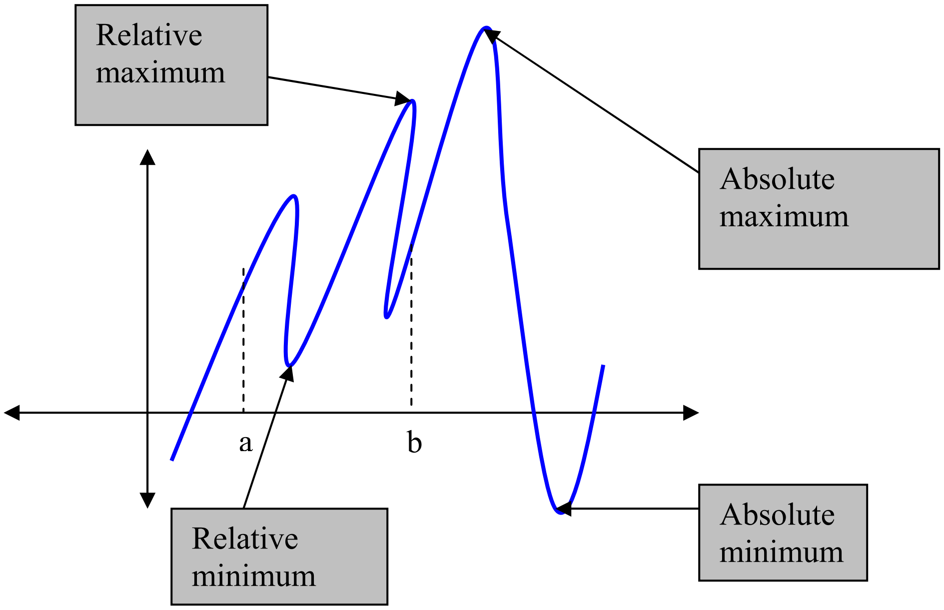
By observing the following graph:

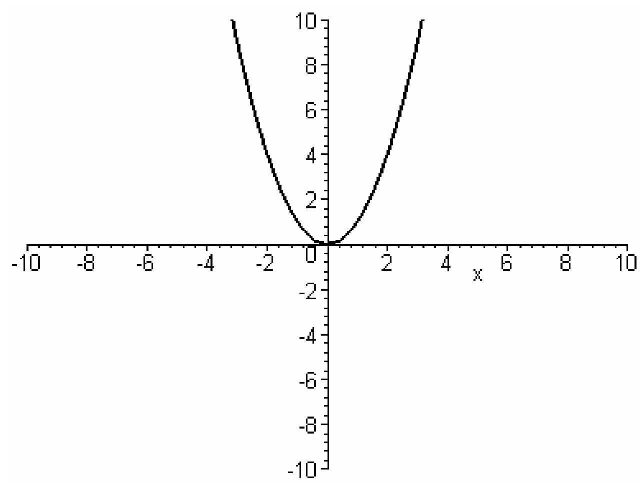


From the above figure the maximum point in the interval (a,b) is called **relative (local) maximum**, and the minimum point in the interval (a,b) is called **relative (local) minimum**. However the highest point in the total domain of the function is called **absolute maximum** and the lowest point in the total domain of the function is called **absolute minimum**.

If the point is either relative maximum or relative minimum, then it is called **relative extremum** and its plural is "**relative extrema**".

Now, we can say that, If $f(x_0) \geq f(x)$ for all x in (a,b), then $f(x_0)$ is called local maximum value. If $f(x_0) \leq f(x)$ for all x in (a,b), then $f(x_0)$ is called local minimum value. On the other hand, if $f(x_0) \geq f(x)$ for all x in the domain of f, then $f(x_0)$ is called absolute maximum value, and if $f(x_0) \leq f(x)$ for all x in the domain of f, then $f(x_0)$ is called absolute minimum value.

Example (1) The function $f(x)=x^2$ has absolute minimum point $(0,0)$, (see the graph):



Definition (1) The value x_0 in the domain of f(x) is called a ***critical number*** of f, if either $f'(x_0)=0$ or $f'(x_0)$ is not defined. The point $(x_0, f(x_0))$ is then called a ***critical point***.

