Rule (1) the chain (1)

If y is differentiable of u and u is differentiable of x, then y is differentiable of x, i.e.

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx} .$$

Example (1) If $y = u^2 - 2u$ and $u = x^2 - x$, find $\frac{dy}{dx}$.

Solution

Since $\frac{dy}{du} = 2u - 2$, $\frac{du}{dx} = 2x - 1$,

But, $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$ then $\frac{dy}{dx} = (2u-2)(2x-1) = 2(u-1)(2x-1)$, by substituting the expression of x instead of u we finally obtain:

$$\frac{dy}{dx} = 2(x^2 - x - 1)(2x-1) .$$

Example (2) If $y = \frac{1}{w^2}$ and $w = 2 - x$, find $\frac{dy}{dx}$.

Solution

Since $y = w^{-2}$ then $\frac{dy}{dw} = -2w^{-3} = -\frac{2}{w^3}$, $\frac{dw}{dx} = -1$,

But, $\frac{dy}{dx} = \frac{dy}{dw} . \frac{dw}{dx}$ then $\frac{dy}{dx} = (-\frac{2}{w^3})(-1) = \frac{2}{w^3}$, by substituting the expression of x instead of w we finally obtain:

$$\frac{dy}{dx} = \frac{2}{(2-x)^3} .$$

Example (3) If $y = \sqrt{w}$ and $w = 7 - t^3$, find $\frac{dy}{dt}$.

Solution

Since $y = w^{\frac{1}{2}}$ then $\frac{dy}{dw} = \frac{1}{2} w^{-\frac{1}{2}} = \frac{1}{2\sqrt{w}}$, $\frac{dw}{dt} = -3t^2$, But, $\frac{dy}{dt}=\frac{dy}{dw}.\frac{dw}{dt}$ then $\frac{dy}{dt}=(\frac{1}{2\sqrt{w}})(-3t^2)=-\frac{3t^2}{2\sqrt{w}}$, by substituting the expression of t instead of w we finally obtain:

$$\frac{dy}{dt}=-\frac{3t^2}{2\sqrt{7-t^3}}.$$

Example (4) If $y=3w^2-8w+4$ and $w=2x^2+1$, find $\frac{dy}{dx}$ when x = 0.

Solution

$\frac{dy}{dw}=6w-8,$ $\frac{dw}{dx}=4x$,

But, $\frac{dy}{dx}=\frac{dy}{dw}.\frac{dw}{dx}$ then $\frac{dy}{dx}=(6w-8)(4x)$, hence

$$\left.\frac{dy}{dx}\right|_{x=0}=(6(2(0)^2+1)-8)(4(0))=0.$$

Example (5) If $y=u^n$ (where u is any real number) and $u=f(x)$, find $\frac{dy}{dx}$.

Solution

Then $\frac{dy}{du}=nu^{n-1}$,

But, $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$ then $\frac{dy}{dx}=nu^{n-1}(\frac{du}{dx})$. Thus, we obtain the following rule:

Rule (2) Power rule

If u is differentiable of x and n is any real number, then

$$\frac{d}{dx}(u)^n=nu^{n-1}(\frac{du}{dx})$$

In other words, we say that: the <u>derivative of a bracket raised to the power (n) equals the derivative of the bracket multiply the derivative of what inside the bracket</u>.

Example (6) If $y=(x^2+1)^8$ find $\dfrac{dy}{dx}$.

Solution

$$\frac{dy}{dx}=\frac{d}{dx}(x^2+1)^8=8(x^2+1)^7(2x)$$
$$=16x(x^2+1)^7$$

Example (7) If $y=3(2x^2-3x-1)^{-\frac{10}{3}}$ find $\dfrac{dy}{dx}$.

Solution

$$\frac{dy}{dx}=\frac{d}{dx}3(2x^2-3x-1)^{-\frac{10}{3}}=3\frac{d}{dx}(2x^2-3x-1)^{-\frac{10}{3}}=3(-\frac{10}{3})(2x^2-3x-1)^{-\frac{10}{3}-1}(4x-3)$$
$$=(-10)(2x^2-3x-1)^{-\frac{13}{3}}(4x-3).$$

Example (8) If $y=\sqrt{5x^2-x}$ find y'.

