Section 13.2 Absolute extrema on a closed interval

Extreme-value theorem:

If a function is continuous on a closed interval, then the function has both a maximum value and minimum value on that interval.

Steps for finding the absolute extrema on a closed interval $[a,b]$:

- Step (1) Find the critical values
- Step (2) Evaluate f at a and b and also at the critical values.
- Step (3) The maximum value of f is the greatest of the values found in the second step, and the minimum value of f is the minimum value in step (2).

Example (1) Find the absolute extrema for $f(x) = x^2 - 2x + 3$ on the closed interval $[0,3]$.

Solution:

$f' = 2x - 2, \quad f' = 0 \Rightarrow \quad 2x - 2 = 0, \quad \Rightarrow x = 1.$

Thus, the critical value is $x = 1$. Also,

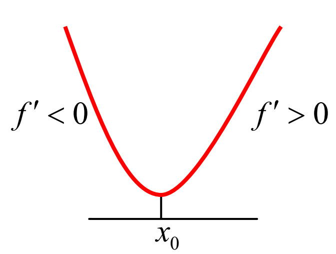
$f(0) = (0)^2 - 2(0) + 3 = 3,$

$f(3) = (3)^2 - 2(3) + 3 = 9 - 6 + 3 = 6,$

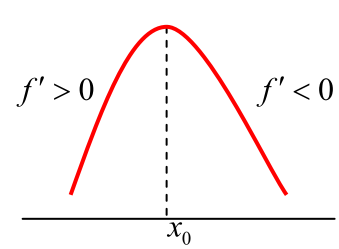
$f(\text{the critical value}) = f(1) = (1)^2 - 2(1) + 3 = 2.$

Now, using the extreme-value theorem, we find that the maximum is $f(3) = 6$ and the minimum is $f(1) = 2$.

Section 13.3 Concavity



(a) concave up



(b) concave down

From the figure (a) we can see that when f' changes when it passes through x_0 from negative to positive (f' is increasing) then f is concave up. On the other hand, figure (b) show that when f' changes when it passes through x_0 from positive to negative (f' is decreasing) then f is concave down.

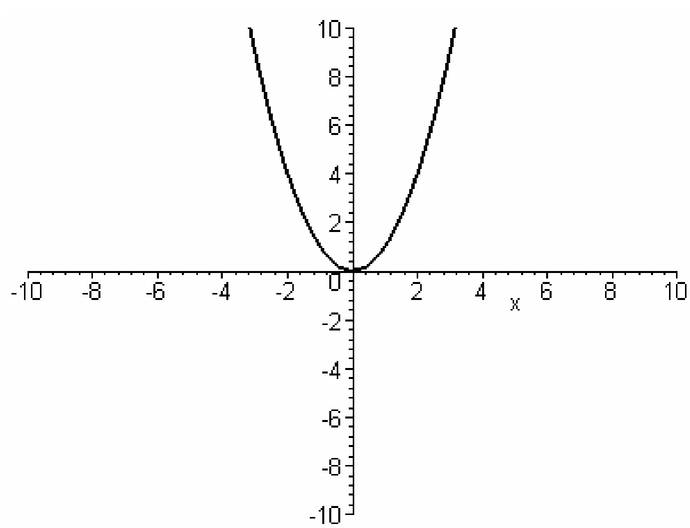
Definition (1) If f is differentiable on interval (a,b), then f is said to be concave up on (a,b) if f' is increasing and concave down if f' is decreasing.

Since f' is the derivative of f'', this implies that if f' is increasing then $f''>0$ and that if f' is decreasing then $f''<0$.

Consequently, based on the above definition and statement we have the following criteria of testing concavity:

If $f''>0$ for all $x\in(a,b)$ then f is concave up, and if $f''<0$ for all $x\in(a,b)$ then f is concave down.

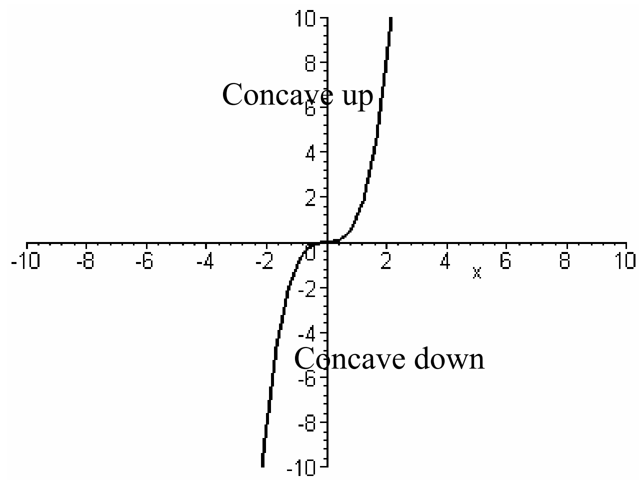
For example, the function $f(x)=x^2$, has $f'=2x$ and $f''=2$. Thus $f''>0$ for all values of x and therefore f is always concave up. This can be shown easily from the graph of the parabola



