Linear approximation of a given function

We know that $f'(a) = \lim_{t \to a} \frac{f(x) - f(a)}{x - a}$, when x is very close to (a) we can omit the limit sign and the last equation can be rewritten as:

$f'(a) \approx \frac{f(x) - f(a)}{x - a}$, this implies that $f'(a)(x - a) \approx f(x) - f(a)$, and by solving this equation for f(x) we obtain:

$$f(x) \approx f(a) + f'(a)(x - a)$$

The last equation is called a linearization of the function of f(x) with respect to a fixed point $(a, f(a))$. The linearization is symbolized by $L(x)$.

Example (1) Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate $\sqrt{3.98}$ and $\sqrt{4.05}$.

Solution

Since $f(x) = \sqrt{x+3}$ then $f'(x) = \frac{1}{2\sqrt{x+3}}$ $\Rightarrow f(1) = \sqrt{1+3} = 2,$ $f'(1) = \frac{1}{2\sqrt{1+3}} = \frac{1}{4}$

and,

$$L(x) = f(a) + f'(a)(x - a)$$
$$= 2 + \frac{1}{4}(x - 1) = \frac{7}{4} + \frac{x}{4}$$

Now, $\sqrt{3.98} = \sqrt{3 + x} = f(x)$ $\Rightarrow x = 0.98$, since $f(x) \approx L(x)$ then

$\sqrt{3.98} = L(0.98) = \frac{7}{4} + \frac{0.98}{4} = 1.995$.

Similarly, $\sqrt{4.05} = L(1.05) = \frac{7}{4} + \frac{1.05}{4} = 2.0125$.

Example (2) Find the linearization of the function $f(x) = \sin x$ when x is very small.

Solution

When x is very small then $x \approx 0$, so we may choose the fixed point $a = 0$.

Then $f(a) = \sin 0 = 0$, and $f'(a) = \cos(0) = 1$.

Now, $\sin x \approx L(x) = f(a) + f'(a)(x - a) = 0 + 1(x - 0) = x$, i.e. when x is very small then $\sin x \approx x$

Example (3) Find the linearization of the function $f(x) = \cos x$ when x is very small.

Solution

When x is very small then $x \approx 0$, so we may choose the fixed point $a = 0$.

Then $f(a) = \cos 0 = 1$, and $f'(a) = -\sin(0) = 0$.

Now, $\cos x \approx L(x) = f(a) + f'(a)(x-a) = 1 + 0(x-0) = 1$, i.e. when x is very small then $\cos x \approx 1$

Example (4) Find an estimation of $\ln 1.07$

Solution

Let $f(x) = \ln(1+x)$, then we may choose $x = 0.07$ and the point (a) very close to (x), i.e. $x = 0$.

Therefore, $f(0) = \ln(1+0) = \ln(1) = 0$ and $f'(x) = \dfrac{1}{x+1} \Rightarrow f'(0) = \dfrac{1}{0+1} = 1$.

By approximating f(x) near the point a = 0, we obtain

$\ln(1+x) \approx L(x) = f(0) + f'(0)(x-0) = 0 + 1(x) = x$.

Now, $\ln(1.07) \approx L(0.07) = 0.07$.

Differentials

Let $y = f(x)$ be a differentiable function, then the differential of y is dy and is defined as:

$$dy = f'(x)\Delta x,$$

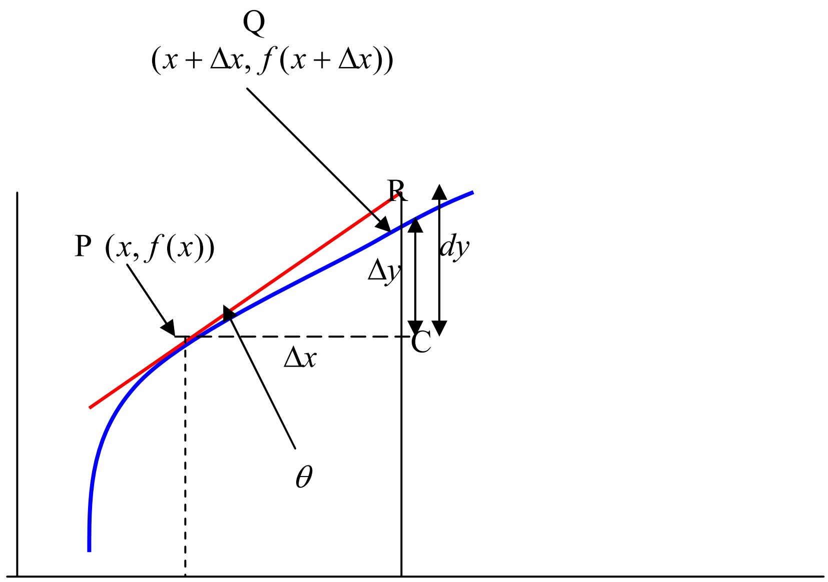
When $y = x$, $\Rightarrow f'(x) = 1$ then $dy = d(x) = dx = 1.\Delta x$, Thus, the differential of can be redefined as

