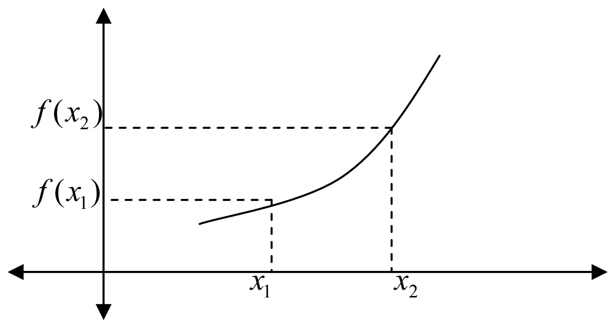
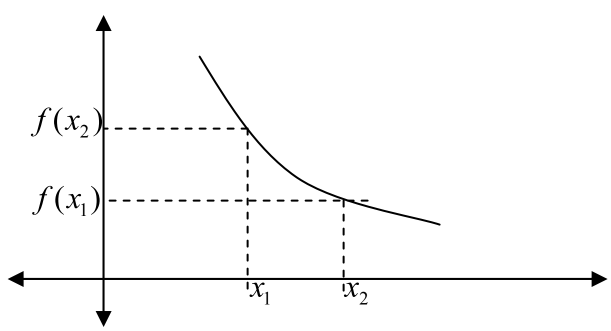
The function is said to be increasing if for all $x_2 > x_1$ then $f(x_2) > f(x_1)$



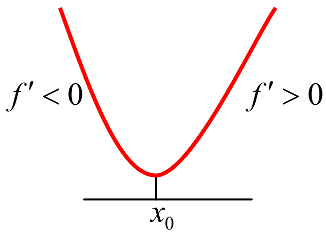
From the last figure if we assumed that the distance between x_1 and x_2 is very small then the curve of the function can be treated as a line segment and its slope $\dfrac{f(x_2)-f(x_1)}{x_2-x_1} > 0$ in the case of increasing function. But from our previous knowledge the slope at the point x_1 is the first derivative x_1. Thus, **If** $f'(x) > 0$ **for all** $x \in (x_1, x_2)$**, then** $f(x)$ **is increasing**.

The function is said to be decreasing if for all $x_2 > x_1$ then $f(x_2) < f(x_1)$

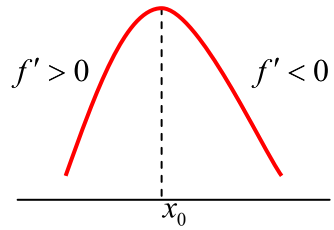


From the above figure if we assumed that the distance between x_1 and x_2 is very small then the curve of the function can be treated as a line segment and its slope $\dfrac{f(x_2)-f(x_1)}{x_2-x_1} < 0$ in the case of decreasing function. But from our previous knowledge the slope at the point x_1 is $f'(x_1)$. Thus**, If** $f'(x) < 0$ **for all** $x \in (x_1, x_2)$**, then** $f(x)$ **is decreasing**.

f' test:



x_0 is relative minimum



x_0 is relative maximum

From the above two figures, we can easily see that:

- If f' changes negative to positive as x increases through x_0, then f has a relative minimum at x_0.
- If f' changes positive to negative as x increases through x_0, then f has a relative maximum at x_0.
- If f' doesn't change its sign through x_0, then x_0 is not relative extremum.

A strategy for finding relative extrema

To find the relative extrema of the function f(x):

Step (1) Find f',

Step (2) Find the critical numbers (Points at which $f'(x) = 0$ and $f'(x)$ not defined),

Step (3) Use f' test to determine which of the candidate critical numbers are relative extrema and which are not.

Example (1) Find the relative extrema of $f(x) = x^4 - 2x^2$.

Solution

Step (1) $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1)$

Step (2) To find the critical numbers put $f'(x) = 0 \Rightarrow 4x(x-1)(x+1) = 0$,

Thus the critical numbers are $x = 0, \quad x = -1, \quad x = 1$.

Step (3) f' test for the points $x=0, \quad x=-1, \quad x=1$:

As $x<-1 \Rightarrow f' = 4(-)(-)(-) = -$ then $f'<0$ and f is decreasing in $(-\infty,-1)$.

As $-1<x<0 \Rightarrow f' = 4(-)(-)(+) = +$ then $f'>0$ and f is increasing in $(-1,0)$.

As $0<x<1 \Rightarrow f' = 4(+)(-)(+) = -$ then $f'<0$ and f is decreasing in $(0,1)$.

