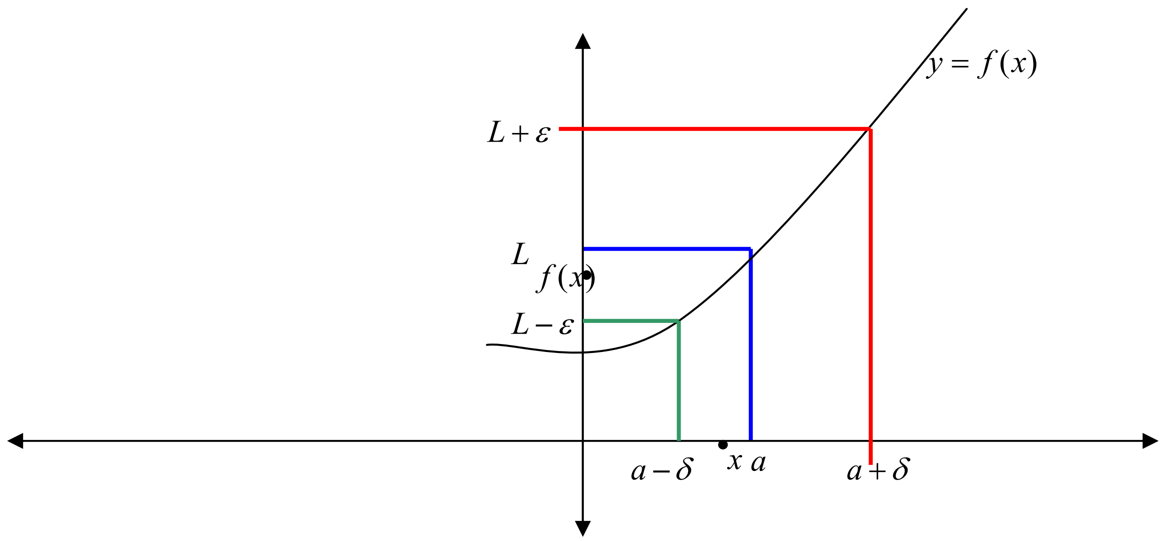
The precise definition of a limit

The following figure shows the interval contains x with center (a) and length small positive number δ on the x axis and an interval contains f(x) with center (L) and length small positive number ε on the y axis.



From the concept of limit we can use the above figure to have a definition of $\lim_{x\to a} f(x) = L$. The geometrical meaning of this limit means that f(x) is contained in the interval $(L-\varepsilon, L+\varepsilon)$ whenever (as long as) x is contained in the interval $(a-\delta, a+\delta)$. In other words the inequality $L-\varepsilon < f(x) < L+\varepsilon$ satisfies whenever the inequality $a-\delta < x < a+\delta$ satisfies. This implies that, the inequality $-\varepsilon < f(x)-L < \varepsilon$ satisfies whenever the inequality $-\delta < x-a < \delta$ satisfies. But, from our previous knowledge, we know that $-\varepsilon < f(x)-L < \varepsilon$ is equivalent to the absolute value notation $|f(x)-L| < \varepsilon$, and similarly the inequality $-\delta < x-a < \delta$ is the same with $|x-a| < \delta$. Now, have the following definition of the limit of a function:

Definition 1 ($\varepsilon-\delta$ definition of the limit):

We say that $\lim_{x\to a} f(x) = L$, If for every $\varepsilon > 0$ there exists $\delta > 0$, such that $|f(x)-L| < \varepsilon$ whenever $|x-a| < \delta$.

Another way of writing the definition of the limit of the function is given as:

We say that $\lim_{x\to a} f(x) = L$, If for every $\varepsilon > 0$ there exists $\delta > 0$, such that

If $|x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

Using the definition we conclude that $\lim_{x\to a} f(x) = L$ means that f(x) is very close to L by taking x very close to (a) (but not equal).

Example (1) prove that $\lim_{x\to 3}(4x-5) = 7$ using $\varepsilon - \delta$ definition of the limit.

Solution: we need to prove that for every $\varepsilon > 0$ there exists $\delta > 0$, such that $|(4x-5)-7| < \varepsilon$ whenever $|x-3| < \delta$??

Or we need to prove that $|4x - 12| < \varepsilon$ whenever $|x-3| < \delta$??