$$dy = f'(x)dx.$$

The last equation indicates that the derivative of a function can be written as quotient of two differentials. Since both of dy and dx represents algebraic quantities we may write:

$$\frac{dx}{dy} = \frac{1}{f'(x)}.$$

The question now is what is the difference between dy and Δy?



From the figure $\Delta y = f(x+\Delta x) - f(x) = \overline{CQ}$

From the above figure $\tan\theta = \dfrac{\overline{CR}}{\overline{PC}} = \dfrac{\overline{CR}}{\Delta x} = \dfrac{\overline{CR}}{dx}$,

Since $\tan\theta$ equal to the slope of the tangent line to the curve at the point $(x, f(x))$, therefore

$\tan\theta = f'(x)$. So using the above relation, then $f'(x) = \dfrac{\overline{CR}}{dx}$ this implies that $\overline{CR} = f'(x)dx$

Finally we have $\overline{CR} = dy$.

Now, the difference between dy and Δy is clear (see the above figure), but when Δx is very small then $\Delta y \approx dx$.

Example (5) If $y = x^3 + x^2 - 2x$ and x changes from 2 to 2.05, compare between dy and Δy.

Solution

Let $x = 2$, as x changes from 2 to 2.05, then $\Delta x = dx = 0.05$

$f(2) = 9$ and $f(2.05) = 9.7176$, then

$\Delta y = f(x+\Delta x) - f(x) = f(2.05) - f(2) = 9.7176 - 9 = 0.7176$.

Also, $f'(x) = 3x^2 + 2x - 2$ $\quad f'(2) = 3(2)^2 + 2(2) - 2 = 14$.

Now, $dy = f'(2)dx = (14)(0.05) = 0.7$.

Remark from the above example we can see that $\Delta y = 0.7176$ is close to $dy = 0.7$, because we have the change in x is very small ($\Delta x = 0.05$).

Using differential for estimating numbers:

The linear approximation of the function $y = f(x)$ about the point $(a, f(a))$ is given by:

$$f(x) \approx f(a) + f'(a)(x-a)$$

Let $\Delta x = x - a$, $\because \Delta x$ is always equal to dx $\Rightarrow dx = x - a$ $\Rightarrow x = a + dx$. Substituting in the above equation we have:

$$f(a+dx) \approx f(a) + f'(a)dx,$$

And this is reduced to

$$f(a+dx) \approx f(a) + dy$$

The last equation is the equation of linearization of any nonlinear function f(x) using differentials.

Example (6) Use the differential to estimate $\sqrt{4.05}$.

Solution

Let $f(x) = \sqrt{x+3}$, by comparing $\sqrt{x+3}$ and $\sqrt{4.05}$, we have $x = 1.05$.

But $x = a + dx$, $a + dx = 1.05$. Hence we may choose $a = 1$ and $dx = 0.05$.

Thus, $f(1) = 2$ and $dy = f'(a)dx = f'(1)(0.05) = \frac{1}{4}(0.05) = 0.0125$

$\because$ $f(a+dx) \approx f(a) + dy$ is the linearization of f(x) about the point $(a, f(a))$.

Then, $\sqrt{4.05} = f(x) \approx f(1+dx) = f(1) + dy = 2 + 0.0125 = 2.0125$.

Example (7) Use the differential to estimate $\ln 1.07$.

Solution

Let $f(x) = \ln x$, by comparing $\ln x$ and $\ln 1.07$, we have $x = 1.07$.

But $x = a + dx$, $a + dx = 1.07$. Hence we may choose $a = 1$ and $dx = 0.07$.

Also, $f(1) = \ln(1) = 0$ and $\because f'(x) = \frac{1}{x}$ $\Rightarrow$ $f'(a) = f(1) = \frac{1}{1} = 1$.

Thus, $dy = f'(a)dx = f'(1)(0.07) = \frac{1}{1}(0.07) = 0.07$

$\because$ $f(a+dx) \approx f(a) + dy$ is the linearization of f(x) about the point $(a, f(a))$.

Then, $\ln(1.07) = f(x) \approx f(1+dx) = f(1) + dy = 0 + 0.07 = 0.07$.

i.e. $\ln(1.07) \approx 0.07$.