Solution

$$\frac{dy}{dx}=\frac{d}{dx}(5x^2-x)^{\frac{1}{2}}=\frac{1}{2}(5x^2-x)^{\frac{1}{2}-1}(10x-1)$$
$$=\frac{1}{2}(5x^2-x)^{-\frac{1}{2}}(10x-1)$$
$$=\frac{(10x-1)}{2\sqrt{5x^2-x}}.$$

Example (9) If $y=\sqrt{\dfrac{x-2}{x+3}}$ find y'.

Solution

$$\frac{dy}{dx}=\frac{d}{dx}(\frac{x-2}{x+3})^{\frac{1}{2}}=\frac{1}{2}(\frac{x-2}{x+3})^{\frac{1}{2}-1}\frac{d}{dx}(\frac{x-2}{x+3})$$
$$=\frac{1}{2}(\frac{x-2}{x+3})^{-\frac{1}{2}}[\frac{(x+3)(1)-(x-2)(1)}{(x+3)^2}]=\frac{1}{2}(\frac{x+3}{x-2})^{\frac{1}{2}}[\frac{(x+3-x+2)}{(x+3)^2}]$$
$$=\frac{1}{2}(\sqrt{\frac{x+3}{x-2}})[\frac{5}{(x+3)^2}]=\frac{5}{2}\sqrt{\frac{x+3}{(x-2)(x+3)^4}}=\frac{5}{2}\sqrt{\frac{1}{(x-2)(x+3)^3}}=\frac{5}{2\sqrt{(x-2)(x+3)^3}}.$$

Example (9) Suppose $p = 50 - \sqrt{q^3 + 10}$ is a demand equation for a manufacturer's product.

(a) Find the rate of change of (p) with respect to q.
(b) Find the relative rate of change of (p) with respect to q.
(c) Find the marginal-revenue function.

Solution

The rate of change of (p) with respect to q is $\frac{dp}{dq}$. Then,

$$\frac{dp}{dq} = \frac{d}{dq}(50 - \sqrt{q^3 + 10}) = \frac{d}{dq}(50 - (q^3 + 10)^{\frac{1}{2}})$$
$$= 0 - (\frac{1}{2})(q^3 + 10)^{\frac{1}{2}-1}(3q^2) = -(\frac{3}{2})(q^3 + 10)^{-\frac{1}{2}}(q^2)$$
$$= \frac{-3q^2}{2\sqrt{q^3 + 10}}.$$

The relative rate of change of (p) with respect to q is $\frac{\frac{dp}{dq}}{p}$. Then,

$$\frac{\frac{dp}{dq}}{p} = (\frac{dp}{dq}).(\frac{1}{p}) = (\frac{-3q^2}{2\sqrt{q^3 + 10}}).(\frac{1}{50 - \sqrt{q^3 + 10}})$$
$$= -\frac{3q^2}{2(50\sqrt{q^3 + 10} - q^3 - 10)}.$$

To evaluate the marginal revenue function $\frac{dr}{dq}$ we first recall the relation between the revenue (r) and the price (p);

revenue = (price).(quantity),

i.e. r = (p).(q)

Since $p = 50 - \sqrt{q^3 + 10}$ then $r = pq = (50 - \sqrt{q^3 + 10})q = 50q - q\sqrt{q^3 + 10}$

Now, the marginal revenue $\frac{dr}{dq} = \frac{d}{dq}(50q - q\sqrt{q^3+10}) = 50 - (\sqrt{q^3+10} + \frac{3q^3}{2\sqrt{q^3+10}})$.

Home work: solve the following problems (page 582 in the book).

[8] If $y = 3u^3 - u^2 + 7u - 2$ and $u = 5x - 2$, find $\frac{dy}{dx}$ when x = 1.

[18] If $y = 4(7x - x^4)^{-\frac{3}{2}}$ find y'.

[44] If $y = \sqrt[3]{\frac{8x^2-3}{x^2+2}}$ find y'.

[46] If $y = \frac{(4x-2)^4}{3x^2+7}$ find y'.

[62] Find the equation of the tangent line to the curve $y = \frac{-3}{(3x^2+1)^3}$ at the point $(0,-3)$.

[69] Suppose $p = 100 - \sqrt{q^2+20}$ is a demand equation for a manufacturer's product.

(1) Find the rate of change of (p) with respect to q.

(2) Find the relative rate of change of (p) with respect to q.

(3) Find the marginal-revenue function.

[73] If the total cost function for a manufacturer is given by

$$c = \frac{5q^2}{\sqrt{q^2+3}} + 5000,$$

Find the marginal cost.