Limits laws:

- $\underset{x\to a}{Lim}\ C = C,$ where C is constant.
- $\underset{x\to a}{Lim}\ x^n = a^n,$ where n is positive integer.
- If f(x) is a polynomial function, then $\underset{x\to a}{Lim}\ f(x) = f(a),$
- $\underset{x\to a}{Lim}\ (1+x)^{\frac{1}{x}} = e,$ where e is the base of the natural logarithm.
- If $\underset{x\to a}{Lim}\ f(x)$ exist and $\underset{x\to a}{Lim}\ g(x)$ exist, then
  - $\underset{x\to a}{Lim}\ (f(x) \pm g(x)) = \underset{x\to a}{Lim}\ f(x) \pm \underset{x\to a}{Lim}\ g(x),$
  - $\underset{x\to a}{Lim}\ f(x).g(x) = \underset{x\to a}{Lim}\ f(x).\underset{x\to a}{Lim}\ g(x),$
  - $\underset{x\to a}{Lim}\ C.f(x) = C.\underset{x\to a}{Lim}\ f(x),$
  - $\underset{x\to a}{Lim}\ \dfrac{f(x)}{g(x)} = \dfrac{\underset{x\to a}{Lim}\ f(x)}{\underset{x\to a}{Lim}\ g(x)},$ provided that g(x) $\neq 0$,
  - $\underset{x\to a}{Lim}\ C.f(x) = C.\underset{x\to a}{Lim}\ f(x),$
  - $\underset{x\to a}{Lim}\ \sqrt[n]{f(x)} = \sqrt[n]{\underset{x\to a}{Lim}\ f(x)}.$

Example (1) find $\underset{r\to 9}{Lim}\ \dfrac{4r-3}{11}$

Solution: $\underset{r\to 9}{Lim}\ \dfrac{4r-3}{11} = \dfrac{4(9)-3}{11} = \dfrac{36-3}{11} = \dfrac{33}{11} = 3.$

Example (2) find $\underset{x\to -6}{Lim}\ \dfrac{x^2+6}{x-6}$

Solution $\underset{x\to -6}{Lim}\ \dfrac{x^2+6}{x-6} = \dfrac{36+6}{-12} = -\dfrac{7}{2}.$

Remark: Notice that the quantities $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\times\infty$ is <u>indeterminate quantities</u>, and we should avoid them in our calculation by using any mathematical trick for the problem like factorizing or multiplying by the conjugate or expanding the brackets or using the theorem that will be given.

Example (3) $\lim_{x\to 2}\dfrac{x^2-x-2}{x-2}$

Solution $\lim_{x\to 2}\dfrac{x^2-x-2}{x-2}=\lim_{x\to 2}\dfrac{(x-2)(x+1)}{(x-2)}=\lim_{x\to 2}(x+1)=2+1=3.$

Example (4) find $\lim_{x\to 0}\dfrac{(x+2)^2-4}{x}$

Solution $\lim_{x\to 0}\dfrac{(x+2)^2-4}{x}=\lim_{x\to 0}\dfrac{x^2+4x+4-4}{x}=\lim_{x\to 0}\dfrac{x(x+4)}{x}=\lim_{x\to 0}(x+4)=0+4=4.$

Example (5) find $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$, where $f(x)=x^2-3.$

Solution $\because f(x)=x^2-3$

$\therefore f(x+h)=(x+h)^2-3=x^2+2xh+h^2-3.$

Now the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ will be simplified as:

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{(x^2+2xh+h^2-3)-(x^2-3)}{h}\\&=\frac{x^2+2xh+h^2-3-x^2+3}{h}\\&=\frac{2xh+h^2}{h}=\frac{h(2x+h)}{h}=2x+h.\end{aligned}$$

Thus, $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim_{h\to 0}(2x+h)=(2x+0)=2x.$

Example (6) find $\lim_{h\to 0}\dfrac{(4+h)^2-16}{h}$,

Solution

$$\lim_{h\to 0}\frac{(4+h)^2-16}{h}=\lim_{h\to 0}\frac{16+8h+h^2-16}{h}=\lim_{h\to 0}(8+h)=8+0=8.$$

Example (7) find $\lim_{t\to 9}\dfrac{9-t}{3-\sqrt{t}}$,

Solution

$$\lim_{t\to 9}\frac{9-t}{3-\sqrt{t}}=\lim_{t\to 9}\frac{9-t}{3-\sqrt{t}}.\frac{(3+\sqrt{t})}{(3+\sqrt{t})}=\lim_{t\to 9}\frac{(9-t)(3+\sqrt{t})}{(9-t)}=\lim_{t\to 9}(3+\sqrt{t})=3+\sqrt{9}=3+3=6.$$

Example (8) find $\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}$,

Solution

$$\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}=\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}.\frac{(\sqrt{x+2}+3)}{(\sqrt{x+2}+3)}=\lim_{x\to 7}\frac{(x+2)-9}{(x-7)(\sqrt{x+2}+3)}=\lim_{x\to 7}\frac{(x-7)}{(x-7)(\sqrt{x+2}+3)}$$

$$=\lim_{x\to 7}\frac{1}{(\sqrt{x+2}+3)}=\frac{1}{(\sqrt{7+2}+3)}=\frac{1}{(\sqrt{9}+3)}=\frac{1}{3+3}=\frac{1}{6}.$$

Theorem 1 the well known limit $\lim_{x\to a}\frac{x^n-a^n}{x-a}=(n)(a)^{n-1}$.