Example (2) test the concavity of $f(x) = x^3$

Solution

$f' = 3x^2$ and $f'' = 6x$, thus, if $x > 0 \Rightarrow f'' > 0$ and hence, f is concave up. If $x < 0 \Rightarrow f'' < 0$ and hence, f is concave down. This is shown clearly from the graph of f(x).



Definition (2) A function has an inflection point when $x = x_0$ if and only if f is continuous at x_0 and f changes concavity at x_0.

Thus, the inflection point at $x = x_0$ must satisfy the following two conditions:

- $f''(x_0) = 0$ or $f''(x_0)$ is undefined,
- f is continuous at x_0.

The condition of continuity is necessary for the inflection point, see the following example:

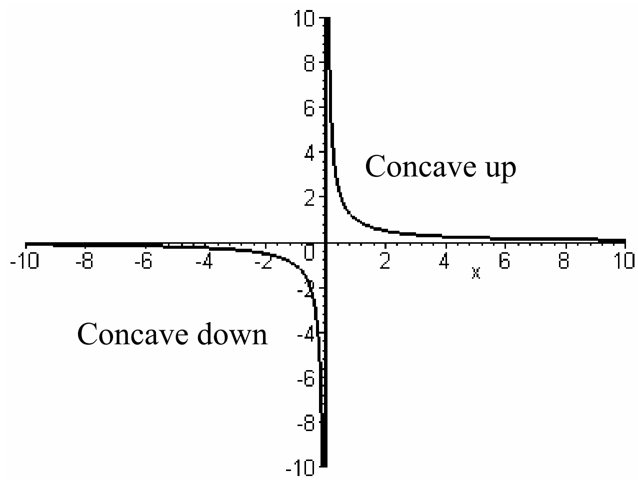
Example (3) Test the function $f(x) = \frac{1}{x}$ for inflection points.

Solution

$f' = -\frac{1}{x^2} \quad \Rightarrow f'' = -\frac{2}{x^3}$, and it is clear that f'' is not defined at $x_0 = 0$.

Since f'' changes its concavity when it passes through $x_0 = 0$, then the first condition is satisfied. But the second condition is not satisfied because f(x) is not continuous at $x_0 = 0$.

Therefore, the value $x = x_0$ is not corresponding to an inflection point of the function $f(x) = \frac{1}{x}$, see the following graph:



Example (4) Test $y = x^4 - 3x^3 + 7x - 5$ for concavity and inflection points.

Solution

$y' = 4x^3 - 9x^2 + 7 \quad \Rightarrow \quad y'' = 12x^2 - 18x = 6x(2x-3)$.

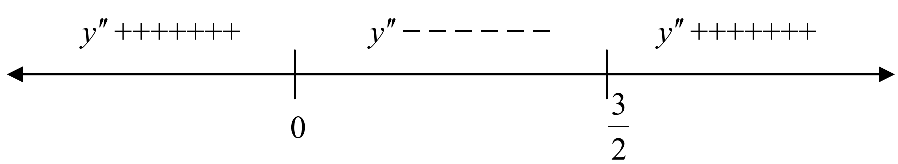
To find the inflection points put $y'' = 0$, then $6x(2x-3) = 0$, this implies that $x = 0$ *or* $x = \frac{3}{2}$.

Thus, the points $x = 0$ *and* $x = \frac{3}{2}$ are candidate for inflection points. It is clear that f(x) is continuous at $x = 0$ *and* $x = \frac{3}{2}$. Now, we test concavity by determining the sign of y'':

If $x < 0$ then $y'' = 6(-)(-) = +$, so the curve is concave up.

If $0 < x < \frac{3}{2}$ then $y'' = 6(+)(-) = -$, so the curve is concave down.

If $x > \frac{3}{2}$ then $y'' = 6(+)(+) = +$, so the curve is concave up.



Now, the inflection points are $(0,-5)$ and $(\frac{3}{2}, 0.4375)$.

Section 13.4 The second derivative test

The second derivative test for relative extrema is given below:

> Suppose that $f'(x_0)=0$:
>
> If $f''(x_0)<0$, then f has a relative maximum at x_0.
>
> If $f''(x_0)>0$, then f has a relative minimum at x_0.

Remark (1): The second derivative test fails when $f''(x_0)=0$.