Or $4|x-3| < \varepsilon$ whenever $|x-3| < \delta$??

i.e., $|x-3| < \frac{\varepsilon}{4}$ whenever $|x-3| < \delta$??, This result gives us the relation between ε and δ, i.e., we can choose $\delta = \frac{\varepsilon}{4}$. Thus, we can adjust our proof as follows:

For given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{4}$, if $|x-3| < \delta$ then:

$$|f(x) - L| = |(4x-5)-7| = |4x-12| = 4|x-3| < 4\delta = (4)(\frac{\varepsilon}{4}) = \varepsilon .$$

Thus, $|f(x) - L| < \varepsilon$ whenever $|x-a| < \delta$. Using the definition, this implies that $\lim_{x\to 3}(4x-5) = 7$.

Example (2) prove that $\lim_{x\to -5}(4 - \frac{3x}{5}) = 7$ using $\varepsilon - \delta$ definition of the limit.

Solution: we need to prove that for every $\varepsilon > 0$ there exists $\delta > 0$, such that $\left|(4 - \frac{3x}{5}) - 7\right| < \varepsilon$ whenever $|x-(-5)| < \delta$??

Or we need to prove that $\left|\frac{-15}{5} - \frac{3x}{5}\right| < \varepsilon$ whenever $|x+5| < \delta$??

Or $\frac{3}{5}|x+5| < \varepsilon$ whenever $|x+5| < \delta$??

i.e., $|x+5| < \frac{5\varepsilon}{3}$ whenever $|x+5| < \delta$??, This result gives us the relation between ε and δ, ie., we can choose $\delta = \frac{5\varepsilon}{3}$. Thus, we can adjust our proof as follows:

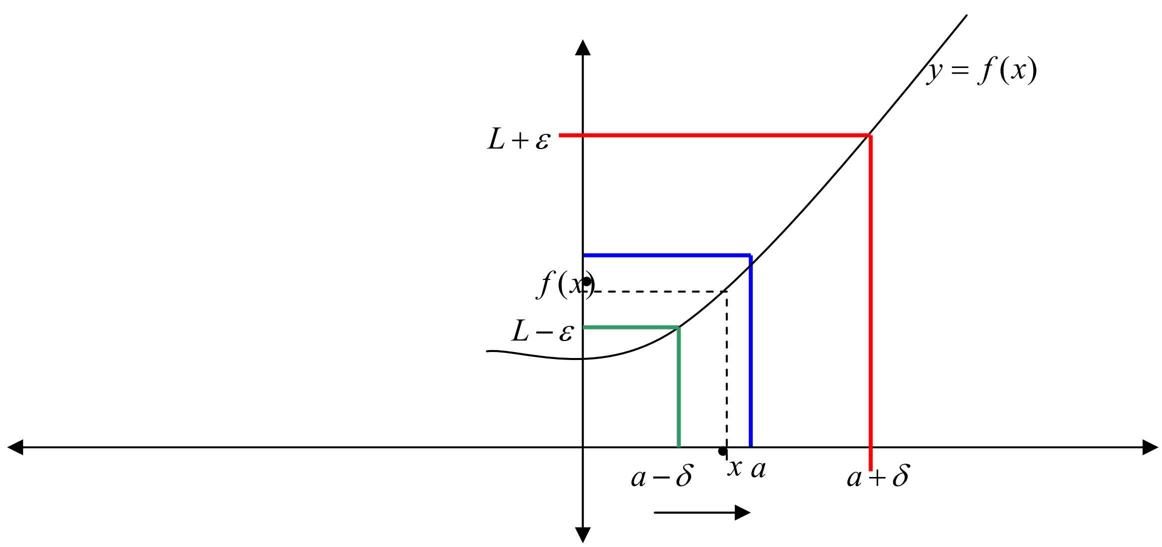
For given $\varepsilon > 0$, choose $\delta = \frac{5\varepsilon}{3}$, if $|x-(-5)| < \delta$ then:

$$|f(x) - L| = \left|(4 - \frac{3x}{5}) - 7\right| = \frac{3}{5}|x+5| < \frac{3}{5}\delta = (\frac{3}{5})(\frac{5\varepsilon}{3}) = \varepsilon.$$

Thus, $|f(x) - L| < \varepsilon$ whenever $|x-a| < \delta$. Using the definition, this implies that $\lim_{x \to -5}(4 - \frac{3x}{5}) = 7$.

