4}{4}2\pi$$
$$I=\frac{M}{\pi R^2}\frac{R^4}{4}2\pi$$
$$I=\frac{MR^2}{2}$$

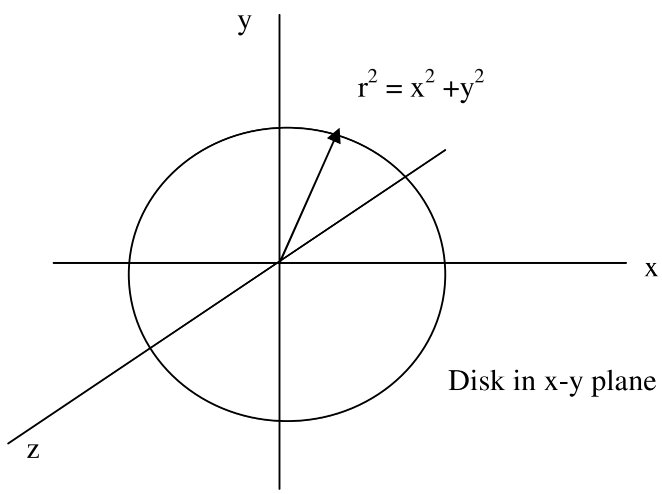
1.2.4. Parallel Axis Theorem

The parallel axis theorem is used to obtain the moment of inertia of a body about an axis which is parallel to the axis which passes through its center of mass. If the moment of inertia about the axis through the center of mass is I_{cm} , the mass of the object M and the distance between the two axes is h, then the moment of inertia about the new axis is
$I = I_{cm} + M h^2$



1.2.5. Perpendicular Axis Theorem

The perpendicular axis theorem is used to obtain the moment of inertia about an axis perpendicular to another axis of which the moment of inertia about it is known. It is usually applied to a lamina of symmetrical shape, ie disk, ring, rectangular/square plates.



The distance r from the axis of rotation (z-axis) is given as $r^2 = x^2 + y^2$, thus $I = \Sigma m_i r_i^2$ can be rewritten as $I = \Sigma m_i\,(x_i^2 + y_i^2) = \Sigma m_i\, x_i^2 + \Sigma m_i\, y_i^2 = I_y + I_x$
Thus $I_z = I_x + I_y$..
For disk and ring, $I_x = I_y$, because of the symmetrical shape.
Then $I_x = I_y = I_z / 2$

1.2.6. Moment Of Inertia Of A Composite Body

The moment of inertia of a composite body about an axis is the sum of the moment of inertia of each components/parts of the body about that axis. This is also called the superposition principle for moment of inertia.

$I_{composite} = \Sigma I_i$

1.2.7. Radius of Gyration

The moment of inertia of any body about an axis can be written as;

$I = mk^2$, where $k = \sqrt{\frac{I}{m}}$ is called the radius of gyration about that particular axis.

1.3. Kinetic energy of a rolling body

1.3.1. Rotational Kinetic Energy

The kinetic energy due to the rotation. Each Particles on the body moves in a circular path about the axis of rotation, the sun of KE of each particles gives

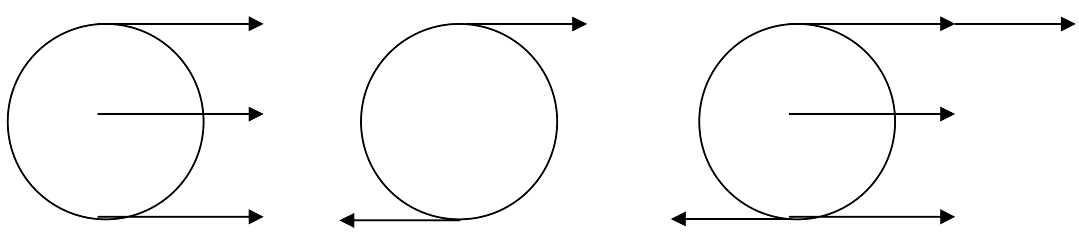
$E = \frac{1}{2} I \omega^2$

1.3.2. Total Kinetic Energy

The sum of KE of the CM and the rotational KE of the body.

1.3.3. Rolling Without Slipping

All points move with translational velocity.
All points have rotational velocity (tangential velocity on the rim of the wheel).
The lowest point is in contact with ground.(rel.velocity = 0)
CM has translational velocity only
Top point has rotational and translational motion.



Purely Translational | Purely Rotational | Combined motion

Consider the point in contact with ground.
If the translational component is larger in magnitude than the rotational component (v_t), then the resultant velocity is to the right. The bottom point is moving to the right with respect to the ground, a skidding condition occurs.
Example – A car skidding with the wheels even though the brakes are on.

If the translational component is smaller in magnitude than the rotational component (v_t), then the resultant velocity is to the left. The bottom point is moving to the left with respect to the ground, a slipping condition occurs. Example – The wheels spinning when a car is stuck in mud, etc.

When the translational component is equal in magnitude with the rotational component (v_t), then the resultant velocity is zero. The bottom point is stationary with respect to the ground. This is rolling without slipping.

1.3.4. Problems Relating To Energy