Example (5) Use the second derivative test to test the relative extrema of the following function:

$$y=18x-\frac{2}{3}x^3.$$

Solution

$f'=y'=18-2x^2$. To obtain the critical values put $y'=0$, then $18-2x^2=0 \Rightarrow 2(9-x^2)=0$. So the critical values are $x=-3$ and $x=3$. To apply y'' test we first find $y''=-4x$. Then

$y''(-3)=-4(-3)=12>0$, so the critical value $x=-3$ is relative minimum, and the point $(-3,f(-3))$ is relative minimum point.

$y''(3)=-4(3)=-12<0$, so the critical value $x=3$ is relative maximum, and the point $(3,f(3))$ is relative maximum point.

Example (6) Use the second derivative test to test the relative extrema of the following function:

$$y=6x^4-8x^3+1.$$

Solution

$f'=y'=24x^3-24x^2=24x^2(x-1)$. To obtain the critical values put $y'=0$, then

$24x^2(x-1)=0$. So the critical values are $x=0$ and $x=1$. To apply y'' test we first find

$y''=72x^2-48x$.

$y''(0)=0$, then y'' test fails to test the critical value $x=0$. We then use y' test for $x=0$

:

If $x<0$, then $y'<0$,

If $0<x<1$, then $y'<0$,

The sign of y' doesn't change while it passes through $x=0$, so $x=0$ doesn't correspond to relative extrema.

$y''(1)=72(1)^2-48(1)=+$, then the critical value $x=1$ is relative minimum.

Remark (2): If the function is continuous and has exactly relative extremum on an interval, then it is absolute extremum on that interval.

In the last example, the function f(x) is continuous on the set of all real numbers R and has only relative minimum when $x=1$. Thus, this relative minimum is absolute minimum, i.e. the point $(1,-1)$ is absolute minimum of f(x).

Example (7) Sketch the graph of $y=2x^3-9x^2+12x$.

Solution

Intercept:

y intercept (put x = 0), $\Rightarrow$ $(0,0)$ is y intercept,

x intercept (put y = 0), $\Rightarrow$ $(0,0)$ is x intercept,

Symmetry:

The function f(x) is neither even nor odd function, therefore f(x) is not symmetric with respect to y axis and also not symmetric with respect with the origin.

Extrema:

$y'=6x^2-18x+12=6(x^2-3x+2)=6(x-1)(x-2)$. So the critical values are $x=1$ and $x=2$.

$y''=12x-18$. Then, $y''(1)=12(1)-18=-6<0$, so the critical value $x=1$ is relative maximum, and the point $(1,5)$ is relative maximum point.

$y''(2)=12(2)-18=6>0$, so the critical value $x=2$ is relative minimum, and the point $(2,4)$ is relative minimum point.

Concavity:

To find the inflection point put $y''=0 \quad \Rightarrow 12x-18=0$. this implies that $x=\frac{3}{2}$. Thus, the point $x=\frac{3}{2}$ is candidate for inflection point. It is clear that f(x) is continuous at $x=\frac{3}{2}$. Now, we test concavity by determining the sign of y'':

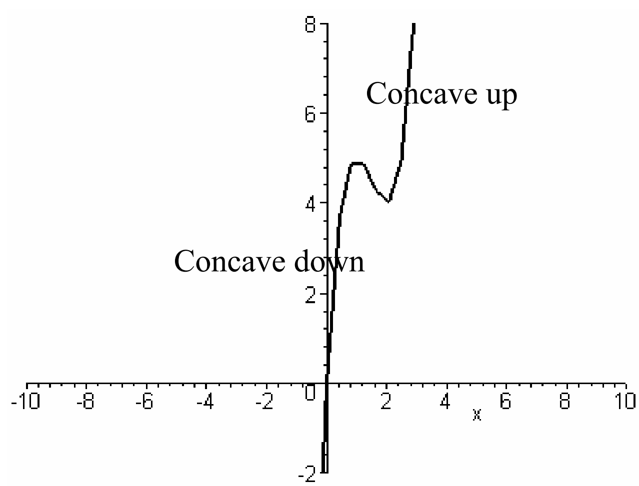
If $x<\frac{3}{2}$ then $y''<0$, so the curve is concave down,

If $x>\frac{3}{2}$ then $y''>0$, so the curve is concave up.

By summarizing all of the above results in the following table:

x	0	1	1.5	2
y	0	5	4.5	4

Now, the graph of f(x) is given as follows:



Home work

section	problems
Section 13.2	Exc. 13.2 problems : 2, 10, 12
Section 13.3	Exc. 13.3 problems : 14, 30, 40, 46, 68
Section 13.4	Exc. 13.4 problems : 6, 8, 12