We know that $f'(x)$ is the derivative of the function f(x), the derivative of the function $f'(x)$ is Called the second derivative of f(x) and symbolized by $f''(x)$ or $\frac{d^2 f(x)}{dx^2}$ *or* $\frac{d^2 y}{dx^2}$. Similarly $f'''(x)$ is the derivative of the function $f''(x)$ and symbolized by $f'''(x)$ or $\frac{d^3 f(x)}{dx^3}$ *or* $\frac{d^3 y}{dx^3}$ and is called the third derivative of f(x). Also the fourth derivative of f(x) is $f''''(x) = \frac{d^4 f(x)}{dx^4} = \frac{d^4 y}{dx^4}$ and so on … . The derivatives $f''(x)$, $f'''(x)$, $f''''(x)$, … are called the higher order derivatives of f(x).

Example (1) Find all higher order derivatives of the function $f(x) = 4x^3 - 12x^2 + 6x + 2$.

Solution

$$f'(x) = \frac{d}{dx} f(x) = 4(3)x^3 - 12(2x) + 6 = 12x^2 - 24x + 6.$$

$$f''(x) = \frac{d}{dx} f'(x) = 12(2)x - 24(1) = 24x - 24.$$

$$f'''(x) = \frac{d}{dx} f''(x) = 24(1) = 24.$$

$$f''''(x) = \frac{d}{dx} f'''(x) = 0.$$

Hence, $f'''''(x) = 0$ and any higher order derivative more than 4 is vanished.

Example (2) If $y = (2x-1)^5$, find y''.

Solution:

$$y' = \frac{d}{dx}(2x-1)^5 = 5(2x-1)^4 . \frac{d}{dx}(2x-1) = 5(2x-1)^4.(2) = 10(2x-1)^4 ,$$

$$\therefore \quad y'' = \frac{d}{dx} 10(2x-1)^4 = 4(10)(2x-1)^3 . \frac{d}{dx}(2x-1) = 40(2x-1)^3.(2) = 80(2x-1)^3.$$

Example (3) If $x^2 + 8y = y^2$, find $\frac{d^2 y}{dx^2}$ when $x = 3, \;\; y = -1$.

Solution: By diff. both sides of the equation w. r. to x,

$$\Rightarrow\ 2x+8\frac{dy}{dx}=2y.\frac{dy}{dx},$$

$\Rightarrow 8\frac{dy}{dx}-2y.\frac{dy}{dx}=-2x \ \Rightarrow 4\frac{dy}{dx}-y.\frac{dy}{dx}=-x$. Then solving the equation for $\frac{dy}{dx}$ we obtain:

$$\frac{dy}{dx}(4-y)=-x \quad \Rightarrow \quad \frac{dy}{dx}=-\frac{x}{(4-y)}=\frac{x}{(y-4)}.$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{x}{(y-4)}=\frac{(y-4)(1)-x.\frac{d}{dx}(y-4)}{(y-4)^2}=\frac{(y-4)-x.\frac{dy}{dx}}{(y-4)^2}=\frac{(y-4)-x.\frac{x}{(y-4)}}{(y-4)^2}=\frac{(y-4)-\frac{x^2}{(y-4)}}{(y-4)^2}$$

$$\Rightarrow\ \frac{d^2y}{dx^2}=\frac{\frac{(y-4)^2-x^2}{(y-4)}}{(y-4)^2}=\frac{(y-4)^2-x^2}{(y-4)^3}.$$

$$\therefore\ \left.\frac{d^2y}{dx^2}\right|_{\substack{x=3\\y=-1}}=\frac{(-1-4)^2-(3)^2}{(-1-4)^3}=\frac{25-9}{-125}=-\frac{16}{125}.$$

Example (4) If $f(x)=(3x-5)e^{-2x}$, show that $f''(x)+4f'(x)+4f(x)=0$.

Solution

$$f'(x)=\frac{d}{dx}(3x-5)e^{-2x}=e^{-2x}.\frac{d}{dx}(3x-5)+(3x-5).\frac{d}{dx}e^{-2x}$$
$$=e^{-2x}.(3)+(3x-5).\frac{d}{dx}e^{-2x}=3e^{-2x}+(3x-5).e^{-2x}.\frac{d}{dx}(-2x)$$
$$=3e^{-2x}+(3x-5).e^{-2x}.(-2)=e^{-2x}[3-6x+10]$$
$$=e^{-2x}[13-6x].$$

$$f''(x)=\frac{d}{dx}f'(x)=\frac{d}{dx}e^{-2x}[13-6x]=[13-6x]\frac{d}{dx}e^{-2x}+e^{-2x}\frac{d}{dx}[13-6x]$$
$$=(13-6x).e^{-2x}.(-2)+e^{-2x}.(-6)=e^{-2x}.(-26+12x-6)=e^{-2x}.(-32+12x)$$
$$=4e^{-2x}.(-8+3x).$$

Now,

$$f''(x)+4f'(x)+4f(x)=4e^{-2x}.(-8+3x)+4e^{-2x}(13-6x)+4e^{-2x}(3x-5)$$
$$=4e^{-2x}(-8+3x+13-6x+3x-5)$$
$$=4e^{-2x}(-13+13+6x-6x)$$
$$=4e^{-2x}(0)$$
$$=0.$$

Finding rate of change of higher derivatives:

Since $f''(x) = \frac{df'(x)}{dx}$, then $f''(x)$ represents the rate of change of the function $f'(x)$ with respect to x. Similarly, then $f'''(x)$ represents the rate of change of the function $f''(x)$ with respect to x (and so on).

Example (5) If $p = 500 - 50q - q^2$ is a demand equation. How fast is the marginal revenue changing when q = 5?

Solution

Since $r = pq$ then

$$\begin{aligned} r &= (500 - 50q - q^2)q \\ &= (500q - 50q^2 - q^3) \end{aligned}$$

The marginal revenue is $\frac{dr}{dq} = r' = 500 - 100q - 3q^2$.

The rate of change of the marginal revenue with respect to q is $\frac{d}{dq} r'$

$$\begin{aligned} \Rightarrow \quad \frac{d}{dq} r' &= r'' \\ &= \frac{d}{dq}(500 - 100q - 3q^2) \\ &= -100 - 6q. \end{aligned}$$

When q = 5 then $\left.\frac{dr'}{dq}\right|_{q=5} = \left.\frac{d^2 r}{dq^2}\right|_{q=5} = -100 - 6(5) = -130.$

This means that when q is changed by 1 unit from 5 to 6 the change in marginal revenue will be decreased by 130.

Home Work: Solve your book pages 626 and 627, problems number: 2, 14, 30, 34, 37.