10.1 Limits

Partial review:

The function:

The function is a relation between two sets A and B, such that each element x in A has a unique image y in B, we then say that f(x) = y.

In the function f(x) = y, the letter x is called <u>independent variable</u>, and y is called <u>dependent variable</u>.

The set of all independent variables-A here-is called the <u>domain</u> of the function, and the set of all dependent variables-B here-is called the <u>range</u> of the function.

Types of functions:

- <u>The polynomial function:</u>

  The general form of the polynomial function of degree n is

  $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$

  When n = 1, the polynomial function becomes linear function and when n = 2, it becomes the quadratic function and so on.

  The domain of the polynomial function of any degree is the set of all real numbers R.

- <u>The rational function:</u>

  The rational function is always written as a quotient of two function, for instance

  $f(x) = \dfrac{g(x)}{h(x)}$, the domain of the rational function is set of all real numbers except zeros of denominator. i.e, domain = $R - \{zeros\ of\ \ h(x)\}$.

- <u>The radical function:</u>

  The radical function always equal expression under the radical sign, for instance

  $f(x) = \sqrt{E(x)}, \quad where\ \ E(x)\ is\ a\ mathematical\ \ \exp ression.$ To find its domain, put the expression under the radical ≥ 0 , then the domain will be $\{x|\ x \in R,\ and\ \ E(x) \geq 0\}$.

- <u>The logarithmic function</u>:

  For instance, $f(x) = \log_a^x$, and $f(x) = \ln_e^x$, … .

  The number $e \approx 2.71828$ is always the base of the ln function, therefore we briefly write f(x) = ln(x).

- Trigonometric function:

For instance,

$$f(x)=\sin(x),\quad f(x)=\cos(x),\quad f(x)=\tan(x)=\frac{\sin(x)}{\cos(x)},\quad f(x)=\sec(x)=\frac{1}{\cos x},\ f(x)=\csc(x)=\frac{1}{\sin x}$$

- Case defined function:

For instance the modulus function $f(x)=|x|,$ which is defined as follows,

$$|x|=\begin{cases} x, & if \quad x>0 \\ -x & if \quad x<0. \end{cases}$$

Example the rational function $f(x)=\dfrac{x^2-1}{x-1}$ has the domain $R-\{1\}$, thus, 1 is not in the domain of our function f(x), therefore we will study the values of this function for values very close (near) to 1 but doesn't equal 1.

The function $f(x)=\dfrac{x^2-1}{x-1}=\dfrac{(x-1)(x+1)}{(x-1)}=x+1.$ To do so, we form the following table:

x	0.9	0.99	0.999	$\xrightarrow{\cdots\cdots}$	1	$\xleftarrow{\cdots\cdots}$	1.01	1.5	2
F(x)	1.9	1.99	1.999	$\xrightarrow{\cdots\cdots}$	2	$\xleftarrow{\cdots\cdots}$	2.01	2.5	3

From the table we observe that f(x) approaches to the value 2 to as x approaches to the value 1, i.e f(x) $\to$ 2 as x$\to 1$,

We then say that "***the limit of the function f(x) equal 2 as x approaches to 1***". We write this as:

$$\lim_{x\to 1} f(x)=2.$$

In general, we say that $\lim_{x\to a} f(x)=L,$ when f(x) approaches L as x approaches a.

In the above table we find that when $x<1 \quad f(x)\to 2$, this is called the Left-hand limit of f(x), and is written as $\lim_{x\to 1^-} f(x)=2,$ and also when $x>1 \quad f(x)\to 2$, this is called the Right-hand limit of f(x), and is written as $\lim_{x\to 1^+} f(x)=2.$

Rule: ***The limit of a function exists if and only if the Left-hand limit equal the Right-hand limit of this function.***

In our example the left hand limit equal the right hand limit $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = 2.$

Then based on the above rule the limit of $f(x) = \dfrac{x^2-1}{x-1}$ exists.

Example (1) show if the limit of the function drawn in the following graph exists or not (give your reasons).

Properties of limits:

- $\lim_{x\to a} C = C,$ where C is constant.
- $\lim_{x\to a} x^n = a^n,$ where n is positive integer.
- If f(x) is a polynomial function, then $\lim_{x\to a} f(x) = f(a),$
- $\lim_{x\to a} (1+x)^{\frac{1}{x}} = e,$ where e is the base of the natural logarithm.
- If $\lim_{x\to a} f(x)$ exist and $\lim_{x\to a} g(x)$ exist, then
    - $\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x),$
    - $\lim_{x\to a} f(x).g(x) = \lim_{x\to a} f(x).\lim_{x\to a} g(x),$
    - $\lim_{x\to a} C.f(x) = C.\lim_{x\to a} f(x),$
    - $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)},$ provided that g(x) $\neq 0,$
    - $\lim_{x\to a} C.f(x) = C.\lim_{x\to a} f(x),$
    - $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}.$

Example (2) find $\lim_{r\to9} \dfrac{4r-3}{11}$

Solution: $\lim_{r\to9} \dfrac{4r-3}{11} = \dfrac{4(9)-3}{11} = \dfrac{36-3}{11} = \dfrac{33}{11} = 3.$

Example (3) find $\lim_{x\to-6} \dfrac{x^2+6}{x-6}$

Solution $\lim\limits_{x\to -6} \dfrac{x^2+6}{x-6} = \dfrac{36+6}{-12} = -\dfrac{7}{2}.$

Example (4) $\lim\limits_{x\to 2} \dfrac{x^2-x-2}{x-2}$

Solution $\lim\limits_{x\to 2} \dfrac{x^2-x-2}{x-2} = \lim\limits_{x\to 2} \dfrac{(x-2)(x+1)}{(x-2)} = \lim\limits_{x\to 2}(x+1) = 2+1 = 3.$

Example (5) find $\lim\limits_{x\to 0} \dfrac{(x+2)^2-4}{x}$

Solution $\lim\limits_{x\to 0} \dfrac{(x+2)^2-4}{x} = \lim\limits_{x\to 0} \dfrac{x^2+4x+4-4}{x} = \lim\limits_{x\to 0} \dfrac{x(x+4)}{x} = \lim\limits_{x\to 0}(x+4) = 0+4 = 4.$

Example (6) find $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$, where $f(x) = x^2-3.$

Solution $\because f(x) = x^2-3$

$\therefore f(x+h) = (x+h)^2-3 = x^2+2xh+h^2-3.$

Now the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ will be simplified as:

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x^2+2xh+h^2-3)-(x^2-3)}{h} \\ &= \frac{x^2+2xh+h^2-3-x^2+3}{h} \\ &= \frac{2xh+h^2}{h} = \frac{h(2x+h)}{h} = 2x+h.\end{aligned}$$

Thus, $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}(2x+h) = (2x+0) = 2x.$