One sided limits:

(1) <u>Left-hand limit</u>: the limit $\lim_{x \to a^-} f(x) = L$ means that f(x) approaches L as x < a (i.e, x approaches a from the left).

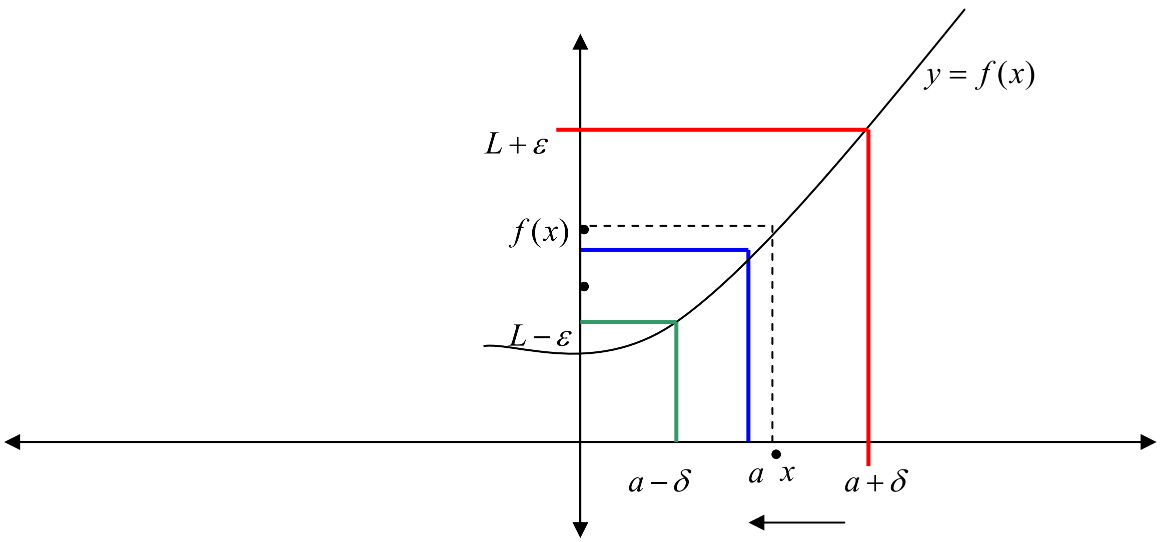


Using the above figure we can summarize the fact that f(x) approaches L as x < a (i.e, x approaches a from the left), by showing that f(x) is contained in the interval $(L-\varepsilon, L+\varepsilon)$ whenever (as long as) x satisfies the inequality $a - \delta < x < a$. As we have proceeded in the above definition, we have now the definition of the left hand limit:

Definition 2 ($\varepsilon - \delta$ definition of the left hand limit):

We say that $\lim_{x \to a^-} f(x) = L$, If for every $\varepsilon > 0$ there exists $\delta > 0$, such that $|f(x) - L| < \varepsilon$ whenever $a - \delta < x < a$.

(2) Right-hand limit: the limit $\lim_{x \to a^+} f(x) = L$ means that f(x) approaches L as x > a (i.e, x approaches a from the right).



Using the above figure we can summarize the fact that f(x) approaches L as x > a (i.e, x approaches a from the right), by showing that f(x) is contained in the interval $(L - \varepsilon, L + \varepsilon)$ whenever (as long as) x satisfies the inequality $a < x < a + \delta$. Now the definition of the right hand limit is given as follows:

Definition 3 ($\varepsilon - \delta$ definition of the right hand limit):

We say that $\lim_{x \to a^+} f(x) = L$, If for every $\varepsilon > 0$ there exists $\delta > 0$, such that $|f(x) - L| < \varepsilon$ whenever $a < x < a + \delta$.

Example (3) use the definition of the right hand limit to prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Solution: we need to prove that for every $\varepsilon > 0$ there exists $\delta > 0$, such that $|\sqrt{x} - 0| < \varepsilon$ whenever $0 < x < 0 + \delta$??

Or we need to prove that $\left|\sqrt{x}\right|^2 < \varepsilon^2$ whenever $0 < x < \delta$??

