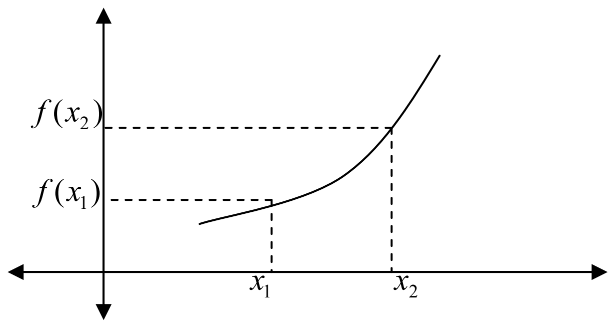
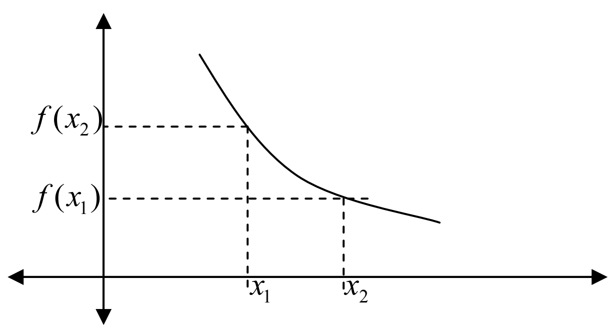
The function is said to be increasing if for all $x_2 > x_1$ then $f(x_2) > f(x_1)$

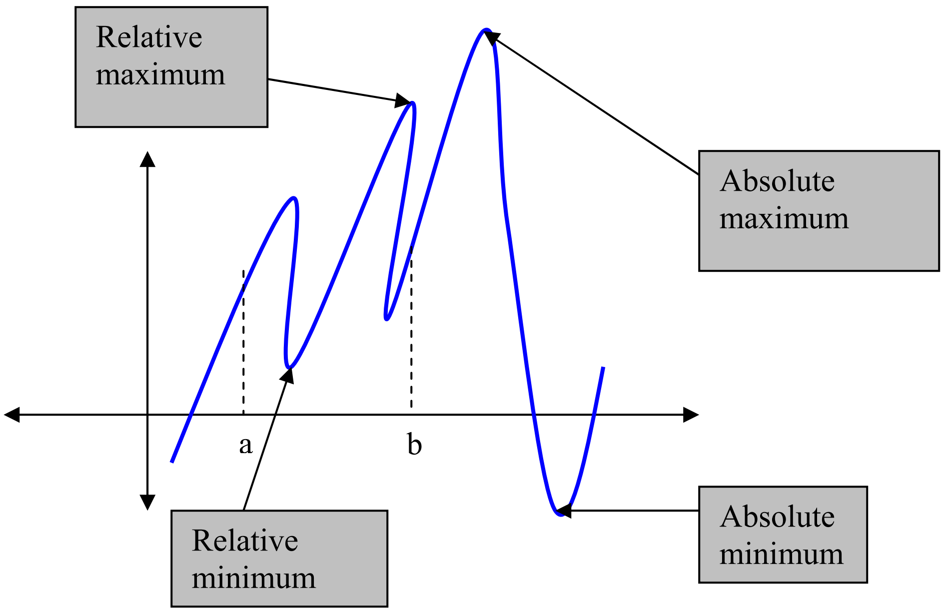


From the last figure if we assumed that the distance between x_1 and x_2 is very small then the curve of the function can be treated as a line segment and its slope $\dfrac{f(x_2)-f(x_1)}{x_2-x_1} > 0$ in the case of increasing function. But from our previous knowledge the slope at the point x_1 is the first derivative x_1. Thus, **If** $f'(x) > 0$ **for all** $x \in (x_1, x_2)$**, then** $f(x)$ **is increasing**.

The function is said to be decreasing if for all $x_2 > x_1$ then $f(x_2) < f(x_1)$



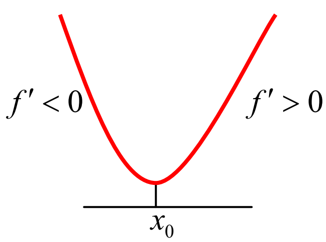
From the above figure if we assumed that the distance between x_1 and x_2 is very small then the curve of the function can be treated as a line segment and its slope $\dfrac{f(x_2)-f(x_1)}{x_2-x_1} < 0$ in the case of decreasing function. But from our previous knowledge the slope at the point x_1 is $f'(x_1)$.
Thus**, If** $f'(x) < 0$ **for all** $x \in (x_1, x_2)$**, then** $f(x)$ **is decreasing**.