As $x>1 \Rightarrow f' = 4(+)(+)(+) = +$ then $f'>0$ and f is increasing in $(1,\infty)$.

By summarizing the above results on the following intervals:

$$\begin{array}{ccccccc} -\,-\,-\,-\,- & | & +\,+\,+\,+\,+\,+ & | & -\,-\,-\,-\,-\,-\,- & | & +\,+\,+\,+\,+ \\ \longleftarrow & -1 & \text{———} & 0 & \text{———} & 1 & \longrightarrow \end{array}$$

Thus, at $x=-1$ there is relative minimum, at the point x = 0 there is relative maximum and at x = 1 there is relative minimum.

Example (2) Find the relative extrema of $f(x) = x^2 e^x$.

Solution

Step (1) $f'(x) = x^2 e^x + e^x(2x) = xe^x(x+2)$

Step (2) To find the critical numbers put $f'(x)=0 \Rightarrow xe^x(x+2)=0$,

Thus the critical numbers are $x=-2, \quad x=0$.

Step (3) f' test for the points $x=-2, \quad x=0$:

As $x<-2 \Rightarrow f' = (-)(+)(-) = +$ then $f'>0$ and f is increasing in $(-\infty,-2)$.

As $-2<x<0 \Rightarrow f' = (-)(+)(+) = -$ then $f'<0$ and f is decreasing in $(-2,0)$.

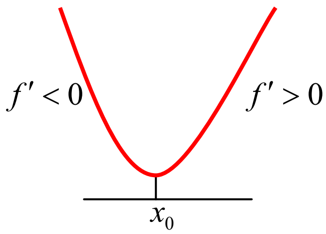
As $x>0 \Rightarrow f' = (+)(+)(+) = +$ then $f'>0$ and f is increasing in $(0,\infty)$.

By summarizing the above results on the following intervals:

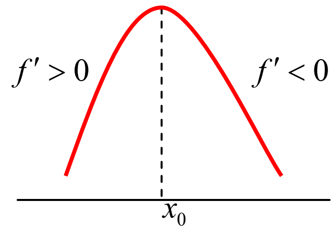
$$\begin{array}{ccccc} +\,+\,+\,+\,+\,+ & | & -\,-\,-\,-\,- & | & +\,+\,+\,+\,+\,+\,+\,+ \\ \longleftarrow & -2 & \text{———} & 0 & \longrightarrow \end{array}$$

Thus, at $x=-2$ there is relative maximum and at x = 0 there is relative minimum.

Concavity test:



(a) concave up



(b) concave down

From the figure (a) we can see that when f' changes when it passes through x_0 from negative to positive (f' is increasing) then f is concave up. On the other hand, figure (b) show that when f' changes when it passes through x_0 from positive to negative (f' is decreasing) then f is concave down.

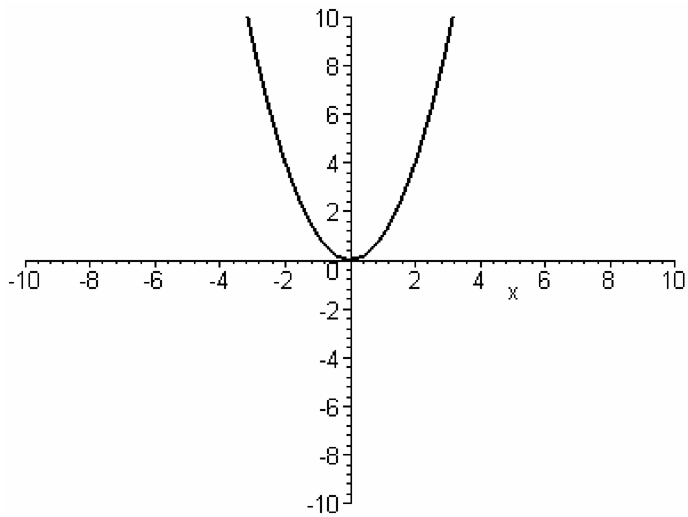
Definition (1) If f is differentiable on interval (a,b), then f is said to be concave up on (a,b) if f' is increasing and concave down if f' is decreasing.

Since f' is the derivative of f'', this implies that if f' is increasing then $f'' > 0$ and that if f' is decreasing then $f'' < 0$.

Consequently, based on the above definition and statement we have the following **concavity test**:

If $f'' > 0$ for all $x \in (a,b)$ then f is concave up, and if $f'' < 0$ for all $x \in (a,b)$ then f is concave down.