Or $0 < x < \varepsilon^2$ whenever $0 < x < \delta$??, This result gives us the relation between ε and δ, i.e., we can choose $\delta = \varepsilon^2$. Thus, we can adjust our proof as follows:

For given $\varepsilon > 0$, choose $\delta = \varepsilon^2$, such that if $0 < x < \delta$ then:

$\sqrt{x} < \sqrt{\delta}$ this implies that $\sqrt{x} < \sqrt{\varepsilon^2} = \varepsilon$,

$\therefore \left|\sqrt{x} - 0\right| < \varepsilon$

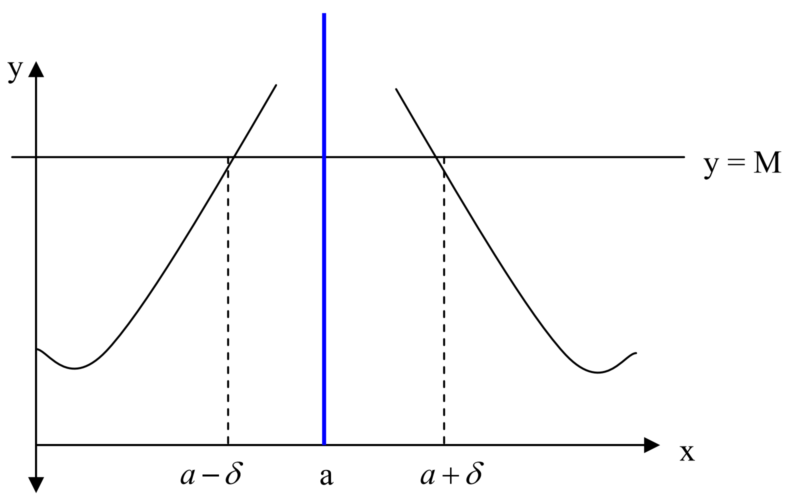
Thus, $|f(x) - 0| < \varepsilon$ whenever $0 < x < 0 + \delta$. Using the definition, this implies that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Infinite limits:

Definition 4

We say that $\lim_{x \to a} f(x) = \infty$, If for every positive number $M > 0$ there exists $\delta > 0$, such that $f(x) > M$ whenever $0 < |x - a| < \delta$.

The definition (4) can be understood easily using the following graph:

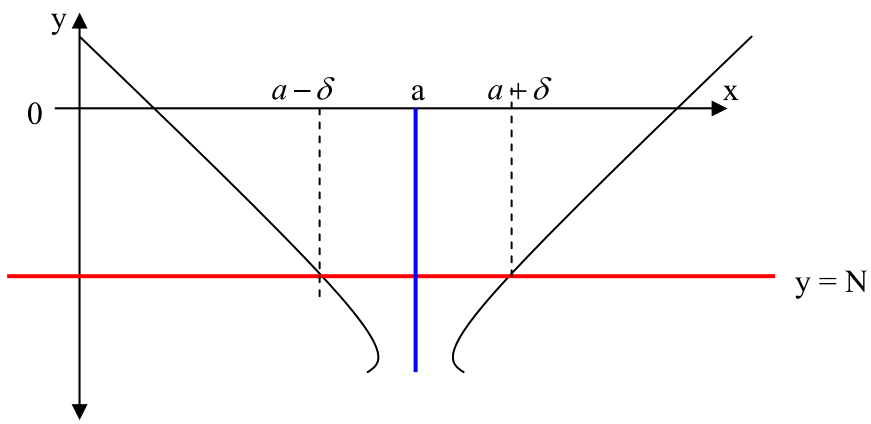


The figure shows that whatever a vertical line y = M exists and cut the graph of the function f(x) there exits a value of the function f(x) that <u>exceeds</u> this vertical line as long as x approaches to the point a.

Definition 5

We say that $\lim_{x \to a} f(x) = -\infty$, If for negative positive number N there exists $\delta > 0$, such that $f(x) < N$ whenever $0 < |x - a| < \delta$.

The definition (5) can be understood easily using the following graph:



The figure shows that whatever a vertical line y = N exists and cut the graph of the function f(x), there exits a value of the function f(x) that is <u>below</u> this vertical line as long as x approaches to the point a.

Important notification: You should <u>memorize</u> all these definitions from 1 to 5.

Home work:

use the $\varepsilon - \delta$ definition of the limit to prove the following limits:

- Problem number (15) in the book, $\lim_{x \to 1}(2x + 3) = 5$,
- Problem number (20) in the book, $\lim_{x \to 6}(\frac{x}{4} + 3) = \frac{9}{2}$,
- Problem number (24) in the book, $\lim_{x \to a} C = C$.