Sec. 12.4 Implicit differentiation

Example (1) If $x^2 + y^2 = 9$, find $\frac{dy}{dx}$.

Solution

By differentiating both sides of this equation with respect to x. Then

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}9$$

$$\Rightarrow \qquad \frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 0$$

$$\Rightarrow \qquad 2x + 2y\frac{dy}{dx} = 0$$

Now solving the last equation for $\frac{dy}{dx}$, then

$$2y\frac{dy}{dx} = -2x \quad \Rightarrow \frac{dy}{dx} = -\frac{x}{y}.$$

Example (2) If $x^2 + y^2 = 2xy + 3$, find $\frac{dy}{dx}$.

Solution

By differentiating both sides of this equation with respect to x. Then

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(2xy + 3)$$

$$\Rightarrow \qquad \frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}2xy + \frac{d}{dx}3$$

$$\Rightarrow \qquad 2x + 2y\frac{dy}{dx} = 2y\frac{d}{dx}x + 2x\frac{d}{dx}y + 0$$

$$\Rightarrow \qquad 2x + 2y\frac{dy}{dx} = 2y(1) + 2x\frac{dy}{dx}$$

Now solving the last equation for $\frac{dy}{dx}$, then

$$+2y\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - 2x$$

$$2\frac{dy}{dx}(y-x)=2(y-x),$$

Therefore $\frac{dy}{dx}=\frac{2(y-x)}{2(y-x)}=1$, for $y\neq x$.

Example (3) If $y\ln x=xe^{y}$, find $\frac{dy}{dx}$.

Solution

By differentiating both sides of this equation with respect to x. Then

$$\frac{d}{dx}(y\ln x)=\frac{d}{dx}(xe^{y})$$

$$\Rightarrow \quad \ln x.\frac{d}{dx}(y)+y\frac{d}{dx}(\ln x)=e^{y}.\frac{d}{dx}(x)+x.\frac{d}{dx}(e^{y})$$

$$\Rightarrow \quad \ln x.\frac{dy}{dx}+y(\frac{1}{x})=e^{y}.(1)+x.(e^{y})\frac{dy}{dx}$$

$$\Rightarrow \quad \ln x.\frac{dy}{dx}-x.(e^{y})\frac{dy}{dx}=e^{y}-\frac{y}{x}$$

Now solving the last equation for $\frac{dy}{dx}$, then

$$\Rightarrow \quad \frac{dy}{dx}(\ln x-xe^{y})=e^{y}-\frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{e^{y}-\frac{y}{x}}{(\ln x-xe^{y})}=\frac{x[e^{y}-\frac{y}{x}]}{x(\ln x-xe^{y})}=\frac{xe^{y}-y}{(x\ln x-x^{2}e^{y})}.$$

Example (4) If $xy-y-11x=5$, find $\frac{dy}{dx}$ at $(0,-5)$.

Solution

By diff. both sides of the equation w. r. to x

$$\frac{d}{dx}[xy-y-11x]=\frac{d}{dx}5$$

$$\Rightarrow \frac{d}{dx}xy-\frac{d}{dx}y-\frac{d}{dx}11x=\frac{d}{dx}5$$

$\Rightarrow\; y\dfrac{d}{dx}x + x\dfrac{dy}{dx} - \dfrac{dy}{dx} - 11(1) = 0,$ and by solving the equation for $\dfrac{dy}{dx}$ we obtain:

$\Rightarrow\; y(1) + \dfrac{dy}{dx}(x-1) - 11 = 0$, then $\dfrac{dy}{dx}(x-1) = 11 - y$, this implies that $\dfrac{dy}{dx} = \dfrac{11-y}{(x-1)}$

Now, $\dfrac{dy}{dx}$ at $(0,-5)$ is $\left.\dfrac{dy}{dx}\right|_{\substack{x=0\\y=-5}} = \dfrac{11-(-5)}{0-1} = \dfrac{11+5}{-1} = -16$.

Example (5) Find the slope of the curve $x^3 = (y-x^2)^2$ at $(1, 2)$

Solution

By diff. both sides of the equation w. r. to x

$$\frac{d}{dx}x^3 = \frac{d}{dx}(y-x^2)^2$$

$$\Rightarrow\quad 3x^2 = 2(y-x^2)\frac{d}{dx}(y-x^2)$$

$$\Rightarrow\quad 3x^2 = 2(y-x^2).\left(\frac{d}{dx}y - \frac{d}{dx}x^2\right)$$

$$\Rightarrow\quad 3x^2 = 2(y-x^2).\left(\frac{dy}{dx} - 2x\right)$$

$\Rightarrow\quad 3x^2 = 2(y-x^2).\dfrac{dy}{dx} - 4x(y-x^2)\;\; \Rightarrow 3x^2 = 2(y-x^2).\dfrac{dy}{dx} - 4xy + 4x^3$, by solving for $\dfrac{dy}{dx}$:

$\Rightarrow 3x^2 + 4xy - 4x^3 = 2(y-x^2).\dfrac{dy}{dx}$, then

$$.\frac{dy}{dx} = \frac{3x^2+4xy-4x^3}{2(y-x^2)}.$$

The slope at the point is at (1, 2) is $\left.\dfrac{dy}{dx}\right|_{\substack{x=1\\y=2}} = \dfrac{3(1)^2+4(1)(2)-4(1)^3}{2(2-1)} = \dfrac{3+8-4}{2(1)} = \dfrac{7}{2}$.

Example (6) Find the rate of change of q with respect to p, if $p = \dfrac{20}{(q+5)^2}$

Solution

The rate of change of q with respect to p is $\dfrac{dq}{dp}$

Since $p = \dfrac{20}{(q+5)^2}$ then $(q+5)^2 = \dfrac{20}{p} \Rightarrow\; (q+5)^2 = 20p^{-1}$

Now, by diff. both sides of the last equation w. r. to p, we obtain:

$$\frac{d}{dp}(q+5)^2 = \frac{d}{dp}20p^{-1}$$

$$\Rightarrow\ 2(q+5)\frac{d}{dp}(q+5) = (-1)20p^{-2}$$

$$\Rightarrow\ 2(q+5)\frac{dq}{dp} = \frac{-20}{p^2}$$ which implies that $\frac{dq}{dp} = \frac{-10}{p^2(q+5)}$,

Now, since $p = \frac{20}{(q+5)^2}$ then $p^2 = [\frac{20}{(q+5)^2}]^2 = \frac{400}{(q+5)^4}$. Substituting for p^2 in $\frac{dq}{dp}$, we obtain

$$\frac{dq}{dp} = \frac{-10}{\frac{400}{(q+5)^4}(q+5)} = \frac{-1}{\frac{40}{(q+5)^3}} = -\frac{(q+5)^3}{40}.$$

Home work: Solve the book page 613, the following problems:

[10] Find $\frac{dy}{dx}$ if $x^2 + xy - 2y^2 = 0$.

[18] Find $\frac{dy}{dx}$ if $y^2 + y = \ln x$.

[24] Find $\frac{dy}{dx}$ if $y^2 = \ln(x+y)$.

[26] If $x\sqrt{y+1} = y\sqrt{x+1}$, then find $\frac{dy}{dx}$ at (3, 3).

[34] Find the rate of change of q with respect to p, if $p = \frac{10}{q^2+3}$.