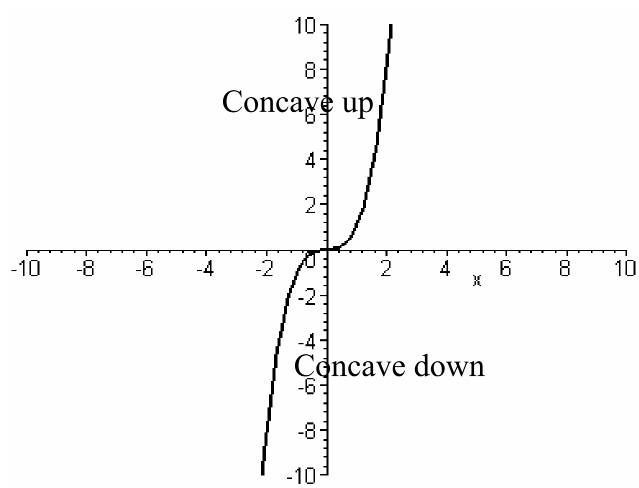
For example, the function $f(x) = x^2$, has $f' = 2x$ and $f'' = 2$. Thus $f'' > 0$ for all values of x and therefore f is always concave up. This can be shown easily from the graph of the parabola



Example (3) test the concavity of $f(x) = x^3$

Solution

$f' = 3x^2$ and $f'' = 6x$, thus, if $x > 0 \Rightarrow f'' > 0$ and hence, f is concave up. If $x < 0 \Rightarrow f'' < 0$ and hence, f is concave down. This is shown clearly from the graph of f(x).



Definition (2) A function has an inflection point when $x = x_0$ if and only if f is continuous at x_0 and f changes concavity at x_0.

Thus, the inflection point at $x = x_0$ must satisfy the following two conditions:

- $f''(x_0) = 0$ or $f''(x_0)$ is undefined,
- f is continuous at x_0.

The condition of continuity is necessary for the inflection point, see the following example:

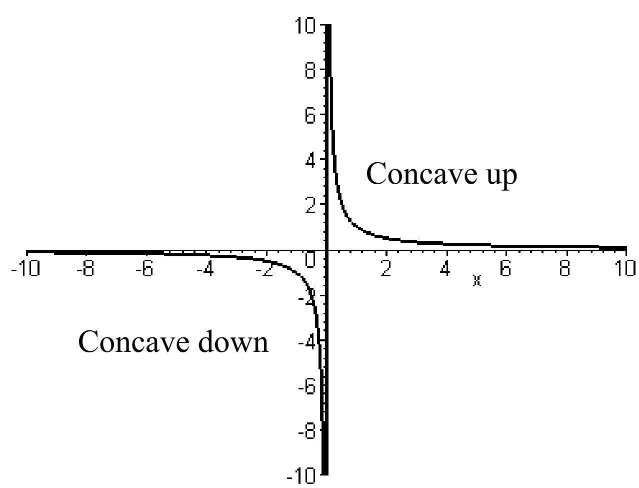
Example (4) Test the function $f(x)=\dfrac{1}{x}$ for inflection points.

Solution

$f'=-\dfrac{1}{x^2} \qquad \Rightarrow f''=-\dfrac{2}{x^3}$, and it is clear that f'' is not defined at $x_0=0$.

Since f'' changes its concavity when it passes through $x_0=0$, then the first condition is satisfied. But the second condition is not satisfied because f(x) is not continuous at $x_0=0$.

Therefore, the value $x=x_0$ is not corresponding to an inflection point of the function $f(x)=\dfrac{1}{x}$, see the following graph:



Example (5) Test $y=x^4-3x^3+7x-5$ for concavity and inflection points.

Solution

$y'=4x^3-9x^2+7 \quad \Rightarrow \quad y''=12x^2-18x=6x(2x-3)$.

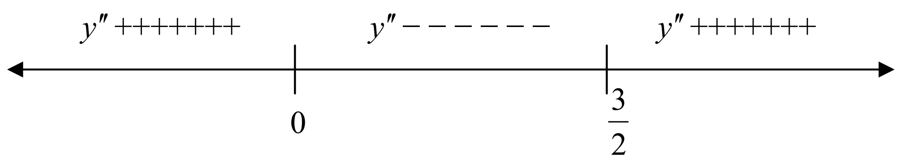
To find the inflection points put $y''=0$, then $6x(2x-3)=0$, this implies that $x=0$ *or* $x=\frac{3}{2}$.