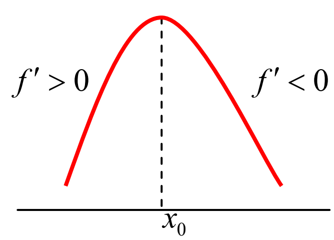
From the above figure the maximum point in the interval (a,b) is called **relative maximum**, and the minimum point in the interval (a,b) is called **relative minimum**. However the highest point in the total domain of the function is called **absolute maximum** and the lowest point in the total domain of the function is called **absolute minimum**.

If the point is either relative maximum or relative minimum, then it is called **relative extremum** and its plural is "**relative extrema**".

f' test:



x_0 is relative minimum



x_0 is relative maximum

From the above two figures, we can easily see that:

- If f' changes negative to positive as x increases through x_0, then f has a relative minimum at x_0.
- If f' changes positive to negative as x increases through x_0, then f has a relative maximum at x_0.

- If f' doesn't change its sign through x_0, then x_0 is not relative extremum.

Definition (1) The value x_0 in the domain of f(x) is called a ***critical value*** of f, if either $f'(x_0) = 0$ or $f'(x_0)$ is not defined. The point $(x_0, f(x_0))$ is then called a ***critical point***.

Rule (1) If f has a relative extremum at $x = x_0$ then x_0 is a critical value, in other words if f has a relative extremum at $x = x_0$ then $f'(x_0) = 0$ or $f'(x_0)$ is not defined.

Remark: Every relative extremum is a critical point but not every critical point is relative extremum.

The following example show a critical point which is not corresponds to relative extremum:

For the function $f(x) = x^3$, $f'(x) = 3x^2$ to find the critical values put $f'(x) = 0$, this implies that $3x^2 = 0 \Rightarrow$ the critical value is x = 0. The question now is"Is the critical value x = 0 corresponds to relative extremum?". To answer this question use f' test and calculate the sign of f' through x = 0. When x < 0 then $f'(x) = 3(-)^2 = +$ and when x > 0 then $f'(x) = 3(+)^2 = +$.

Thus f' doesn't change its sign through x = 0 and consequently x = 0 is not relative extremum.

A strategy for finding relative extrema

To find the relative extrema of the function f(x):

Step (1) Find f',

Step (2) Find the critical values (Points at which $f'(x) = 0$ and $f'(x)$ not defined),

Step (3) Use f' test to determine which of the candidate critical values are relative extrema and which are not.

Example (1) Find the relative extrema of $f(x) = x^4 - 2x^2$.

Solution

Step (1) $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1)$

Step (2) To find the critical values put $f'(x) = 0 \Rightarrow 4x(x-1)(x+1) = 0$,

Thus the critical values are $x = 0, \quad x = -1, \quad x = 1$.

Step (3) f' test for the points $x = 0, \quad x = -1, \quad x = 1$:

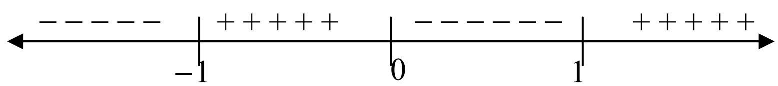
As $x < -1 \Rightarrow f' = 4(-)(-)(-) = -$ then f' is decreasing in $(-\infty, -1)$.

As $-1 < x < 0 \Rightarrow f' = 4(-)(-)(+) = +$ then f' is increasing in $(-1, 0)$.

As $0 < x < 1 \Rightarrow f' = 4(+)(-)(+) = -$ then f' is decreasing in $(0, 1)$.

As $x > 1 \Rightarrow f' = 4(+)(+)(+) = +$ then f' is increasing in $(1, \infty)$.

By summarizing the above results on the following intervals:



Thus, $x = -1$ is relative minimum value, the point x = 0 is relative maximum value and the point x = 1 is relative minimum value.

Example (2) Find the relative extrema of $f(x) = x^2 e^x$.

Solution

Step (1) $f'(x) = x^2 e^x + e^x (2x) = xe^x (x+2)$

Step (2) To find the critical values put $f'(x) = 0 \Rightarrow xe^x (x+2) = 0$,

Thus the critical values are $x = -2, \quad x = 0$.

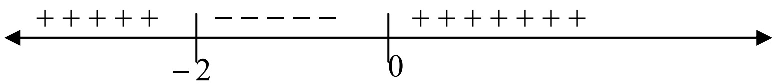
Step (3) f' test for the points $x = -2, \quad x = 0$:

As $x < -2 \Rightarrow f' = (-)(+)(-) = +$ then f' is increasing in $(-\infty, -2)$.

As $-2 < x < 0 \Rightarrow f' = (-)(+)(+) = -$ then f' is decreasing in $(-2, 0)$.

As $x > 0 \Rightarrow f' = (+)(+)(+) = +$ then f' is increasing in $(0, \infty)$.

By summarizing the above results on the following intervals:



Thus, $x = -2$ is relative maximum value and the point x = 0 is relative is relative minimum value.

See also example (4) in the book page 640.

Home work: Solve your book page 642 problems number: 18, 30, 46, 48, 60.