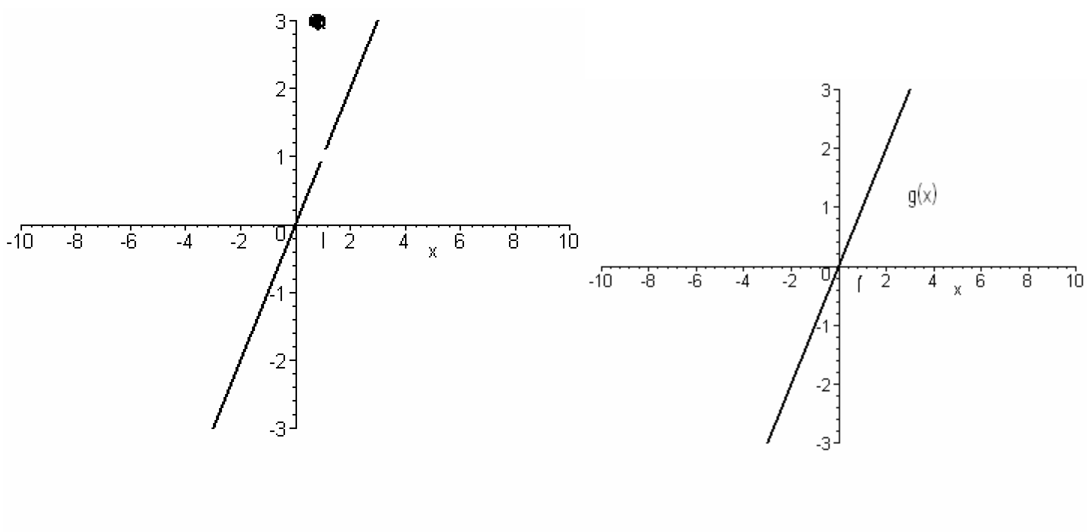
Lecture 3

Example (1) The function f(x) is defined as follows:

$$f(x) = \begin{cases} x & if \quad x \neq 1, \\ 3 & if \quad x = 1. \end{cases}$$

By sketching the graph of this function:

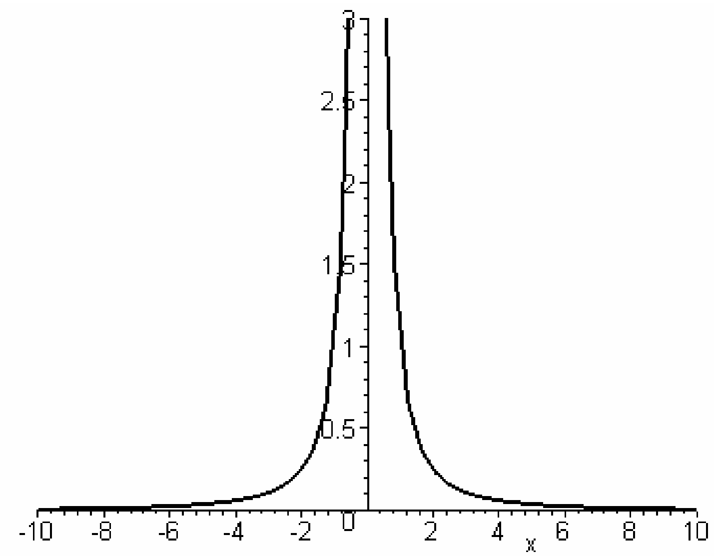


We observe that the graph of f(x) is not continuous at x = 1. However the graph of g(x) = x is continuous at each point in its domain.

The function is defined at x = 1 and equal 3,

$\lim_{x \to 1} f(x) = \lim_{x \to 1} x = 1$.

Example (2) The graph of the function $f(x) = \dfrac{1}{x^2}$ is not continuous at x = 0 (see below)



The function is not defined at x = 0,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2} = \infty..$$

Now, we have the following definition of a continuous function:

Definition:

A function f(x) is continuous at a if and only if the following conditions are follows:

1- f(x) is defined at x = a, i.e., a is in the domain of f(x),

2- $\lim_{x\to a} f(x)$ exists,

3- $\lim_{x\to a} f(x) = f(a)$.

If f(x) doesn't satisfy one of the above conditions, it is said to be <u>discontinuous</u> at a.

Example (3) use the definition of continuity to show that the given function $f(x) = x^3 - 5x$ is continuous at x = 2.

Solution

1- $f(2) = (2)^3 - 5(2) = 8 - 10 = -2$,

2- $\lim_{x\to 2} f(x) = \lim_{x\to 2}(x^3 - 5x) = (2)^3 - 5(2) = 8 - 10 = -2$,

3- $\lim_{x\to 2} f(x) = f(2)$.

Now, all conditions are satisfied, i.e., f(x) is continuous at x = 2.

In general : A polynomial function is continuous at every point.

Example (4) use the definition of continuity to show that the function $f(x)=\dfrac{x-2}{4x}$ is continuous at x = −2.

Solution

1- $f(-2)=\dfrac{(-2)-2}{4(-2)}=\dfrac{-4}{-8}=\dfrac{1}{2}$,

2- $\lim\limits_{x\to -2} f(x)=\lim\limits_{x\to -2}\dfrac{x-2}{4x}=\dfrac{-2-2}{4(-2)}=\dfrac{-4}{-8}=\dfrac{1}{2}$,

3- $\lim\limits_{x\to -2} f(x)=f(-2)$.

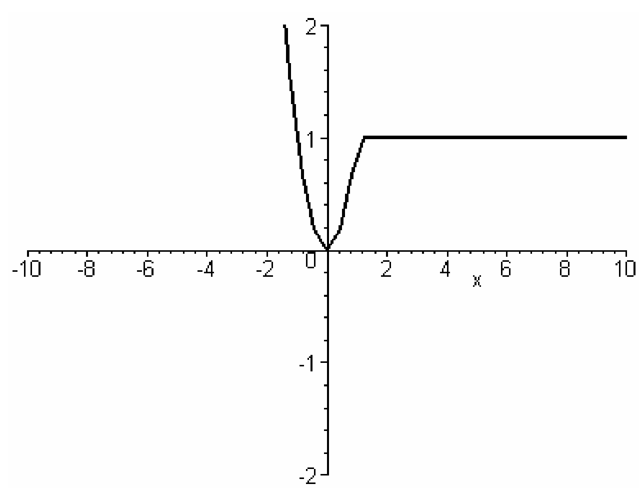
Now, all conditions are satisfied, i.e., f(x) is continuous at x = −2.

Example (5) study the continuity of the following function:

$$f(x)=\begin{cases}1 & if \quad x\ge 1,\\ x^2 & if \quad x<1.\end{cases}$$

At the point x = 1.

Solution



1- $f(1)=1$, i.e., the function is defined at x = 1.

2- $\lim\limits_{x\to 1} f(x)=\lim\limits_{x\to 1} x^2=(1)^2=1$.

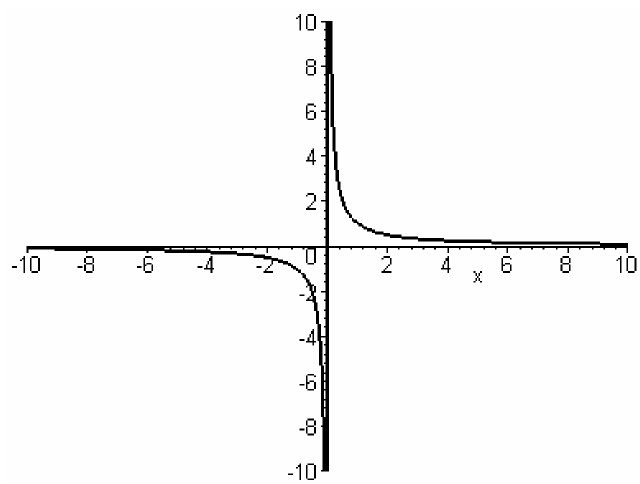
3- $\lim\limits_{x\to 1} f(x)=f(1)=1$.

Now, all conditions are satisfied, i.e., f(x) is continuous at x = 1.

Discontinuities

A rational function is discontinuous at zero of the denominator and continuous otherwise.

Example (6) the function $f(x)=\frac{1}{x}$ has a zero denominator when we put x = 0, so this function has a discontinuity at x = 0 (i.e., $f(x)=\frac{1}{x}$ is not continues at x = 0) and continuous at all other points. This fact can also be shown from the graph of f(x):



Example (7) Find all points of discontinuity of the following function:

$$f(x)=\frac{x^2+6x+9}{x^2+2x-15}.$$

Solution

First we calculate the zeros of the denominator by putting, $x^2+2x-15=0$,

This implies that (x – 3)(x + 5) = 0. Then zeros of the denominator are $3 \quad and \quad -5$. Thus, f(x) is discontinuous at the points $3 \quad and \quad -5$ and continuous otherwise.

Example (8) Find all points of discontinuity of the following function:

$$f(x)=\frac{x-4}{x^2+1}.$$

Solution

First we calculate the zeros of the denominator by putting, $x^2 + 1 = 0$, this implies that $x^2 = -1$

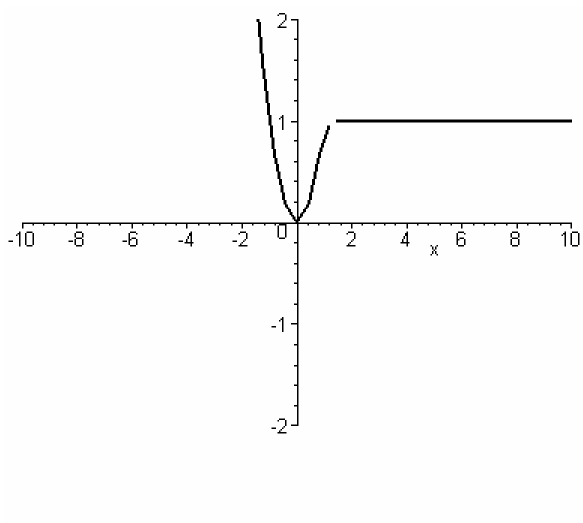
So, no zeros in the real numbers, therefore there is <u>no</u> points of discontinuity of this function.

Example (9) Find all points of discontinuity of the following function:

$$f(x) = \begin{cases} 1 & if \quad x \geq 1, \\ x^2 & if \quad x < 1. \end{cases}$$

Solution

Since f(x) is not defined at x = 1, then there is a discontinuity at x = 1. and f(x) is continuous otherwise. This can also be clear fro the following figure:



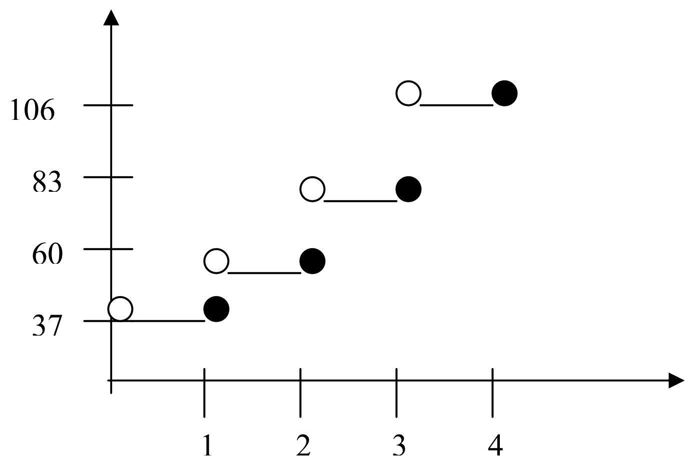
Example (10) "Post office function"

$$f(x) = \begin{cases} 37 & if \quad 0 < x \leq 1, \\ 60 & if \quad 1 < x \leq 2. \\ 83 & if \quad 2 < x \leq 3, \\ 106 & if \quad 3 < x \leq 4. \end{cases}$$

Find all points of discontinuity of the following function.

Solution:

From the following graph, it appears that the function has discontinuities at the points 1, 2 and 3.


106
83
60
37
1
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4