Thus, the points $x=0$ *and* $x=\frac{3}{2}$ are candidate for inflection points. It is clear that f(x) is continuous at $x=0$ *and* $x=\frac{3}{2}$. Now, we test concavity by determining the sign of y'':

If $x<0$ then $y''=6(-)(-)=+$, so the curve is concave up.

If $0<x<\frac{3}{2}$ then $y''=6(+)(-)=-$, so the curve is concave down.

If $x>\frac{3}{2}$ then $y''=6(+)(+)=+$, so the curve is concave up.



Now, the inflection points are $(0,-5)$ and $(\frac{3}{2},0.4375)$.

The second derivative test

The second derivative test for relative extrema is given below:

> Suppose that $f'(x_0)=0$:
>
> If $f''(x_0)<0$, then f has a relative maximum at x_0.
>
> If $f''(x_0)>0$, then f has a relative minimum at x_0.

Remark (1): The second derivative test fails when $f''(x_0)=0$.

Example (6) Use the second derivative test to test the relative extrema of the following function:

$$y = 18x - \frac{2}{3}x^3 .$$

Solution

$f' = y' = 18 - 2x^2$. To obtain the critical numbers put $y' = 0$, then $18 - 2x^2 = 0 \Rightarrow 2(9 - x^2) = 0$.
So the critical values are $x = -3$ and $x = 3$. To apply y'' test we first find $y'' = -4x$. Then
$y''(-3) = -4(-3) = 12 > 0$, so the critical number $x = -3$ is relative minimum, and the point
$(-3, f(-3))$ is relative minimum point.
$y''(3) = -4(3) = -12 < 0$, so the critical number $x = 3$ is relative maximum, and the point
$(3, f(3))$ is relative maximum point.

Example (7) Use the second derivative test to test the relative extrema of the following function:

$$y = 6x^4 - 8x^3 + 1 .$$

Solution

$f' = y' = 24x^3 - 24x^2 = 24x^2(x-1)$. To obtain the critical numbers put $y' = 0$, then
$24x^2(x-1) = 0$. So the critical numbers are $x = 0$ and $x = 1$. To apply y'' test we first find
$y'' = 72x^2 - 48x$.
$y''(0) = 0$, then y'' test fails to test the critical number $x = 0$. We then use y' test for $x = 0$:
If $x < 0$, then $y' < 0$,
If $0 < x < 1$, then $y' < 0$,
The sign of y' doesn't change while it passes through $x = 0$, so $x = 0$ doesn't correspond to relative extrema.
$y''(1) = 72(1)^2 - 48(1) = +$, then the critical number $x = 1$ is relative minimum.

Remark (2): If the function is continuous and has exactly relative extremum on an interval, then it is <u>absolute</u> extremum on that interval.

In the last example, the function f(x) is continuous on the set of all real numbers R and has only relative minimum when $x=1$. Thus, this relative minimum is absolute minimum, i.e. the point $(1,-1)$ is absolute minimum of f(x).

Example (8) Sketch the graph of $y=2x^3-9x^2+12x$.

Solution

Intercept:

y intercept (put x = 0), $\Rightarrow$ $(0,0)$ is y intercept,

x intercept (put y = 0), $\Rightarrow$ $(0,0)$ is x intercept,

Symmetry:

The function f(x) is neither even nor odd function, therefore f(x) is not symmetric with respect to y axis and also not symmetric with respect with the origin.

Extrema:

$y'=6x^2-18x+12=6(x^2-3x+2)=6(x-1)(x-2)$. So the critical numbers are $x=1$ and $x=2$.

$y''=12x-18$. Then, $y''(1)=12(1)-18=-6<0$, so the critical number $x=1$ is relative maximum, and the point $(1,5)$ is relative maximum point.

$y''(2)=12(2)-18=6>0$, so the critical number $x=2$ is relative minimum, and the point $(2,4)$ is relative minimum point.

Concavity:

To find the inflection point put $y''=0$ $\Rightarrow 12x-18=0$. this implies that $x=\frac{3}{2}$. Thus, the point $x=\frac{3}{2}$ is candidate for inflection point. It is clear that f(x) is continuous at $x=\frac{3}{2}$. Now, we test concavity by determining the sign of y'':

If $x<\frac{3}{2}$ then $y''<0$, so the curve is concave down,

If $x>\frac{3}{2}$ then $y''>0$, so the curve is concave up.

By summarizing all of the above results in the following table:

x	0	1	1.5	2
y	0	5	4.5	4

Now, the graph of f(x) is given as follows: