Rule (1) If f and g are both differentiable, then

$$\frac{d}{dx}[f(x).g(x)] = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x).$$

In other words,

"*The derivative of the product =(second) (derivative of the first)+(first)(derivative of the first)*"

Proof

Let $T(x) = f(x).g(x)$, Then

$$T'(x) = \lim_{h\to 0}\frac{T(x+h)-T(x)}{h} = \lim_{h\to 0}\frac{f(x+h).g(x+h)-f(x).g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h).g(x+h)-f(x).g(x+h)+f(x).g(x+h)-f(x).g(x)}{h}$$

$$= \lim_{h\to 0}\frac{g(x+h).[f(x+h)-f(x)]+f(x).[g(x+h)-g(x)]}{h}$$

$$= \lim_{h\to 0}\frac{g(x+h).[f(x+h)-f(x)]}{h} + \lim_{h\to 0}\frac{f(x).[g(x+h)-g(x)]}{h}$$

$$= \lim_{h\to 0} g(x+h).\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} f(x).\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0} g(x+h).(\frac{df(x)}{dx}) + f(x).(\frac{dg(x)}{dx})$$

Since f and g are differentiable, this implies that both f and g are continuous functions and then:

$$\lim_{h\to 0} g(x+h) = g(x),$$

This implies that $$\frac{d}{dx}T(x) = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x),$$

Thus, the proof is completed.

Example (1) If $f(x) = xe^x$, find $f'(x)$

Solution

$$f'(x) = \frac{d}{dx}xe^x = e^x.\frac{d}{dx}x + x.\frac{d}{dx}e^x$$

$$= e^x.(1) + xe^x = e^x(1+x).$$

Rule (2) (Quotient rule) if f, g are differentiable functions and $g(x) \neq 0$, then

$$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x).\frac{df(x)}{dx} - f(x).\frac{dg(x)}{dx}}{[g(x)]^2}.$$

In other words, the derivative of quotient =

$$\frac{(denominator).(derivative\ of\ numerator) - (denominator).(derivative\ of\ numerator)}{(denominator)^2}.$$

Proof

Let $T(x) = \frac{f(x)}{g(x)}$, then $f(x) = T(x).g(x)$,

Using the product rule, then $f'(x) = g(x).T'(x) + T(x).g'(x)$,

$$\Rightarrow \quad g(x).T'(x) = f'(x) - T(x).g'(x)$$
$$= f'(x) - \frac{f(x)}{g(x)}.g'(x)$$
$$= \frac{g(x).f'(x) - f(x).g'(x)}{g(x)}$$

$$\therefore \quad T'(x) = \frac{g(x).f'(x) - f(x).g'(x)}{[g(x)]^2}$$

Now the proof is completed.

Example (2) Find an equation of the tangent line to the curve $y = \frac{e^x}{1+x^2}$ at the point $(1, \frac{e}{2})$.

Solution

$$\frac{dy}{dx} = \frac{(1+x^2).\frac{d}{dx}e^x - e^x.\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$
$$= \frac{(1+x^2).e^x - e^x.(2x)}{(1+x^2)^2} = \frac{e^x.[1+x^2-2x]}{(1+x^2)^2} = \frac{e^x.(x-1)^2}{(1+x^2)^2}.$$

The slope at the point $(1, \frac{e}{2})$ is $m = \left.\frac{dy}{dx}\right|_{x=1} = \frac{e^1.(1-1)^2}{(1+1^2)^2} = 0.$

Therefore, $m = \dfrac{y - \frac{e}{2}}{x-1} = 0 \;\Rightarrow\; y - \frac{e}{2} = 0$, then the equation of the tangent line is;

$$y = \frac{e}{2}.$$

Example (3) Suppose that $f(5)=1,\;\; f'(5)=6,\;\; g(5)=-3 \quad and \quad g'(5)=2$. Find the following:

(a) $(fg)'(5)$, (b) $(\frac{f}{g})'(5)$.

Solution

(a) Since $(f.g)'(x) = [f(x).g(x)]' = g(x).f'(x) + f(x).g'(x)$, then

$$(f.g)'(5) = g(5).f'(5) + f(5).g'(5)$$

$$= (-3)(6) + (1)(2) = -18 + 2 = -16.$$

(b) Since $\therefore \quad (\frac{f}{g})'(x) = [\frac{f(x)}{g(x)}]' = \dfrac{g(x).f'(x) - f(x).g'(x)}{[g(x)]^2}$, then

$$(\frac{f}{g})'(5) = [\frac{f(x)}{g(x)}]' = \frac{g(5).f'(5) - f(5).g'(5)}{[g(5)]^2}$$

$$= \frac{(-3)(6) - (1)(2)}{(-3)^2} = \frac{-18-2}{9}$$

$$= -\frac{20}{9}.$$