Example (9) find $\lim_{x\to 2}\frac{x^5-32}{x-2}$

Solution: we can compare our problem with the general form of the above theorem. And then have a = 2 and n = 5. Now,

$$\lim_{x\to 2}\frac{x^5-32}{x-2}=\frac{x^5-2^5}{x-2}=5(2)^4=5(16)=80.$$

Example (10) find $\lim_{x\to 2}\frac{x^4-16}{x-2}$

Solution $\lim_{x\to 2}\frac{x^4-16}{x-2}=\lim_{x\to 2}\frac{x^4-2^4}{x-2}$ using the theorem, then

$$\lim_{x\to 2}\frac{x^4-16}{x-2}=\lim_{x\to 2}\frac{x^4-2^4}{x-2}=4(2)^3=(4)(8)=32.$$

Example (11) find $\lim_{x\to 9}\frac{x^2-81}{\sqrt{x}-3}$,

Solution we can solve this limit directly by multiplying conjugate $(\sqrt{x}+3)$ in both the numerator and denominator of the quantity $\frac{x^2-81}{\sqrt{x}-3}$ and then simplifying. Another way of solving this problem is by using the theorem; where you may notice that:

$$\underset{x\to 9}{Lim}\frac{x^2-81}{\sqrt{x}-3}=\underset{\sqrt{x}\to\sqrt{9}}{Lim}\frac{(\sqrt{x})^4-3^4}{\sqrt{x}-3}=\underset{\sqrt{x}\to 3}{Lim}\frac{(\sqrt{x})^4-3^4}{\sqrt{x}-3}$$

Let $y=\sqrt{x}$, then $\underset{x\to 9}{Lim}\frac{x^2-81}{\sqrt{x}-3}=\underset{\sqrt{x}\to 3}{Lim}\frac{(\sqrt{x})^4-3^4}{\sqrt{x}-3}=\underset{y\to 3}{Lim}\frac{(y)^4-3^4}{y-3}$

Now, it is clear that we can apply the theorem where n = 4 and a = 3, therefore:

$$\underset{x\to 9}{Lim}\frac{x^2-81}{\sqrt{x}-3}=\underset{y\to 3}{Lim}\frac{(y)^4-3^4}{y-3}=4(3)^3=4(27)=108.$$

Theorem 2: If $f(x)\le g(x)$ when x is near a point a (except possibly at a) and the limits of f(x) and g(x) both exist as x approaches a, then

$$\lim_{x\to a}f(x)\le\lim_{x\to a}g(x).$$

Theorem 3 (**The squeeze theorem**) If $f(x)\le g(x)\le h(x)$ when x is near a point a (except possibly at a), and

$$\lim_{x\to a}f(x)=\lim_{x\to a}h(x)=L.$$

Then,

$$\lim_{x\to a}g(x)=L.$$

The previous theorem (squeeze theorem) is useful in calculating the limit of the following example.

Example (12) prove that $\underset{x\to 0}{Lim}\ x^2\sin\frac{1}{x}=0.$,

Solution as we have previously discussed $\underset{x\to 0}{Lim}\ \sin\frac{1}{x}$ doesn't exist, therefore we can't use limit laws to calculate $\underset{x\to 0}{Lim}\ x^2\sin\frac{1}{x}$. Now, we know that

$$-1\le\sin\frac{1}{x}\le 1$$

Multiplying each part of this inequality by x^2, then we obtain

$$-x^2\le x^2\sin\frac{1}{x}\le x^2$$

Taking the limit to each part of this inequality as x approaches to zero we obtain:

$$\lim_{x\to 0}(-x^2) \le \lim_{x\to 0} x^2 \sin\frac{1}{x} \le \lim_{x\to 0} x^2$$

It is clear that $\lim_{x\to 0}(-x^2) = \lim_{x\to 0}(x^2) = 0$.

Assume that $f(x) = -x^2$, $g(x) = x^2 \sin\frac{1}{x}$ and $h(x) = x^2$. Then we have:

$$f(x) \le g(x) \le h(x)$$

And $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = 0$. Using the squeeze theorem, we obtain $\lim_{x\to a} g(x) = 0$.

i.e., $\mathop{Lim}_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

Home work: Solve pages 111 and 112 in your book, problems No. 3, 11, 16, 18, 22, 34, 37, 41.