Example (2) Find the critical numbers of the function $f(x)=x^{\frac{1}{3}}-x^{-\frac{2}{3}}$

Solution

$$f'(x)=\frac{d}{dx}(x^{\frac{1}{3}}-x^{-\frac{2}{3}})=\frac{1}{3}x^{-\frac{2}{3}}+\frac{2}{3}x^{-\frac{5}{3}}=\frac{1}{3x^{\frac{5}{3}}}\{x^{\frac{5}{3}}.x^{-\frac{2}{3}}+2x^{\frac{5}{3}}.x^{-\frac{5}{3}}\}=\frac{1}{3x^{\frac{5}{3}}}(x+2),$$

Now, put $f'(x)=0$, this implies that $x=-2$, and $f'(x)$ is not defined when $x=0$.

Thus, the critical numbers are -2 and 0.

Example (3) Find the critical numbers of the function $g(t)=\sqrt{t}(1-t)$

Solution

$$g(t)=t^{\frac{1}{2}}(1-t)=t^{\frac{1}{2}}-t^{\frac{3}{2}}$$

$$\Rightarrow \quad g'(t)=\frac{1}{2}t^{-\frac{1}{2}}-\frac{3}{2}t^{\frac{1}{2}}=\frac{1}{2}\frac{t^{\frac{1}{2}}}{t^{\frac{1}{2}}}(t^{-\frac{1}{2}}-3t^{\frac{1}{2}})=\frac{1}{2t^{\frac{1}{2}}}(1-3t),$$

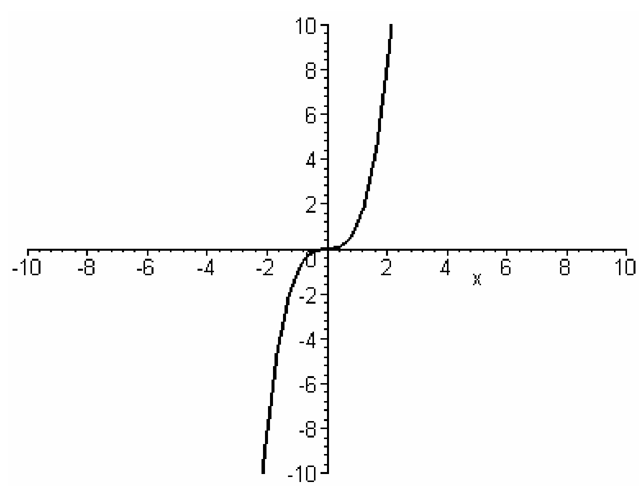
If $g'(t) = 0 \quad \Rightarrow \quad 1 - 3t = 0$, and this implies that $t = \frac{1}{3}$. However, $g'(t)$ is undefined at the following set of points $T = \{t^*, such\ that\ t^* = (-\infty, 0]\}$. Thus, the critical numbers are the set T and $\frac{1}{3}$.

Fermat's theorem If f has a relative extremum at $x = x_0$ then x_0 is a critical number of f.

In other words, if f has a relative extremum at $x = x_0$ then $f'(x_0) = 0$ or $f'(x_0)$ is not defined.

Remark: Every relative extremum is a critical point but **not** every critical point is relative extremum.

The following example show a critical point which is not corresponds to relative extremum: For the function $f(x) = x^3$, $f'(x) = 3x^2$ to find the critical numbers put $f'(x) = 0$, this implies that $3x^2 = 0 \Rightarrow$ the critical number is x = 0. From the graph of the function $f(x) = x^3$, it is clear the point $x = 0$ doesn't correspond to relative extremum.



Extreme-value theorem:

If a function is continuous on a closed interval, then the function has both absolute maximum value and minimum value on that interval.

Steps for finding the absolute extrema on a closed interval $[a,b]$:

- Step (1) Find the critical numbers
- Step (2) Evaluate f at a and b and also at the critical numbers.
- Step (3) The maximum value of f is the greatest of the values found in the second step, and the minimum value of f is the minimum value in step (2).

Example (4) Find the absolute extrema for $f(x) = x^2 - 2x + 3$ on the closed interval $[0,3]$.

Solution:

$f' = 2x - 2, \quad f' = 0 \Rightarrow \quad 2x - 2 = 0, \quad \Rightarrow x = 1.$

Thus, the critical number is $x = 1$. Also,

$f(0) = (0)^2 - 2(0) + 3 = 3,$

$f(3) = (3)^2 - 2(3) + 3 = 9 - 6 + 3 = 6,$

$f(the\,critical\ number) = f(1) = (1)^2 - 2(1) + 3 = 2.$

Now, using the extreme-value theorem, we find that the maximum value is $f(3) = 6$ and the minimum value is $f(1) = 2$.